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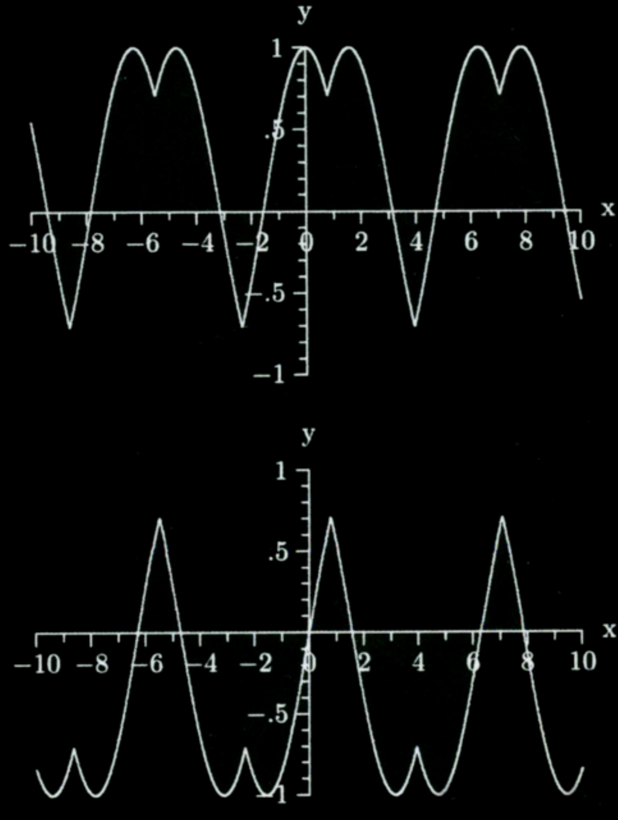

RICHARD J. ROSSI

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Theorems, Corollaries, Lemmas, and Methods of Proof

Richard J. Rossi
Department of Mathematical Sciences
Montana Tech
Butte, MT

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To my parents, my wife Debbie, and in memory of Maggie and Bayes

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Preface

I have written this textbook to help students who are studying mathematics make the transition from the calculus courses to the typical advanced core courses found in an undergraduate math program. Specifically, this book has been written to prepare students for rigorous mathematical reasoning of junior/senior-level courses on advanced calculus, real analysis, and modern algebra. Furthermore, in writing this book it is my hope that students taking a course from this textbook will begin to appreciate the beauty of the axiomatic structure of modern mathematics.

The topics chosen for this book were chosen for pedagogical reasons and have been tried, tested, and adjusted over the last 12 years of teaching a course on "methods of proof." In particular, the following topics are presented in this text.

- Chapter 1 provides an introduction to the axiomatic nature of modern mathematics, key terminology, and commonly used symbols.
- Chapter 2 presents an introduction to symbolic logic and is used to help the student understand why the methods of proof by contrapositive and proof by contradiction are valid methods of proof in Chapter 3.
- Chapter 3 discusses the method of forward direct proof, proof by contrapositive, and proof by contradiction. Also included in this chapter are specialized proofs for uniqueness and existence theorems, the methods of mathematical induction, proof by cases, proofs of biconditional theorems, and disproving a conjecture by using a counterexample.
- Chapter 4 provides a gentle introduction to numbers and number theory. Specifically, this chapter includes topics on binary operators, the natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers, properties of numbers, divisibility, prime numbers, and recursively defined numbers.
- Chapter 5 introduces the students to real analysis through the study of sequences and convergence, limits of real-valued functions, continuity, and differentiability. This chapter also introduces the students to convergence proofs of the ϵ-N and ϵ-δ forms.
- Chapter 6 introduces the students to sets and set theory, indexed families of sets, countable and uncountable sets, and group theory.

This book is not meant to cover the foundations of mathematics; therefore, topics such as relations, equivalence classes, and functions as relations have not been included. Furthermore, this text is not meant to be a book on discrete mathematics, and thus topics such as combinatorics and graph theory have not been included. The topics and the ordering of their presentation have been chosen for purely pedagogical reasons.

It is also my experience that the order of presentation is appropriate for the nurture and development of the student's confidence and mathematical maturity. These topics also provide the student with the necessary mathematical tools required to succeed in advanced math courses such as advanced calculus, modern algebra, number theory, and real analysis.

