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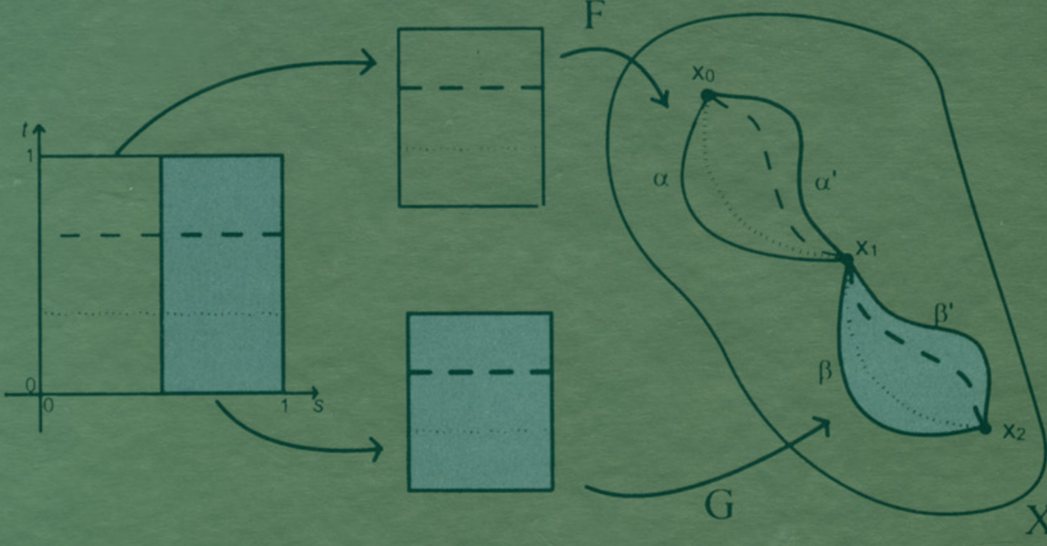

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Topology

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Topology

Point-Set and Geometric

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To Sue, with all my love

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FOREWORD

This book is intended as a text for a first course in topology, modeled on a junior/senior level course offered at John Carroll University. This particular course is required of all our mathematics majors, and is generally taken after the students have had a sophomore-level abstract algebra course. After teaching this course countless times over the last 20 years, and after endless discussions with my colleagues about it, I've become convinced that

1. An introductory topology course has to cover enough point-set topology to prepare the students adequately for ideas they're likely to encounter in analysis, geometry and other areas.

2. An introductory topology course can't do *just point-set topology, for two reasons: (a) this leaves out the more intuitive geometric aspects of the field in favor of the more classical point-set areas, ignoring the portions of topology most applicable to other fields of mathematics; and (b) often students completing a point-set topology course feel that they've learned a number of topics, but that the course (and field) lacks a "big theorem" that forms a capstone for their study.*

3. *An introductory topology course should start with the axiomatic definition of a topology on a set, rather than using metric spaces or the topology of subsets of* $\mathbb{R}^n$ *to "ease" the students into the subject, for three reasons: (a) the metric approach leaves out too many important examples (such as function spaces); (b) students who see only the metric approach, or who see this first, tend to develop more simplistic intuition than students who learn the more general definition first; and (c) the more general approach allows the student to learn how to write precise proofs in a brand new context, an invaluable experience for math majors. The important examples or* $\mathbb{R}$ *and* $\mathbb{R}^n$ *(in their usual topologies) should be presented only after the general definitions. Metric spaces should be covered (much) later.*

This text is designed with these beliefs in mind. It's intended to be covered in one semester by typical math majors (as opposed to some "one-semester" texts that are so dense that typical classes can cover only a small portion in one term). However, the text is intended to be quite rigorous, modeling for the students how one writes precise proofs. It covers the essentials of point-set topology in a relatively terse presentation, with lots of examples and motivation along the way. The introductory chapter attempts to explain what topology is in the context of what math majors have usually seen in their previous coursework. Along with the standard point-set topology topics (connected spaces, compact spaces, separation axioms and metric spaces), we include path-connectedness and a chapter on constructing spaces from other spaces (including products, quotients, etc.). Each chapter has a short introduction designed to motivate the ideas and place them into an appropriate context. Each section has a set of exercises, ranging in difficulty from easy to fairly challenging. The text culminates in two "capstone" chapters, each independent of the other, enabling instructors to choose which subject best suits their views and students. These capstone chapters are:

