The History of Mathematics
A Brief Course
Second Edition

Roger Cooke
University of Vermont

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Preface

This second edition of *The History of Mathematics: A Brief Course* must begin with a few words of explanation to all users of the first edition. The present volume constitutes such an extensive rewriting of the original that it amounts to a considerable stretch in the meaning of the phrase *second edition*. Although parts of the first edition have been retained, I have completely changed the order of presentation of the material. A comparison of the two tables of contents will reveal the difference at a glance: In the first edition each chapter was devoted to a single culture or period within a single culture and subdivided by mathematical topics. In this second edition, after a general survey of mathematics and mathematical practice in Part 1, the primary division is by subject matter: numbers, geometry, algebra, analysis, mathematical inference.

For reasons that mathematics can illustrate very well, writing the history of mathematics is a nearly impossible task. To get a proper orientation for any particular event in mathematical history, it is necessary to take account of three independent "coordinates": the time, the mathematical subject, and the culture. To thread a narrative that is to be read linearly through this three-dimensional array of events is like drawing one of Peano's space-filling curves. Some points on the curve are infinitely distant from one another, and the curve must pass through some points many times. From the point of view of a reader whose time is valuable, these features constitute a glaring defect. The problem is an old one, well expressed eighty years ago by Felix Klein, in Chapter 6 of his *Lectures on the Development of Mathematics in the Nineteenth Century*:

> I have now mentioned a large number of more or less famous names, all closely connected with Riemann. They can become more than a mere list only if we look into the literature associated with the names, or rather, with those who bear the names. One must learn how to grasp the main lines of the many connections in our science out of the enormous available mass of printed matter without getting lost in the time-consuming discussion of every detail, but also without falling into superficiality and dilettantism.

Klein writes as if it were possible to achieve this laudable goal, but then his book was by intention only a collection of essays, not a complete history. Even so, he used more pages to tell the story of one century of European mathematics than a modern writer has available for the history of all of mathematics. For a writer who hates to leave any threads dangling the necessary sacrifices are very painful. My basic principle remains the same as in the first edition: not to give a mere list of names and results described in general terms, but to show the reader what important results were achieved and in what context. Even if unlimited pages were available, time is an important consideration for authors as well as readers. To switch metaphors, there were so many times during the writing when tempting digressions arose which I could not resist pursuing, that I suspected that I might be traversing the boundary of a fractal snowflake or creating the real-life example of Zeno's dichotomy. Corrections and supplementary material relating to this book can be found at my website at the University of Vermont. The url is:

`http://www.cem.uvm.edu/~cooke/history/seconded.html`

Fortunately, significant mathematical events are discrete, not continuous, so that a better analogy for a history of mathematics comes from thermodynamics. If the state of mathematics at any given time is a system, its atoms are mathematical problems and propositions, grouped into molecules of theory. As they evolve, these molecules sometimes collide and react chemically, as happened with geometry and algebra in the seventeenth century. The resulting development of the mathematical system resembles a Brownian motion; and while it is not trivial to describe a Brownian motion in detail, it is easier than drawing a space-filling curve.

Now let me speak more literally about what I have tried to do in the present book. As mentioned above, Part 1 is devoted to a broad survey of the world of mathematics. Each of the six subsequent parts, except Part 3, where the color plates are housed, concentrates on a particular aspect of mathematics (arithmetic, geometry, algebra, analysis, and mathematical inference) and discusses its development in different cultures over time. I had two reasons for reorganizing the material in this way.

First, I am convinced that students will remember better what they learn if they can focus on a single area of mathematics, comparing what was done in this area by different cultures, rather than studying the arithmetic, geometry, and algebra of each culture by turns. Second, although reviewers were for the most part kind, I was dissatisfied with the first edition, feeling that the organization of the book along cultural lines had caused me to omit many good topics, especially biographical material, and sources that really ought to have been included. The present edition aims to correct these omissions, along with a number of mistakes that I have noticed or others have pointed out. I hope that the new arrangement of material will make it possible to pursue the development of a single area of mathematics to whatever level the instructor wishes, then turn to another area and do the same. A one-semester course in mostly elementary mathematics from many cultures could be constructed from Chapters 1–7, 9–11, and 13 14. After that, one could use any remaining time to help the students write term papers (which I highly recommend) or go on to read other chapters in the book. I would also point out that, except for Chapters 8–12, and 15–19, the chapters, and even the sections within the chapters, can be read independently of one another. For a segment on traditional Chinese mathematics, for example, students could be assigned Section 3 of Chapter 2, Subsection 3.4 of Chapter 5, Section 2 of Chapter 6, Section 3 of Chapter 7, Section 3 of Chapter 9, Section 4 of Chapter 13, and Section 2 of Chapter 14.