Three special features of this book are (1) a basic discussion of the axiomatic nature of modern mathematics, (2) presentation of algorithms for several different types of proofs, and (3) the idea that scratchwork must occur as part of the proof process. In Chapters 1 and 3, the basic structure of modern mathematics is discussed and each of the key components of modern mathematics is defined. In particular, the following terms are defined and examples of each are presented: definition, axiom, conjecture, proof, theorem, corollary, and lemma.

Throughout the text, algorithms are given providing the students with an outline for attacking a particular type of proof. It is my experience that proving a mathematical result is a very difficult skill for an undergraduate math student to master. For this reason, I have provided the students with a clear approach to attack several different types of proof. In particular, algorithms are provided for forward direct proofs, proof by contrapositive, proof by contradiction, mathematical induction, uniqueness proofs, existence proofs, proof by cases, closure proofs, convergence proofs for both sequences and limits of functions, element chasing proofs, and group theoretical proofs. These algorithms are not intended to present proofs in a cookbook fashion, rather, these algorithms are presented as guides for the student to use when faced with the problem of proving a theorem.

Another distinctive feature of this book is the idea of scratchwork. It is important to emphasize to the students that proving a mathematical result is unlike any problem they have encountered in their previous algebra and calculus classes. Furthermore, it is unlikely that the typical sophomore math student will be able to quickly and easily prove most of the problems in this text. Thus, I emphasize that the process of proving a theorem generally involves creative work other than that presented in the proofs included in this text. My goal is to convince the students to do their scratchwork and creative thinking as a first step in their attempts to prove a theorem; once they are satisfied that their scratchwork successfully demonstrates the truth of the theorem, they can then proceed to begin writing their proof up in a clear and concise fashion. Throughout the text there are several theorems whose proof will be preceded by my scratchwork in an attempt to get the student thinking about the thought processes that went into developing the actual proof.

Numerous exercises have been included in each chapter of this text. I believe that the exercises accompanying this text do indeed cover a wide range of topics and levels of difficulty. I believe that the successful completion of these exercises will help the student gain the confidence necessary to be successful in junior/senior-level mathematics courses, which, of course, is the goal of this book.

When teaching from this book I have used it for a one-semester transition course by covering Chapters 1–3, and parts of Chapters 4–6; for a two-semester sequence I cover Chapters 1–4 in the first semester and Chapters 5 and 6 in the second semester. While I have not taught a course from this book on the quarter system, I believe that Chapters 1–3, Sections 4.1 and 4.2 of Chapter 4, and Sections 5.1 and 5.2 of Chapter 5 would make a suitable course to be taught in a single quarter; for a two-quarter sequence, Chapters 1–3 and Sections 4.1 and 4.2 could be covered in the first quarter with the remainder of the book left for the second quarter. However, there are many different ways to teach from this book, and I leave that to the discretion and goals of the particular instructor.

I am grateful to a number of friends, colleagues, and students for their help and motivation during the writing of this book. I am especially indebted to Lloyd Gavin and Dan Brunk, two very inspirational advisors from whom I learned so much and the two people who are indirectly responsible for this book. A great deal of my motivation for writing this book came from long discussions of modern mathematics with two great colleagues, Steve Cherry and Dennis Haley. Special thanks go to Susan Patton, VCAAR at Montana Tech, for supporting a sabbatical to work on this book. Finally, I wish to thank the following individuals who have also contributed in one way or another to this book: Ray Carroll, David Ruppert, Fred Ramsey, Jay Devore, Scott Lewis, Erin Esp, Celeste McGregor, Michelle Johnson, Russ Akers, and Donielle Biers.

Finally, it was my intent to write a book that introduces the students to the philosophy and structure of modern mathematics as well as prepare them for future courses in theoretical mathematics. It is my hope that I have accomplished this task.

R. J. Rossi

Butte, Montana

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Chapter 1

Introduction to Modern Mathematics

The field of mathematics was born out of the human necessity for counting items and determining areas and the desire to explain the natural world. The word *mathematics* is derived from the Greek words *mathema*, which means "science, knowledge, or learning," and *mathematikos*, which means "fond of learning." In addition to being responsible for the roots for the term *mathematics*, the ancient Greeks were also the first people to study pure mathematics and to record their logical arguments in proofs. Furthermore, the ancient Greek mathematicians were the first mathematicians to think abstractly about mathematics; the Babylonians and ancient Egyptians, unlike the Greeks, tended to think of mathematics in only practical terms with applications to trade and other universal problems. Thus, the ancient Greek mathematicians are generally credited with providing the foundation for modern mathematics.