1. The Classification Theorem for Compact, Connected Surfaces.

2. Fundamental groups and classifying spaces, with applications.

As you can see, geometric and algebraic parts of topology are introduced in the two capstone chapters. This is not intended to diminish the importance of these topics, but rather to make the text more flexible.

ACKNOWLEDGMENTS

What appears in this text is strongly influenced by the topologists with whom I've worked or studied over the years. Among these, Bob Kolesar and Mark Mahowald deserve special thanks. Much of my understanding of the topics covered here is due to the classic books of James Munkres and William Massey. My thanks especially go to Sue Simonson Shick for her patient help with the graphics. Bob Kolesar, among others, deserves thanks for patient and careful proofreading. Finally, my heartfelt thanks go to my students, who have helped shape me as well as this book.

P. S.

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CHAPTER 1

INTRODUCTION: INTUITIVE TOPOLOGY

1.1 INTRODUCTION: INTUITIVE TOPOLOGY

What is topology? One would hope that a book entitled *Topology* would provide a simple answer to this question as a starting point. Unfortunately, it's not terribly easy to give a brief answer without first building up some background. Here's a first attempt: Topology is the study of the qualitative properties of certain objects (called *topological spaces*) that are invariant under certain kinds of transformation (called *continuous maps*), especially those properties which are invariant under a certain kind of equivalence (called a *homeomorphism*). This answer probably raises more questions than it answers, but it makes a good deal of sense when put into an appropriate mathematical context.

Euclidean geometry, for example, is a branch of mathematics that nearly everyone has spent some time studying. If asked for a short definition of Euclidean geometry, most students would have a difficult time distilling a year-long high school course into a single sentence. However, there is a very simple theme that unites nearly all of the varied topics studied in high school geometry – one considers the properties of certain objects (triangles, squares and other planar figures) that are preserved under congruence or similarity. Precisely, we consider certain transformations of the plane, called *Euclidean transformations*: rotations (of the coordinate axes by any angle), reflections (across any line in the plane) and translations (moving the "origin" of the coordinate system to any point in the plane), and transformations obtained as some combination of these. Two triangles are said to be congruent if one can be transformed into the other under some Euclidean transformation. If we add "rescalings" (scalar multiplication of the plane by any positive number) to our list of possible transformations, we can define the concept of similar triangles in much the same way. So our short definition of Euclidean geometry is as follows: geometry is the study of the properties of planar figures that are invariant under Euclidean transformations. We'll look at a similar explanation of topology shortly.

This idea ties in very nicely to another area of mathematics that most students have studied before encountering topology – abstract algebra. The set of Euclidean transformations of the plane forms a *group* under the operation of composition. In fact, Felix Klein defined a "geometry on a space X" to be the study of the properties of subsets of X that are invariant under the action of some transformation group G. [1]