Because of limitations of time and space, the present book will show the reader only a few of the major moments in the history of mathematics, omitting many talented mathematicians and important results. This restriction to the important moments makes it impossible to do full justice to what Grattan-Guinness has stated as the question the historian should answer: What happened in the past? We are reconstructing an evolutionary process, but the "fossil record" presented in any general history of mathematics will have many missing links. Unavoidably, history gets distorted in this process. New results appear more innovative than they actually are. To take just one example (not discussed elsewhere in this book), it was a very clever idea of Hermann Weyl to trivialize the proof of Kronecker's theorem that the fractional parts of the multiples of an irrational number are uniformly distributed in the unit interval; Weyl made this result a theorem about discrete and continuous averages of integrable periodic functions. One would expect that in an evolutionary process, there might be an intermediate step—someone who realized that these fractional parts are dense but not necessarily that they are uniformly distributed. And indeed there was: Nicole d'Oresme, 500 years before Kronecker. There are hundreds of results in mathematics with names on them, in many cases incorrectly attributed, and in many more cited in a much more polished form than the discoverer ever imagined. History ought to correct this misimpression, but a general history has only a limited ability to do so.

The other question mentioned by Grattan-Guinness—How did things come to be the way they are?—is often held up in history books as the main justification for requiring students to study political and social history.[1] That job is somewhat easier to do in a general textbook, and I hope the reader will be pleased to learn how some of the current parts of the curriculum arose.

I would like to note here three small technical points about the second edition.

Citations. In the first edition I placed a set of endnotes in each chapter telling the sources from which I had derived the material of that chapter. In the present edition I have adopted the more scholarly practice of including a bibliography organized by author and date. In the text itself, I include citations at the points where they are used. Thus, the first edition of this book would be cited as (Cooke, *1997*). Although I dislike the interruption of the narrative that this practice entails, I do find it convenient when reading the works of others to be able to note the source of a topic that I think merits further study without having to search for the citation. On balance, I think the advantage of citing a source on the spot outweighs the disadvantage of having to block out parenthetical material in order to read the narrative.

Translations. Unless another source is cited, all translations from foreign languages are my own. The reader may find smoother translations in most cases. To bring out significant concepts, especially in quotations from ancient Greek, I have made translations that are more literal than the standard ones. Since I don't know Sanskrit, Arabic, or Chinese, the translations from those languages are not mine; the source should be clear from the surrounding text.

Cover. Wiley has done me the great favor of producing a cover design in four colors rather than the usual two. That consideration made it possible to use a picture that I took at a quilt exposition at Norwich University (Northfield, Vermont) in 2003. The design bears the title "A Number Called Phi," and its creator, Mary Knapp of Watertown, New York, incorporated many interesting mathematical connections through the geometric and floral shapes it contains. I am grateful for her permission to use it as the cover of this second edition.

[1] In a lecture at the University of Vermont in September 2003 Grattan-Guinness gave the name *heritage* to the attempt to answer this question. Heritage is a perfectly respectable topic to write on, but the distinction between history and heritage is worth keeping in mind. See his article on this distinction (Grattan-Guinness, *2004*).

Acknowledgements. I am grateful to the editors at Wiley, Steve Quigley and especially Susanne Steitz, for keeping in touch throughout the long period of preparation for this book. I would also like to thank the copy editor, Barbara Zeiders, who made so many improvements to the text that I could not begin to list them all. I also ask Barbara to forgive my obstinacy on certain issues involving commas, my stubborn conviction that *independently of* is correct usage, and my constitutional inability to make all possessives end in *'s*. I cannot bring myself to write *Archimedes's* or *Descartes's*; and if we are going to allow some exceptions for words ending in a sibilant, as we must, I prefer—in defiance of the *Chicago Manual of Style*—to use an unadorned apostrophe for the possessive of *all* words ending in *s*, *z*, or *x*.

The diagram of Florence Nightingale's statistics on the Crimean War (Plate 5) is in the public domain; I wish to thank *Cabinet* magazine for providing the electronic file for this plate.

Many of the literature references in the chapters that follow were given to me by the wonderful group of mathematicians and historians on the Historia Mathematica e-mail list. It seemed that, no matter how obscure the topic on which I needed information, there was someone on the list who knew something about it. To Julio Gonzalez Cabillon, who maintains the list as a service to the community, I am deeply grateful.

Roger Cooke

January 2005

Part 1

The World of Mathematics and the Mathematics of the World

This first part of our history is concerned with the "front end" of mathematics (to use an image from computer algebra)—its relation to the physical world and human society. It contains some general considerations about mathematics, what it consists of, how it may have arisen, and how it has developed in various cultures around the world. Because of the large number of cultures that exist, a considerable paring down of the available material is necessary. We are forced to choose a few sample cultures to represent the whole, and we choose those that have the best-recorded mathematical history. The general topics studied in this part involve philosophical and social questions, which are themselves specialized subjects of study, to which a large amount of scholarly literature has been devoted. Our approach here is the naive commonsense approach of an author who is not a specialist in either philosophy or sociology. Since present-day governments have to formulate *policies* relating to mathematics and science, it is important that such questions not be left to specialists. The rest of us, as citizens of a republic, should read as much as time permits of what the specialists have to say and make up our own minds when it comes time to judge the effects of a policy.