The term "modern mathematics" is generally used to refer to the current formal axiomatic system of mathematics that is based on rigorous logical foundations. *Mathworld*, a popular Internet Website maintained by Wolfram Research, describes the field of mathematics as follows:

> *Mathematics is a broad-ranging field of study in which the properties and interactions of idealized objects are examined. Whereas mathematics began merely as a calculational tool for computation and tabulation of quantities, it has blossomed into an extremely rich and diverse set of tools, terminologies, and approaches which range from the purely abstract to the utilitarian.*

Whereas the roots of mathematics are based on counting and the study of geometric shapes, modern mathematics is much more than just the study of numbers and shapes. In particular, modern mathematics is the science of operations on collections of arbitrary objects. Modern mathematics, or axiomatic mathematics, is developed according to the following structure:

$$\text{Axioms} \Longrightarrow \text{definitions} \Longrightarrow \text{conjectures} \Longrightarrow \text{proofs}$$
$$\Longrightarrow \text{theorems} \Longrightarrow \text{generalizations and extensions} \Longrightarrow \cdots$$

It is this formal structure, along with the abstract nature of mathematics, that sets modern mathematics apart from the earlier developments in mathematics.

1.1 Inductive and Deductive Reasoning

For the most part, the development of early mathematics was motivated by the study of the physical world and natural phenomena by physical scientists. In fact, early mathematicians made most of their discoveries from their observations of physical phenomena and everyday occurences. The process of making inferences based on observations is called *inductive reasoning.*

Definition 1.1.1: *Inductive reasoning* is the method of reasoning based on making inferences and conclusions from observations.

Inductive reasoning is often used to extrapolate from a particular set of observations to a more general conclusion or future event. An example of inductive reasoning is given below.

> *Since the sun has come up every day of my life, it follows that the sun will come up tomorrow.*

This statement is based completely on making inferences from past experiences to what is to be expected to occur in the future. Much of the primary focus of the earliest development of mathematics was based on observed results, and did not rely on the formal justification of the mathematical conclusions. One reason why the inductive approach was the common theme in the early development of mathematics was that it was motivated primarily by the study of physical phenomenon (i.e., physics).

Inductive reasoning is also used in the developing mathematical conjectures; however, inductive reasoning can never be relied on as concrete proof of the validity of a conjecture. A classic example of the fallibility of inductive reasoning is due to Pierre de Fermat's (1601-1665) conjecture that $2^{2^n}+1$ is prime for all natural numbers n. While it is true that $2^{2^n}+1$ is prime for $n = 1, 2, 3, 4$, Leonhard Euler (1707-1783) disproved Fermat's conjecture by showing that $2^{2^5}+1$ is not a prime number. While large amounts of empirical data are likely to be used as evidence to support an unproven conjecture, data based reasoning can never provide absolute proof that a conjecture is true.

The writings of the ancient Greek mathematician Thales (circa 600 B.C.) provide the first documented use of sound logical reasoning in the justification of a mathematical result. Thales is credited with being the first person to write down a set of postulates, a set of mathematical conclusions, and provide a justification of these conclusions with a sequence of sound logical arguments. Thales' writings provide the first known to use of *deductive reasoning.*

Definition 1.1.2: *Deductive reasoning* is the method of reasoning where a conclusion is reached by logical arguments based on a collection of assumptions.

Following Thales, Greek mathematicians such as Pythagoras (569–500 B.C.), Aristotle (384–322 B.C.), and Euclid (325–265 B.C.) used deductive reasoning in justifying their mathematical results. In fact, it was Euclid who is credited for first proving that there are infinitely many primes and a student of Pythagoras who first proved the irrationality of $\sqrt{2}$.

An example of the use of deductive reasoning to prove a mathematical result is given in Examples 1.1.1 and 1.1.2, which follow.

Example 1.1.1: Let $x = 0.\overline{9}$. Then, using deductive reasoning, it can be shown that the conjecture $x = 1$ is true. In fact, there are many different ways to show deductively that $x = 0.\overline{9} = 1$, including the following deductive argument. Let $x = 0.\overline{9}$. Then

$$10x = 9.\overline{9} \tag{1}$$

$$10x - x = 9x = 9 \tag{2}$$

$$9x = 9 \tag{3}$$

$$x = 1 \tag{4}$$

Example 1.1.2: Conjecture: $\left| \int_1^\infty \frac{\cos(x)}{x^2}\, dx \right| < \infty$.