The various areas of study lumped together as abstract algebra provide another important analogy for understanding what topology is all about. In group theory, for example, one studies groups and group homomorphisms. (See Section 2.6 for a primer on groups.) Precisely, a group is a set G with an operation, $\bullet$, satisfying three properties: $\bullet$ is associative ($(g_1 \bullet g_2) \bullet g_3 = g_1 \bullet (g_2 \bullet g_3)$ for all $g_1, g_2, g_3 \in G$); G contains an identity for the operation $\bullet$ (there exists $e \in G$ such that $g \bullet e = g = e \bullet g$ for every $g \in G$) and each element of G has an inverse in G (for every $g \in G$, there exists $g^{-1} \in G$ such that $g \bullet g^{-1} = e = g^{-1} \bullet g$). For example, the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ forms a group under $+$, but not under multiplication. A group homomorphism is a function $f : G \to H$ from a group G to a group H that respects the group structures: $f(g_1 \bullet_G g_2) = f(g_1) \bullet_H f(g_2)$ for every $g_1, g_2 \in G$, where $\bullet_G$ indicates the operation in G and $\bullet_H$ the operation in H. Group theory can be summarized as the study of the properties of groups that are preserved under group homomorphisms, especially those preserved by isomorphisms (equivalences between groups). For example, the property of being *abelian* [2] (or commutative: $g_1 \bullet g_2 = g_2 \bullet g_1$ for every $g_1, g_2 \in G$) is preserved by group homomorphisms (both trivial and nontrivial). So if G is abelian and H is not, then G cannot be isomorphic to H. Thus, the cyclic group of order 6, $\mathbb{Z}_{/6}$, cannot be isomorphic to the dihedral group on three letters, D_3, despite the fact that they both have six elements. This idea shows up in other areas of algebra; including semigroup theory, ring theory and field

[1] Felix Klein's (1849–1925) Erlangen program (named after the Universitat Erlange, where he held his chair) defined the concept of geometry entirely in terms of abstract algebra, a notable departure from previous mainstream mathematical thought.

theory. In each case, we have a set plus some other structure, and we study properties preserved by functions that respect this extra structure.

The field of topology works similarly. The objects we study are topological spaces, where a *topological spaceis* a set X with an extra structure, called a *topology* on X, which has to do with how "close" points in X are to each other. A function between spaces that respects these structures is called a *continuous function* or *continuous map*, with an equivalence referred to as a *homeomorphism*. Examples of topological spaces that are commonly studied are curves in $\mathbb{R}^2$ or $\mathbb{R}^3$, surfaces in $\mathbb{R}^3$ or $\mathbb{R}^4$, and spaces of functions between surfaces.

It's rather complicated to give a description of what a topology on a set X entails, but it's much easier to give an intuitive idea of what is meant by a continuous function between spaces. For subsets of $\mathbb{R}^n$, a Euclidean transformation is any composition of rotations (of the coordinate axes by any angle), reflections [across any (codimension 1) hyperplane (or subvector space) in $\mathbb{R}^n$] and translations (moving the "origin" of the coordinate system to any point in $\mathbb{R}^n$). Subsets of $\mathbb{R}^n$ are congruent if one can be transformed into the other by a Euclidean transformation. We add "rescalings" (scalar multiplication of $\mathbb{R}^n$ by any positive number) to the list of Euclidean transformations to define when two sets are similar. These concepts are generalized further in defining topological transformations. We add to the list of Euclidean transformations some new ones; we allow any transformations that involve bending or stretching, but not tearing. For example, an ellipse with eccentricity $e < 1$ is not geometrically equivalent to a circle, because such an ellipse has nonconstant curvature, while the circle's curvature is constant. The two planar curves are topologically equivalent (homeomorphic), however, because a composition of Euclidean transformations and bending/stretching can change one into the other. So, while curvature is a nice geometric property (preserved by Euclidean transfomations), it is not a topological property.

Topological properties are more "qualitative" than curvature. Examples of topological properties can be explained only intuitively at this point, but some of the central ideas can be made clear. *Connectedness* is perhaps the easiest topological property to look at. A topological space X is said to be connected if X is in a single "continuous" piece. For example, the two planar figures illustrated here (an "8" and an "81")

[2] The use of the term *abelian* for commutative groups is in honor of Nils H. Abel (1802–1828), a Danish mathematician whose work on the unsolvability of the quintic polynomial was part of a movement in mathematics that helped start the field of group theory. Abel's work is closely tied into that of Evariste Galois, who also died quite young, but in more romantic circumstances (reputedly in a duel over a barmaid, rather than by tuberculosis). These two exemplify the maxim that a mathematician's greatest work is most often accomplished before age 30. We'll see some striking counterexamples to this statement later.