This section consists of four chapters. In Chapter 1 we consider the nature and prehistory of mathematics. In this area we are dependent on archaeologists and anthropologists for the comparatively small amount of historical information available. We ask such questions as the following: "What is the subject matter of mathematics?" "Is new mathematics created to solve practical problems, or is it an expression of free human imagination, or some of each?" "How are mathematical concepts related to the physical world?"

Chapter 2 begins a broad survey of mathematics around the world. This chapter is subdivided according to a selection of cultures in which mathematics has arisen as an indigenous creation, in which borrowings from other cultures do not play a prominent role. For each culture we give a summary of the development of mathematics in that culture, naming the most prominent mathematicians and their works. Besides introducing the major works and their authors, an important goal of this chapter is to explore the question, "Why were these works written?" We quote the authors themselves as often as possible to bring out their motives. Chapters 2 and 3 are intended as background for the topic-based presentation that follows beginning with Chapter 5.

In Chapter 3 we continue the survey with a discussion of mathematical cultures that began on the basis of knowledge and techniques that had been created elsewhere. The contributions made by these cultures are found in the extensions, modifications, and innovations—some very ingenious—added to the inherited materials. In dividing the material over two chapters we run the risk of seeming to minimize the creations of these later cultures. Creativity is involved in mathematical innovations at every stage, from earliest to latest. The reason for having two chapters instead of one is simply that there is too much material for one chapter.

Chapter 4 is devoted to the special topic of women mathematicians. Although the *subject* of mathematics is gender neutral in the sense that no one could determine the gender of the author of a mathematical paper from an examination of the mathematical arguments given, the *profession* of mathematics has not been and is not yet gender neutral. There are obvious institutional and cultural explanations for this fact; but when an area of human endeavor has been polarized by gender, as mathematics has been, that feature is an important part of its history and deserves special attention.

CHAPTER 1

The Origin and Prehistory of Mathematics

In this chapter we have two purposes: first, to consider what mathematics is, and second, to examine some examples of *protomathematics*, the kinds of mathematical thinking that people naturally engage in while going about the practical business of daily life. This agenda assumes that there is a mode of thought called *mathematics* that is intrinsic to human nature and common to different cultures. The simplest assumption is that counting and common shapes such as squares and circles have the same meaning to everyone. To fit our subject into the space of a book of moderate length, we partition mathematical modes of thought into four categories:

Number. The concept of number is almost always the first thing that comes to mind when mathematics is mentioned. From the simplest finger counting by pre-school children to the recent sophisticated proof of Fermat's last theorem (a theorem at last!), numbers are a fundamental component of the world of mathematics.

Space. It can be argued that space is not so much a "thing" as a convenient way of organizing physical objects in the mind. Awareness of spatial relations appears to be innate in human beings and animals, which must have an instinctive understanding of space and time in order to move purposefully. When people began to intellectualize this intuitive knowledge, one of the first efforts to organize it involved reducing geometry to arithmetic. Units of length, area, volume, weight, and time were chosen, and *measurement* of these continuous quantities was reduced to *counting* these imaginatively constructed units. In all practical contexts measurement becomes counting in exactly this way. But in pure thought there is a distinction between what is *infinitely divisible* and what is *atomic* (from the Greek word meaning *indivisible*). Over the 2500 years that have elapsed since the time of Pythagoras this collision between the discrete modes of thought expressed in arithmetic and the intuitive concept of continuity expressed in geometry has led to puzzles, and the solution of those puzzles has influenced the development of geometry and analysis.

Symbols. Although early mathematics was discussed in ordinary prose, sometimes accompanied by sketches, its usefulness in science and society increased greatly when symbols were introduced to mimic the mental operations performed in solving problems. Symbols for numbers are almost the only *ideograms* that exist in languages written with a phonetic alphabet. In contrast to ordinary words, for example, the symbol 8 stands for an idea that is the same to a person in Japan, who reads it as *hachi*, a person in Italy who reads it as *otto*, and a person in Russia, who reads it as *vosem'*. The introduction of symbols such as + and = to stand for the common operations and relations of mathematics has led to both the clarity that mathematics has for its initiates and the obscurity it suffers from in the eyes of the nonmathematical. Although it is primarily in studying algebra that we become aware of the use of symbolism, symbols are used in other areas, and algebra, considered as the study of processes inverse to those of arithmetic, was originally studied without symbols.