Proof: Since $\frac{\cos(x)}{x^2} < \frac{1}{x^2}$ on $[1, \infty)$ and

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx$$

it follows that

$$\left| \int_1^\infty \frac{\cos(x)}{x^2}\, dx \right| \le \int_1^\infty \left| \frac{\cos(x)}{x^2} \right| dx < \int_1^\infty \frac{1}{x^2}\, dx = 1 < \infty$$

Thus, it is true that $\left| \int_1^\infty \frac{\cos(x)}{x^2}\, dx \right| < \infty$.

Whereas deductive reasoning is the method mathematicians must use in the justification of a mathematical result, inductive reasoning still plays an important role in modern mathematics. As mentioned before, mathematical conjectures are often based on empirical data and inductive reasoning. However, empirical data can serve as proof of a conjecture only if (1) there are finitely many cases to consider in the conjecture and (2) all the possible cases are considered and the conjecture is shown to be true in each of these cases. However, except for these rare conjectures involving only finitely many cases, no amount of empirical data is sufficient to prove that a more general mathematical conjecture is actually true; mathematical proof comes only from logically sound deductive reasoning. Thus, new contributions to mathematics are justified using only deductive reasoning. Furthermore, deductive reasoning has made it possible for mathematics to become a formalized axiomatic system of the

Axiom–definition–conjecture–theorem–proof–generalization–extension

form.

1.2 Components of Modern Mathematics

The components of the modern axiomatic mathematical system are the axioms, definitions, conjectures, proofs, theorems, corollaries, lemmas, and counterexamples. The basic components on which the mathematical structure is built are the axioms and the definitions.

Definition 1.2.1: An *axiom* or *postulate* is a mathematical statement that is taken to be self-evidently true without proof.

Definition 1.2.2: A mathematical *definition* is a statement that gives precise meaning to a mathematical concept or word.

Mathematical axioms are the building blocks on which an axiomatic system is built. In fact, the validity of any further implications and mathematical conclusions in an axiomatic system will be based on the basic axioms and deductive reasoning. An example of one of the important axioms in axiomatic set theory is the "axiom of choice," given below.

> ***Axiom:*** Let C be a nonempty set, and if A_α is a nonempty set for each α in C, then it is possible to chose an x_α from the set A_α for each $\alpha \in C$.

The axiom of choice is a very important axiom in the foundation on which axiomatic set theory is based. The following two axioms were stated and used throughout Euclid's *Elements*, the first book of axiomatic mathematics.

> ***Axiom:*** Two things that are equal to the same thing are also equal to one another (i.e., "If $a = c$ and $b = c$, then $a = b$").

> ***Axiom:*** If equals be added to equals, the wholes are equals (i.e., "If $a = b$, then $a + c = b + c$").

One of the most famous axioms is the *parallel-line axiom*, which is also known as the *parallel postulate.*

> ***Parallel-Line Axiom:*** Given any straight line and a point not on it, there exists one and only one straight line that passes through that point and never intersects the first line, no matter how far the lines are extended.

The parallel-line axiom is equivalent to the Fifth Postulate of Book I of Euclid's *Elements* and is an important axiom of Euclidean geometry. In fact, the foundations of non-Euclidean geometry were developed by mathematicians who did not accept the parallel-line axiom. Euclid's fifth postulate is given below:

> ***Euclid's Fifth Postulate:*** If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Along with the axioms, the other basic building block in an axiomatic mathematical system are the definitions. Now, unlike the dictionary definition of a word, mathematical definitions are designed to have one and only one interpretation. Specifically, a mathematical definition is a precise statement that is used to give explicit conditions for the mathematical term being defined. Furthermore, a mathematical definition is designed to prevent two different mathematicians from using the same word to represent different mathematical ideas. For example, two mathematicians discussing the continuity of a function are basing their discussion on the following definition:

> ***Definition:*** A real-valued function $f(x)$ is said to be *continuous* at a point x_0 in the domain of f if and only if
>
> $$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

While there are alternative definitions of the continuity of a function, they are all equivalent to this definition of continuity. On the other hand, consider what might happen with two people discussing the paint color white. Clearly, there can be variations in the actual color of the white paint due to the shade of white or the paint company that produced the paint. In fact, it is not unusual for a person to buy a can of white paint, paint a room, and then be unsatisfied with the resulting shade of white. Therefore, to ensure the consistency of mathematics, it is important that mathematical definitions be clear, precise, and uniformly understood within the mathematical community.