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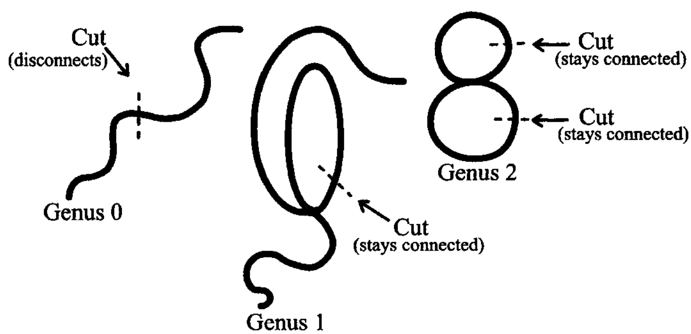
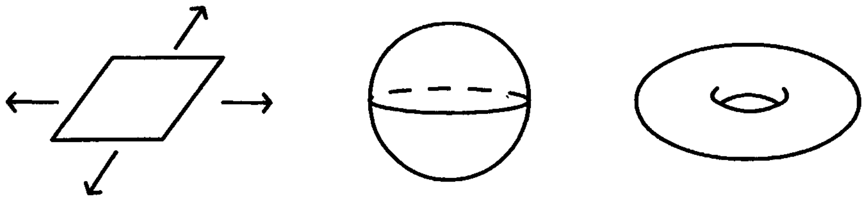
cannot be homeomorphic because one is connected, the other disconnected. A second topological property, closely related to connectivity, is that of *genus*, which is quite easy to explain in the setting of planar curves. For a given connected planar curve X, a "cutpoint" of X is a point $x \in X$ such that X becomes disconnected if the point x is removed. A planar curve X is said to have genus 0 if every point of X (except the endpoints, if X has any) is a cut point. More generally, a connected planar curve X has genus n if some set of n points removed from X leaves it connected, but every set of $n + 1$ "cuts" disconnects X (again, avoiding endpoints). Here are some examples of planar curves and their genera (plural of genus), with some cuts shown:

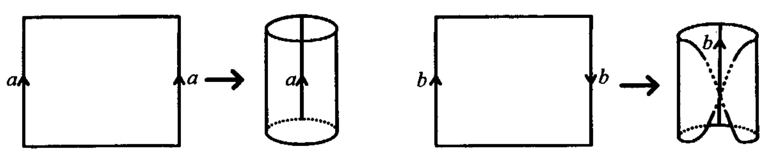


Finally, a very interesting example of a topological property is that of *orientability* in the context of surfaces. A *surface* is a topological space X that is "locally homeomorphic" to the open disk in $\mathbb{R}^2$ (meaning that each point of X has a "neighborhood" that is topologically equivalent to $B^2 = \{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$). Some simple examples of surfaces are:



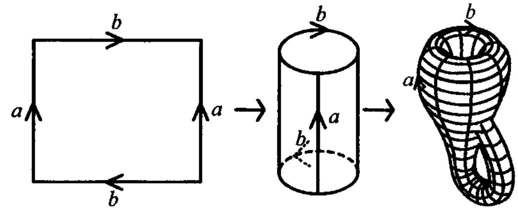
$\mathbb{R}^2$ (plane) S^2 (sphere) T^2 (torus)

These are all *oriented* surfaces, in that there is a consistently defined "outward" direction on each. Precisely, at each point $x \in X$, we have a neighborhood equivalent to the planar disk, so we have two sides, which we'll call "inward" and "outward." If these directions can be defined *consistently* for each $x \in X$, we say that X is oriented. The key word here is "consistently"; if one takes any simple closed curve on the surface (one forming a simple loop), the "outward" direction must be preserved throughout the curve. It may seem counterintuitive, but it's quite easy to construct nonorientable surfaces. For example, a Moebius band is built by identifying edges of a rectangle as indicated in the following diagrams, which construct a cylinder and a Moebius band:



