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MICHAEL C. BERG

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A WILEY-INTERSCIENCE PUBLICATION
John Wiley & Sons, Inc.
NEW YORK / CHICHESTER / BRISBANE / TORONTO / SINGAPORE / WEINHEIM

For ordering and customer service, call 1-800-CALL-WILEY

Library of Congress Cataloging in Publication Data is available.

ISBN 0-471-35830-4

10 9 8 7 6 5 4 3 2 1

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Preface

This is not a book for experts. This is not a book for raw beginners. It is, instead, an exposition of and commentary on a handful of sources, most of them classical by now and at least one of them notoriously austere. The material presented has been compiled as an aid to number theorists seeking to work on the analytic proof of reciprocity laws. This is a notorious affair, of course. The quadratic case is completely settled by Hecke [He23], and resettled by Weil [We64], but for $n > 2$ the matter is still open and ranks as one of the hardest open problems in the field. This book is written for those reckless few who are predisposed to enter this area of research at an early (but not too early) stage of their career, when they don't yet know any better and don't know a lot about the indicated specialized techniques either. The goal is to make entry into this field a little easier by explicitly delineating and comparing the three existing approaches to the (Fourier-) analytic proof of quadratic reciprocity which qualifies as a paradigm for the general case.

Of course, there is really only one Fourier-analytic proof of quadratic reciprocity, traced back to Cauchy's treatment of the classical absolute case and Hecke's treatment of the relative case. Hecke then posed the generalization problem mentioned above, which still remains open today. Not until some four decades after Hecke's work was the quadratic case cast into a form amenable to modern techniques, that is, unitary representation theory. This breakthrough, of paramount importance in the subject, was due to Weil, in a famous (and famously difficult) paper whose objective went considerably beyond a reformulation of the analytic proof of quadratic reciprocity. These two formulations, Hecke's and Weil's, stand in contrast one to the other in that Hecke's approach is classical while Weil's is anything but. But the book goes one step beyond even Weil's formulation and includes a treatment of Kubota's reformulation (and "algebraization") of Weil's work. This reformulation resulted in the context in which some dramatic subsequent achievements of the 1980s and the 1990s (by, e.g., Matsumoto, Kazhdan-Patterson) were carried out—while the proof of the general case remained, and remains, out of reach!

The goal of this book is to make it possible (if not easy) for the reader to proceed from here to the papers by Weil and by Kubota, as well as some of the aforementioned papers of very recent vintage. Beyond this I intend that this book provide incentive to the reader to tie up some loose ends which I leave untied, by going to the indicated supplementary sources to which I refer rather copiously. So, from this perspective, this book is something of a guide to the more peripheral literature, too.

Here are a few examples of these loose ends. In the discussion of Hecke's proof I take a lot of Fourier analysis for granted; the reader should read Hecke or the cited books by Lang [Lang70] and by Garrett [Ga90]. I also leave out consideration of the full general case, treating only the case of a totally real algebraic number field. While nothing dramatic happens in the former case as compared to the latter case, the reader, if he is a Fourier-analysis-in-number-theory rookie, should study the general case also.

And then, in the chapter on the Stone-Von Neumann Theorem, I give only the proof of the finite group case, while I quickly go on to use the theorem as it applies to locally compact groups. The reader is urged to carry out one (or both) of the following projects: learn the proof for the locally compact case; figure out without looking at § 4.9 how Weil's very intricate arguments succeed in achieving what the Stone-Von Neumann Theorem brings about without ever bringing the theorem in explicitly. In any case, § 4.9 gives a relatively detailed sketch of what goes into designing the proof for the more general case.

Rather than present an exhaustive treatment of the full quadratic case, then, I have purposely left the reader some tasks which should go a very long way toward providing him with a feeling for how this kind of number theory interacts with, for example, harmonic analysis on suitable topological groups, unitary representation theory, and so on. I presuppose, therefore, that the reader has a solid foundation along the indicated lines, that he is able to navigate easily through the indicated analysis, topology, and algebra (possibly with standard sources nearby), and that he is reasonably patient. This is dense material, but it is concerned with beautiful and deep mathematics.

Daring to look ahead and prophesy—which is in itself very reckless, of course—I claim that the resolution of the open problem Hecke posed in 1923 will indeed come about by means of the discovery of a generalization of Poisson's formula applicable to an as yet elusive generalization of Weil's Θ-functional fitted into the (algebraic) setting laid out by Kubota. This prophecy is proposed at the end of the book, in the context of some algebraic architecture of my own design (actually nothing more than a careful comparison between Weil's formalism and Kubota's). It is my hope that this will resonate with the reader as he proceeds onward.

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Acknowledgments

I should like to thank an anonymous referee who corrected a number of mistakes in an early version of this work. My wife dissected my prose with a sharp and sure scalpel; the extent to which this book is readable is largely credited to her. (Of course, the extent to which it is not is to be blamed only on me.) I wish also to express my gratitude to Cathy Herrera, my department's world-class secretary, who prepared the entire manuscript from my hand-written pages. Finally, thanks are due to my research assistants, Ryan Brown and Donna Morano, who slaved away arduously on the thankless job of indexing the book.

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Introduction

Aside from its unquestionable novelty, leading to its inclusion in most if not all introductory courses in number theory, the law of quadratic reciprocity stands out as one of the deepest facts of the theory of algebraic number fields. This was certainly already understood by Gauss, who in his lifetime gave six proofs of this beautiful theorem first conjectured by Euler. There are a number of good sources available treating this central theme of Gauss' arithmetical work, among which we recommend *Variationen über ein Zahlentheoretisches Thema von Carl Friedrich Gauss* [Pi78], and the indicated section of Scharlau-Opolka [SO84].

Gauss' work laid bare deep connections between at first glance rather disparate aspects of the behavior of rings of integers of algebraic number fields. Presently it became clear that the splitting of primes in quadratic extensions is completely governed by the fine structure of the Legendre symbol, that is, by quadratic reciprocity, and this set the stage for Gauss' work on the genera of quadratic forms.

If there is a tool *par excellence* in Gauss' armory for these arithmetical investigations it is surely the method of Gauss sums. Their relation to the Legendre symbol is fundamental: it is an easy exercise to show that Gauss sums transform very nicely under the Legendre symbol's natural action. It is a quick step from there to the formulation of quadratic reciprocity as an identity between so-called reciprocal Gauss sums. But where are the quadratic forms?