Now, an axiomatic mathematical system begins with explicitly stated axioms and definitions, and from these initial ideas new mathematical results are added using deductive reasoning. Furthermore, the addition of new mathematical results follows from studying and making hypotheses concerning the implications of the axioms and definitions. When the truth of a hypothesized result is not yet known, the result is called a *conjecture.*

Definition 1.2.3: A *conjecture* is any mathematical statement that has not yet been proved or disproved.

Whereas the truths of many mathematical conjectures remain unknown today, one of the most famous and heavily studied conjectures, *Fermat's Last Theorem*, was more recently proved by Andrew Wiles of Princeton and his former student Richard Taylor. Fermat's Last Theorem is stated below.

Fermat's Last Theorem: The equation $x^n + y^n = z^n$ has solutions in positive integers x, y, z and n only when $n = 2$; but there are no solutions for $n > 2$.

Fermat's Last Theorem had been studied intensively for over 300 years before Wiles and Taylor finally proved this result in 1995. While Wiles' proof of Fermat's Last Theorem was an incredible accomplishment, even more importantly, the 300 years of study on this particular problem has led to many important and useful mathematical results. An interesting book detailing the history of Fermat's Last Theorem and Wiles' work is *Fermat's Enigma* by Simon Singh (1997).

Three of the most famous unproven mathematical conjectures are listed below:

Goldbach's Conjecture: Every even integer greater than 2 can be expressed as the sum of two prime numbers.

The Odd Perfect Number Conjecture: There do not exist any perfect odd numbers.

The Twin Prime Conjecture: There are an infinite number of twin primes.

Now, once a conjecture has been shown to be true with a mathematical *proof*, the conjecture can now be called a *theorem*; on the other hand, when a conjecture is shown not to be true, it is no longer of much interest in the mathematical world and is discarded.

Definition 1.2.4: A *proof* of a mathematical result is a sequence of rigorous mathematical arguments that are presented in a clear and concise fashion, and which convincingly demonstrates the truth of the given result.

Definition 1.2.5: A *theorem* is any mathematical statement that can be shown to be true using accepted logical and mathematical arguments.

Note that inductive reasoning is often used in the development of a conjecture; however, the proof of a conjecture or theorem is always based on deductive reasoning. In Examples 1.1.1 and 1.1.2, a mathematical conjecture was considered and then proved; since the conjectures in these two theorems have been proved, these conjectures can now be called *theorems*. The conjectures in Examples 1.1.1 and 1.1.2 are stated below in the form of theorems along with their respective proofs.

Theorem: Let $x = 0.\overline{9}$. Then, $x = 1$.

Proof: Let $x = 0.\overline{9}$. Then

$$10x = 9.\overline{9} \tag{1}$$

$$10x - x = 9x = 9 \tag{2}$$

$$9x = 9 \tag{3}$$

$$x = 1 \tag{4}$$

■

Theorem: $\left|\int_1^\infty \frac{\cos(x)}{x^2}\,dx\right| < \infty.$

Proof: Since $\frac{\cos(x)}{x^2} < \frac{1}{x^2}$ on $[1, \infty)$ and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

it follows that

$$\left|\int_1^\infty \frac{\cos(x)}{x^2}\,dx\right| \leq \int_1^\infty \left|\frac{\cos(x)}{x^2}\right|\,dx < \int_1^\infty \frac{1}{x^2}\,dx = 1 < \infty$$

Thus, it is true that $\left| \int_1^\infty \frac{\cos(x)}{x^2}\, dx \right| < \infty$.

■

It is important to note that a mathematical proof is very different from an empirical proof, which is often used in the sciences, or proof beyond a reasonable doubt, which is used in our legal system. For example, at one time scientists believed beyond a reasonable doubt, on the basis of empirical data, that the earth was the center of the universe; however, it is now understood that the sun is the center of the universe. A mathematical proof must represent absolute truth, so that a theorem is absolutely true regardless of any and all empirical data. For example, the fact that Euclid proved using deductive reasoning that there are infinitely many prime numbers is irrefutable (i.e., absolutely true). Furthermore, a mathematical proof provides a sequence of rigorous logical arguments where each step of the proof and the connection between steps is completely justified using mathematical and/or logical arguments.