These are not surfaces (since each point on the edge has no neighborhood equivalent to the planar disk) but they are examples of "surfaces with boundary." Note that the cylinder has a consistent orientation, as one can check by looking at the curve that gives the "equator." However, the Moebius band is a "one- sided surface," without a consistently definable outward direction, as one can see by tracing a path along the equator of the band. An example of a true surface that is nonorientable is the *Klein bottle*[3] :

[3]Yes, the same Felix Klein as before. The term "Klein bottle" is used almost universally for this beast, but seems to have been an unintended name. The original German term apparently should have been translated



So the Klein bottle cannot be homeomorphic to any oriented surface, despite its similarities to the torus, for example. This is a very good example of how one uses topological properties to tell spaces apart. We'll explore the Klein bottle in more detail in Section 10.2.

In some sense, the goal of topology should be to come up with an exhaustive set of topological properties that would allow one to "classify" all topological spaces. Such a classification program has been completed, for example, in the setting of finite simple groups, where we recall that a simple group is a group with no nontrivial normal subgroups. The Classification Theorem for Finite Simple Groups was completed around 1980, culminating decades of work by scores of mathematicians. Given a finite simple group, one can identify it, up to isomorphism, by knowing its order (the number of elements in the group) and its character table (a complicated invariant, defined by representation theory). Such a classification theorem is nowhere in sight for topological spaces in general, but we do have such a result for connected, compact surfaces (where the term *compact* will be explained in excruciating detail in Chapter 7). The statement and proof of this theorem will occupy us in Chapter 10.

Exercises

Consider the letters of the English alphabet drawn as follows:

A B C D E F G H I J K L M N O P Q R

S T U V W X Y Z

1. Classify the letters by homeomorphism; that is, group the letters together so that all the letters in each set are homeomorphic to each other. You will need a total of nine sets.

as "Klein surface." An inaccurate translation rendered this as "Klein bottle," which has stuck since, for obvious reasons. The surface is often described as a bottle which can hold no water. For more concrete examples, see the website of the Acme Klein Bottle Corporation, `www.kleinbottle.com`.

2. Classify the letters by genus; that is, group the letters together so that all the letters in a set have the same genus. You will need a total of three sets.

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CHAPTER 2

BACKGROUND ON SETS AND FUNCTIONS

2.1 SETS

What is a set? In a mathematics course, one is expected to supply precise definitions of the terms one uses, so it seems reasonable to ask for a precise definition of the term "set." However, a class of mathematics majors, when asked to come up with such a definition, typically respond with something like "A set is a collection of elements." When asked to define the term "collection," many reply "a group" or "a gathering." When pressed to be precise, most eventually resort to "you know, a set!"

Given that much of modern mathematics is couched in terms of sets, this seems a rather unfortunate state of affairs. However, the last answer supplied ("you know, a set!") is actually pretty close to being exactly correct. When I'm in a playful mood, and my students ask me for a definition of the term "set," I respond "I can't tell you." Eventually, I explain that we *can't* define the term "set", since it's going to be the basis for nearly all of the definitions to follow. More precisely, the term "set" is part of the starting point for mathematics: a commonly understood notion that we can't define. Similarly, in Euclidean geometry, we cannot define the notions of "point," "line," "plane" or "solid"; we must assume the reader/student has an understanding of this basic idea, and we proceed from there. We can *describe* these ideas, but we can't define them precisely.

We can illustrate this fundamental notion with lots of examples:

- $A = \{a, b, c, \ldots, z\}$, the set of letters in the English alphabet.
- $\emptyset = \{\}$, the empty set.
- $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of natural numbers.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of integers. (The notation comes the German word *zahlen*, or number.)
- $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$, the set of rational numbers. This example uses two nice notations: (1) $x \in X$ is pronounced "x is an element of the set X," and (2) we use what's usually called "set-builder notation" to specify a set whose elements would be difficult or impossible to list. Here, we pronounce $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ as "the set of all elements $\frac{m}{n}$, with the property that m is any integer and n is any natural number." Note that we could just as easily have specified that $m, n \in \mathbb{Z}$, with $n \neq 0$.
- $\mathbb{R}$ = the set of all real numbers. We can specify this set, at least intuitively, by saying that $\mathbb{R}$ is the set of all decimals. We'll deal quite extensively with this set later.

We recall that two sets A and B are equal (denoted $A = B$) if they have exactly the same elements. A set C is a subset of A (denoted $C \subset A$) if every element of C is also an element of A. Hence, $A = B$ if and only if $A \subset B$ and $B \subset A$. This observation makes clear why the usual strategy employed in proving two sets are equal is called "double inclusion." Note that our notation for subset ($C \subset A$) does not imply that $C \neq A$. We'll use the notation $C \underset{\neq}{\subset} A$ to indicate that $C \subset A$ and $C \neq A$ (pronounced "C is a proper subset of A"). [*Warning*: Many texts use the convention that $C \subseteq A$ to indicate that C is a (possibly nonproper) subset of A and use $C \subset A$ to indicate inequality (proper inclusion). Our convention is less ambiguous and possibly more standard these days.]

We denote the *complement* (or set difference) of two sets as $X \setminus A = \{x \in X : x \notin A\}$. For example, $\mathbb{Z} \setminus \mathbb{N} = \{\ldots, -3, -2, -1, 0\}$. Note that this is defined even when $A \not\subset X$. For example, let $E = \{1, 2, 3, 4\}$ and $F = \{3, 4, 5, 6\}$. Then $E \setminus F = \{1, 2\}$ and $F \setminus E = \{5, 6\}$. If one is dealing only with subsets of a particular set X, it might be tempting to shorten the notation $X \setminus A$ to A^c, or something like this. However, such notations make sense only in the context of a particular set X, and the notation A^c is getting dangerously close to that of some sort of "set of all sets." We'll be clearer about why this is something to avoid below.

We have two familiar operations on sets, namely, union and intersection. We'll define these, for now, as "binary" operations, but we'll work in more generality shortly. Given sets A and B, we define $A \cup B = \{x : x \in A \text{ or } x \in B\}$, the union of A with B. We define the intersection of A with B as $A \cap B = \{x : x \in A \text{ and } x \in B\}$. For example, for the sets E and F as defined in the previous paragraph, $E \cap F = \{3, 4\}$ and $E \cup F = \{1, 2, 3, 4, 5, 6\}$. We recall that two sets A and B are said to be *disjoint* if $A \cap B = \emptyset$.

We note (a polite way of saying that we won't prove it!) that the operations of $\cup$ and $\cap$ are both commutative and both associative; that is, $A \cup B = B \cup A$ and $A \cap B = B \cap A$, and also $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$ for any sets A, B and C. Further, the two operations distribute across each other, as we hope: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. We'll prove the first of these identities.

Theorem 2.1.1. *For any sets A, B and C, we have*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof: To show that two sets are equal, the usual approach is to show that they are subsets of each other (a "double inclusion" proof). First, we'll show the left-hand side of the equation is a subset of the right-hand side:
$\subset$) Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$; that is, $x \in A$ and ($x \in B$ or $x \in C$). So we see that ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$), another way of saying that $x \in (A \cap B) \cup (A \cap C)$.

The other inclusion:
$\supset$) Let $x \in (A \cap B) \cup (A \cap C)$. Then ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). So it follows that $x \in A$ and ($x \in B$ or $x \in C$), which is another way of saying that $x \in A \cap (B \cup C)$. □

Ↄ[1] Note that for any set Y, $Y \cup \emptyset = Y$. One might ask, then, whether the "collection" of sets is a group under the operation of union. (See Section 2.6 for a