Now, the proof of a theorem may be extremely long and complicated, or it may be very short and easily understood. A theorem that has a complicated or long proof is often referred to as a "deep theorem." A proof that takes a novel or unusual approach is often called an "elegant proof." A common feature in the Mathematical Association of America (MAA) publication *Mathematics Magazine* is "Proofs without Words," where a mathematical result is proved without using any words; a proof without words usually involves only formulas and/or graphical representation of the proof and should be self-explanatory.

Similarly, the proof of a theorem might be described as complicated or deep when the proof is long or difficult to follow because of its complexity. For example, the apparent simplicity of Fermat's Last Theorem is betrayed by the length and complexity of its proof. In fact, the proof of Fermat's Last Theorem, due to Wiles and Taylor, is long and very difficult for most mathematicians to follow. Examples of some very important mathematical theorems are listed below. Note that the theorems in this list contain results that are based on only addition and multiplication.

The Pythagorean Theorem: The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse.

Fermat's Last Theorem: $x^n + y^n = z^n$ has no nonzero integer solutions for x, y, and z when $n > 2$.

The Fundamental Theorem of Arithmetic: Every natural number greater than 1 either is prime or can be uniquely factored as a product of primes.

Among these three theorems, the Pythagorean theorem is one of the oldest and widely used theorems in mathematics, Fermat's Last Theorem is most likely the most famous mathematical theorem, and the Fundamental Theorem of Arithmetic shows that the prime numbers are the atoms from which the natural numbers are formed.

Often, great contributions are made to mathematical knowledge in the study of a difficult problem as occurred in the pursuit to prove Fermat's Last Theorem. Hence, the importance of an individual theorem is based not only on its utility but also on its complexity or the difficulty of its proof. Moreover, a theorem may be referred to as a "revolutionary" theorem when its impact on mathematics is dramatic or far-ranging. Often a revolutionary theorem will be important in opening up new directions in mathematical research. An example of a revolutionary and very important mathematical theorem is due to Kurt Gödel (1906–1978), who proved the following theorem in 1931 (Gödel 1931):

> **Gödel's Incompleteness Theorem:** In any consistent formalization of mathematics that is sufficiently strong to axiomatize the natural numbers — that is, sufficiently strong to define the operations that collectively define the natural numbers — one can construct a true statement that can be neither proved nor disproved within that system itself.

Prior to Gödel's Incompleteness Theorem, several influential mathematicians believed that all mathematical truths could be logically derived. In fact, David Hilbert (1862–1943), Bertrand Russell (1872–1970), and Alfred North Whitehead (1861–1947) believed that mathematics could be expressed as an axiomatic system that is free of inconsistencies and is also complete. Specifically, Hilbert, Russell, and Whitehead believed that an axiomatic mathematical system could be constructed where true statements are always true regardless of method of proof (i.e., a consistent system) and that all mathematical truths could be proved from the basic axioms of the system (i.e., a complete system). Gödel's Incompleteness Theorem shows that no mathematical axiomatic system that axiomatizes the natural numbers can be complete and hence, all of the mathematical truths cannot be proved from the basic axioms.

1.3 Commonly Used Mathematical Notation

In the communication of mathematics it is often useful to write mathematical sentences using symbols rather than words. The reason for this is that it makes it easier to read, shortens the communication while conveying all the information, and in essence creates a language of mathematics. Effectively communicating mathematical ideas is like writing an essay; it requires well-composed sentences, paragraphs, and correct mathematical grammar. Often, a mathematical essay is written using a great deal of mathematical shorthand and thus, the reading of modern mathematics will require the knowledge of the symbolic language of mathematics. Over the years, mathematicians have created their own language on the basis of standard mathematical shorthand (i.e., symbols used to shorten mathematical communications). A summary of the standard mathematical notation that used is in this text follows.

Throughout this text several different sets of numbers will be studied. In particular, the collections of numbers that are discussed in this text are the natural numbers, whole numbers, integers, rational numbers, real numbers, and irrational numbers. The notation used in this text to represent each of these sets of numbers is given below.