[1] We're using this symbol Ↄ as a variant on the roadsign that indicates "dangerous curves ahead," following the tradition of Nicholas Bourbaki. This indicates that the reader should use particular care, because the ideas to follow contain more complexity that one might expect. By the way, Nicholas Bourbaki is not really a person. Bourbaki is the collective pseudonym for a group of (mostly French) mathematicians who began to meet in the 1930s, with the goal of putting much of modern mathematics on a firm logical foundation. The group has published a large number of books, all of which proceed in a very formal definition–theorem–proof format, which give precise proofs of most of the important theorems in algebra, analysis, etc. Ralph Boas, as editor of *Mathematical Reviews*, wrote the annual review of mathematics for *Encyclopedia Brittanica* in the 1950s, and wrote about the Bourbaki program. Ralph later received an

primer on group theory.) Certainly the empty set acts as an identity for this operation. The operation is commutative (since $A \cup B = B \cup A$ for any sets A and B), so the collection, *if* it is a group, is abelian. We observed above that the operation $\cup$ is associative. However, it's easy to see that we have no inverses for nonempty sets. Precisely, if a set A is nonempty, we can never find a set A^{-1} with $A \cup A^{-1} = \emptyset$, since $A \subset A \cup B$ for any set B. So the "collection"[2] of all sets is not a group under $\cup$.

If we're in the situation of dealing with sets that are all subsets of some particular set X, then we have a similar situation: $A \cap X = A$ whenever $A \subset X$. So for subsets of X, the set X acts as an identity under the operation $\cap$. This situation occurs often enough that we adopt the following notation.

Definition 2.1.1. *For a given set X, the power set of X [denoted by $\mathcal{P}(X)$] is the set of all subsets of X.*

irate letter, purportedly from Bourbaki, beginning "You miserable worm, how dare you say that I do not exist?" About this time, the *American Mathematical Society* received an application for membership from Bourbaki, which was rejected on the grounds that the application was for an individual, rather than an institutional membership. The "feud" continued for a while, with members of Bourbaki floating a rumor that Boas did not exist, but was rather a collective pseudonym for the editors of *Mathematical Reviews*. This rumor was perhaps somewhat believable, given how productive Ralph was at that time. The *Seminaire Bourbaki* continues in Paris to this day, bringing rigor to new areas of mathematics (and providing, I'm told, one of the most intimidating seminar audiences around). In any case, we'll use the "dangerous curves ahead" warning from the Bourbaki collective without apology. The details of the Boas/Bourbaki dealings can be found in Ref. [3].

[2]We're avoiding the term "set of all sets" because this is a very problematic term, exposing the difficulty that arises even when dealing with such a simple concept as set. The issue here is what's known as Russell's paradox, which arises when one considers such objects as the $S =$"set of all sets." If such a set S exists, then $S \in S$. (i.e. S is an element of itself.) This property leads to a paradox in the following way: we'll say that a set A is said to be "nice" if $A \notin A$. For example, the set $\mathbb{N}$ is "nice." Let R denote the set of all "nice" sets, which is well defined if we allow the existence of the set S. Now we ask, is R "nice"? If R is "nice," then $R \in R$, since R is the set of all "nice" sets, but this implies R is not "nice." Similarly, if R is not "nice," then $R \notin R$, by the definition of R, so R fulfills the criterion to be in R! So R is "nice" if and only if R is not "nice"!!! We'll stay away from this paradox by scrupulously avoiding any constructions involving a "universal" set.

We can ask the same question: Does the operation $\cap$ give $\mathcal{P}(X)$ the structure of a group? Again, we have an identity and associativity. Do we have inverses? Given $A \subset X$, does there exist a set A^{-1} with $A \cap A^{-1} = X$? The answer is easily seen to be no, since $A \cap B \subset A$ for any set B, so any *proper* subset of X has no inverse.

The familiar set operations of union and intersection relate nicely to complements in what have become known as DeMorgan's[3]laws.

Theorem 2.1.2. *(DeMorgan's laws) For A, B and X any sets, we have*

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$$

and

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B).$$

We will prove one of these, leaving the other for an exercise.
Proof: $\subset$) Let $x \in X \setminus (A \cup B)$. Then $x \in X$ and $x \notin (A \cup B)$; that is, $x \in X$ and x is in neither A nor B. This means, then, that $x \in (X \setminus A)$ and $x \in (X \setminus B)$, so that $x \in (X \setminus A) \cap (X \setminus B)$, as we wish.
$\supset$) Let $x \in (X \setminus A) \cap (X \setminus B)$. So $x \in (X \setminus A)$ and $x \in (X \setminus B)$. We see, then, that $x \in X$, and that x is in neither A nor B, so that $x \in X \setminus (A \cup B)$. Then $x \in X$ and $x \notin (A \cup B)$, as we had hoped. □

Another useful set construction is the Cartesian product ; for sets A and B, the Cartesian product is defined by

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\},$$

where (a, b) is our notation for an ordered pair of points. [Unfortunately, this is also the usual notation for "open" intervals in the set $\mathbb{R}$. The meaning of (a, b) should be clear in context.] Cartesian products interact nicely with unions and intersections:

$$A \times (B \cap C) = (A \times B) \cap (A \times C),$$

and so on. The product of k sets is defined analogously:

$$X_1 \times X_2 \times \cdots \times X_k := \{(x_1, x_2, \ldots, x_k) : x_i \in X_i\},$$

[3] Augustus DeMorgan (1806–1871) helped found the British Association for the Advancement of Science. Aside from the laws stated above, he is best known, perhaps, for bringing the famous four-color problem to the attention of the British Royal Society.

the set of ordered k-tuples. We often use the notation $\prod_{i=1}^{k} X_i$ to denote this product.

Finally, we extend the definitions of union and intersection from binary operations to operations on collections of sets. First, an *indexed collection* of sets is a set $\mathcal{C}$ which has as its elements sets, each labeled by an element of an "indexing set" Λ. More precisely, $\mathcal{C} = \{A_\alpha : \alpha \in \Lambda\} = \{A_\alpha\}_{\alpha \in \Lambda}$. This is a nice shorthand notation for dealing with collections of sets that are too complicated to list out. For example, let $B_n = [-n, n]$ for each $n \in \mathbb{N}$, a "nested" sequence of closed intervals. This specifies an indexed collection $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}} = \{[-n, n]\}_{n \in \mathbb{N}}$.

We define the union of an indexed collection $\mathcal{C}$ as

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \{x : x \in A_\beta \text{ for some } \beta \in \Lambda\}.$$

Sometimes we'll denote this set as $\bigcup \mathcal{C}$ as a shorthand notation. For our example $\mathcal{B}$, we see that

$$\bigcup \mathcal{B} = \bigcup_{n \in \mathbb{N}} [-n, n] = \mathbb{R},$$

since every real number is an element of $[-k, k]$ for some $k \in \mathbb{N}$.

Similarly, we define the intersection of an indexed collection $\mathcal{C}$ as

$$\bigcap_{\alpha \in \Lambda} A_\alpha = \{x : x \in A_\beta \text{ for all } \beta \in \Lambda\},$$

with shorthand notation $\bigcap \mathcal{C}$. For our example $\mathcal{B}$, we see that

$$\bigcap \mathcal{B} = \bigcap_{n \in \mathbb{N}} [-n, n] = [-1, 1],$$

since every real number x with $|x| > 1$ fails to be in $[-n, n]$ for $n = 1$.

Note that an indexed collection $\{A_\alpha : \alpha \in \Lambda\}$ can be considerably more complicated than just a sequence of sets, such as $\{B_n : n \in \mathbb{N}\} = \{B_1, B_2, B_3, \ldots\}$. For example, the usual open interval $(0, 1) \subset \mathbb{R}$ can be written as a union of an indexed collection: $(0, 1) = \bigcup_{x \in (0,1)} \{x\}$. This indexed collection is far more complex than a mere sequence of sets (see Theorem 2.5.4).

DeMorgan's laws also hold true for more general unions.

Theorem 2.1.3. *(DeMorgan's laws) For X any set and for any indexed collection $\{A_\alpha : \alpha \in \Lambda\}$ of sets, we have*

$$X \setminus \bigcup_{\alpha \in \Lambda} A_\alpha = \bigcap_{\alpha \in \Lambda} (X \setminus A_\alpha)$$