The Natural Numbers: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

The Whole Numbers: $\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$

The Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$

The Rationals: $\mathbb{Q} = \left\{q : q = \frac{p}{r} \text{ for } p \text{ and } r \neq 0 \text{ integers}\right\}$

The Reals: $\mathbb{R} = \{x : -\infty < x < \infty\}$

The Positive Reals: $\mathbb{R}^{+} = \{x : 0 < x < \infty\}$

The Negative Reals: $\mathbb{R}^{-} = \{x : -\infty < x < 0\}$

The Irrationals: $\mathbb{I} = \{r : r \text{ is a real number but not rational}\}$

The Complex Numbers: $\mathbb{C} = \left\{\xi : \xi = a + bi,\ a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\right\}$

Note that notation analogous to $\mathbb{R}^{+}$ and $\mathbb{R}^{-}$ can also be used with the sets $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{I}$. For example, $\mathbb{Z}^{+}$ is used to represent the positive integers and $\mathbb{Q}^{-}$ would be used to denote the negative rational numbers. Also, the following standard notation will be used for the intervals of $\mathbb{R}$:

a. **Open Intervals**: The open interval containing the points lying strictly between two endpoints, say, a and b, is denoted by (a, b). This set of numbers can also be represented by $a < x < b$. Note that either a or b, or both a and b may be ∞. In particular, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^{+} = (0, \infty)$, and $\mathbb{R}^{-} = (-\infty, 0)$.

b. **Closed Intervals**: The closed interval containing the points lying between and including two finite endpoints, say, a and b, is denoted by $[a, b]$. This set of numbers can also be represented by $a \leq x \leq b$. Note that a closed interval must have finite endpoints.

c. **Half-Open Intervals**: The half-open/half-closed intervals contain all the points lying strictly between the endpoints a and b and either a or b, but not both a and b. The half-open intervals are denoted by $(a, b]$ and $[a, b)$. These sets of numbers may also be represented by $a < x \leq b$ and $a \leq x < b$, respectively.

Example 1.3.1: Write out, using interval notation, the following sets of real numbers:

a. $-1 < x < 10$

b. $0 \leq x < \infty$

c. $0 < x \leq 10$

d. $-3 \leq x \leq -1.5$

Solutions:

a. $(-1, 10)$

b. $[0, \infty)$

c. $(0, 10]$

d. $[-3, -1.5]$

It is very important that mathematical results be presented using precise and consistent notation. Even the earliest mathematicians began developing and using a symbolic language in their presentations of mathematics. Moreover, in the twentieth century mathematicians began to standardize the symbols and notation used in modern mathematics. Some commonly used mathematical shorthand (i.e., symbols and notation) that is universal within the field of mathematics is given below:

a. $:=$ is often used for "defined to be." For example, the notation $:=$ might be used in defining the set $\mathbb{Z}_E$ that contains the even integers as follows:

$$\mathbb{Z}_E := \{x \in \mathbb{Z} : \ x = 2z \text{ for some integer } z.\}$$

b. s.t. or $\ni$: is often used for "such that." For example, the statement "there exists $x > 0$ such that $f(x) = 0$" could be written as "there exists $x > 0 \ \ni: \ f(x) = 0$.

c. $\in$ is often used for "is an element of" or "is a member of." For example, the statement "x is an element of $\mathbb{R}$" could be written as "$x \in \mathbb{R}$."

d. $x!$ is used to denote "x factorial," where, for a positive integer x, $x! = x(x-1)(x-2)\cdots 3\cdot 2\cdot 1$. For example, $6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.

e. $\sum$ is mathematical shorthand for summation. For example

$$\sum_{n=3}^{7} np^{n-1} = 3p^2 + 4p^3 + 5p^4 + 6p^5 + 7p^6$$

f. Π is mathematical shorthand for product. For example

$$\prod_{i=1}^{10} x^i(1-x)^{10-i} = x(1-x)^9 \times x^2(1-x)^8 \cdots x^9(1-x) \times x^{10}$$

g. $\forall$ is the mathematical shorthand for "for all" or "for every" or "for each." The symbol $\forall$ is referred to as the *universal quantifier*. The symbol $\forall$ was first used by Gerhard Gentzen (1909–1945) in 1934 according to "Earliest Uses of Some of the Words of Mathematics." by Jeff Miller (2006). An example of the usage of $\forall$ is

$$\forall\, i \in \{1, 2, \ldots, 10\},\ g(i) \geq 10$$

This statement is the mathematical shorthand for "for all i ranging from 1 to 10, $g(i) \geq 10$."

h. $\exists$ is the mathematical shorthand for "there exists" or "there is at least one." The symbol $\exists$ is referred to as the *existential quantifier*. The symbol $\exists$ was first used by Giuseppe Peano (1858–1932) in 1895 according to "Earliest Uses of Some of the Words of Mathematics." by Jeff Miller (2006). An example of the usage of $\exists$ is

$$\exists\, i \in \{1, 2, \ldots, 10\} \ni: g(i) \geq 10$$

This statement is the mathematical shorthand for "there exists a value of i between 1 and 10 such that $g(i) \geq 10$."

i. The symbol ∞ is used to represent the concept of the unbounded quantity "infinity." Jon Wallis (1616–1703) is credited for first using the symbol ∞ to represent infinity in 1655 according to *The Penguin Dictionary of Mathematics*, third edition (Nelson 2003). For example, $x \in (0, \infty)$ is the mathematical shorthand for the statement "x is an element of the open interval ranging from 0 to infinity.

j. The symbol $\Longrightarrow$ or $\rightarrow$ is the mathematical shorthand often used for "implies"; for example, $0 < x < y \Longrightarrow 0 < \frac{1}{y} < \frac{1}{x}$.

k. The symbol $\Longleftrightarrow$ or $\leftrightarrow$ is the mathematical shorthand for often used "if and only if"; for example, $ab = 0 \Longleftrightarrow a = 0$ or $b = 0$.

Example 1.3.2: Write the following sentences using as much mathematical notation as possible:

a. If there exists a real number x such that $e^x = 10$, then $x = \ln(10)$.

b. Let a and b be real numbers. The product of a and b is zero if and only if a is zero or b is zero.

Solutions:

a. $\exists\, x \in \mathbb{R} \ni: e^x = 10 \Longrightarrow x = \ln(10)$.

b. Let $a, b \in \mathbb{R}$. $ab = 0 \Longleftrightarrow a = 0$ or $b = 0$.

Example 1.3.3: Using common English sentences, write out each of the statements below.

a. $\exists\, x \in (0, \infty) \ni: e^x = 3$.

b. $\forall\, x \in \mathbb{Z}, \exists\, y \in \mathbb{Z} \ni: x + y = 0$.

c. $\forall\, \epsilon > 0, \exists\, N \in \mathbb{N} \ni: \left|\frac{1}{n}\right| < \epsilon, \forall\, n \geq N$.

Solutions:

a. There exists a nonnegative real number x such that e^x is equal to 3.

b. For every integer x, there exists an integer y such that $x + y = 0$.

c. For every ϵ greater than 0, there exists a natural number N such that $\left|\frac{1}{n}\right| < \epsilon$ whenever n is greater than or equal to N.

In honor of ancient Greek mathematicians, who laid the foundation for modern mathematics, Greek letters are commonly used in mathematical expressions. For example, π is the universal symbol used to express a constant that denotes the ratio of the circumference of a circle to the diameter of that circle; the uppercase version of π is Π and is used as mathematical shorthand to represent products. The Greek alphabet is listed in Table 1.3.1.

TABLE 1.3.1 The Greek Alphabet

Name	Uppercase Symbol	Lowercase	Name	Uppercase Symbol	Lowercase
Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π
Epsilon	E	ϵ	Rho	P	ρ
Zeta	Z	ζ	Sigma	Σ	σ
Eta	H	η	Tau	T	τ
Theta	Θ	θ	Upsilon	Y	υ
Iota	I	ι	Phi	Φ	ϕ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

Example 1.3.4: The Greek letters π, γ, Γ, β, ψ and ζ are commonly used in mathematics as follows:

$$\pi = \text{ratio of circumference of every circle to its diameter} \approx 3.14159$$

$$\gamma = \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{1}{k} - \ln(n) \right) \approx 0.577 \qquad \text{(Euler–Mascheroni constant)}$$

$$\pi(x) = \text{number of prime numbers} \leq x$$

$$\Gamma(k) = \int_0^\infty x^k e^{-x}\, dx \qquad \text{(gamma function)}$$

$$\beta(j,k) = \frac{\Gamma(j)\Gamma(k)}{\Gamma(k+j)} \qquad \text{(beta function)}$$

$$\psi(x) = \frac{d}{dx}\left[\ln(\Gamma(x)\right] = \frac{\Gamma'(x)}{\Gamma(x)} \qquad \text{(digamma function)}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{(Riemann–Zeta function)}$$