Symbol-making has been a habit of human beings for thousands of years. The wall paintings on caves in France and Spain are an early example, even though one might be inclined to think of them as pictures rather than symbols. It is difficult to draw a line between a painting such as the *Mona Lisa*, an animé representation of a human being, and the ideogram for a person used in languages whose written form is derived from Chinese. The last certainly *is* a symbol, the first two usually are not thought of that way. Phonetic alphabets, which establish a symbolic, visual representation of sounds, are another early example of symbol-making. A similar spectrum presents itself in the many ways in which human beings convey instructions to one another, the purest being a computer program. Very often, people who think they are not mathematical are quite good at reading abstractly written instructions such as music, blueprints, road maps, assembly instructions for furniture, and clothing patterns. All these symbolic representations exploit a basic human ability to make correspondences and understand analogies.

Inference. Mathematical reasoning was at first numerical or geometric, involving either counting something or "seeing" certain relations in geometric figures. The finer points of logical reasoning, rhetoric, and the like, belonged to other areas of study. In particular, philosophers had charge of such notions as cause, implication, necessity, chance, and probability. But with the Pythagoreans, verbal reasoning came to permeate geometry and arithmetic, supplementing the visual and numerical arguments. There was eventually a countercurrent, as mathematics began to influence logic and probability arguments, eventually producing specialized mathematical subjects: mathematical logic, set theory, probability, and statistics. Much of this development took place in the nineteenth century and is due to mathematicians with a strong interest and background in philosophy. Philosophers continue to speculate on the meaning of all of these subjects, but the parts of them that belong to mathematics are as solidly grounded (apart from their applications) as any other mathematics.

We shall now elaborate on the origin of each of these components. Since these origins are in some cases far in the past, our knowledge of them is indirect, uncertain, and incomplete. A more detailed study of all these areas begins in Part 2. The present chapter is confined to generalities and conjectures as to the state of mathematical knowledge preceding these records.

1. Numbers

Counting objects that are distinct but similar in appearance, such as coins, goats, and full moons, is a universal human activity that must have begun to occur as soon as people had language to express numbers. In fact, it is impossible to imagine that numbers could have arisen without this kind of counting. Several closely related threads can be distinguished in the fabric of elementary arithmetic. First, there is a distinction that we now make between cardinal and ordinal numbers. We think of cardinal numbers as applying to *sets* of things—the word *sets* is meant here in its ordinary sense, not the specialized meaning it has in mathematical set theory—and ordinal numbers as applying to the individual elements of a set by virtue of an ordering imposed on the set. Thus, the cardinal number of the set $\{a, b, c, d, e, f, g\}$ is 7, and e is the fifth element of this set by virtue of the standard alphabetical ordering. These two notions are not so independent as they may appear in this illustration, however. Except for very small sets, whose cardinality can be perceived immediately, the cardinality of a set is usually determined by *counting*, that is, arranging its elements linearly as first, second, third, and so on, even though it may be the corresponding cardinal numbers—one, two, three, and so on—that one says aloud when doing the counting.

A second thread closely intertwined with counting involves the elementary operations of arithmetic. The commonest actions that are carried out with any collection of things are taking objects out of it and putting new objects into it. These actions, as everyone recognizes immediately, correspond to the elementary operations of subtraction and addition. The etymology of these words shows their origin, *subtraction* having the meaning of *pulling out* (literally pulling up or under) and *addition* meaning *giving to*. All of the earliest mathematical documents use addition and subtraction without explanation. The more complicated operations of multiplication and division may have arisen from comparison of two collections of different sizes (counting the number of times that one collection fits into another, or copying a collection a fixed number of times and counting the result), or perhaps as a shortened way of peforming addition or subtraction. It is impossible to know much for certain, since most of the early documents also assume that multiplication of small integers is understood without explanation. A notable exception occurs in certain ancient Egyptian documents, where computations that would now be performed using multiplication or division are reduced to repeated doubling, and the details of the computation are shown.

1.1. Animals' use of numbers. Counting is so useful that it has been observed not only in very young children, but also in animals and birds. It is not clear just how high animals and birds can count, but they certainly have the ability to distinguish not merely patterns, but actual numbers. The counting abilities of birds were studied in a series of experiments conducted in the 1930s and 1940s by O. Koehler (1889–1974) at the University of Freiburg. Koehler (*1937*) kept the trainer isolated from the bird. In the final tests, after the birds had been trained, the birds were filmed automatically, with no human beings present. Koehler found that parrots and ravens could learn to compare the number of dots, up to 6, on the lid of a hopper with a "key" pattern in order to determine which hopper contained food. They could make the comparison no matter how the dots were arranged, thereby demonstrating an ability to take account of the *number* of dots rather than the *pattern*.

1.2. Young children's use of numbers. Preschool children also learn to count and use small numbers. The results of many studies have been summarized by Karen Fuson (*1988*). A few of the results from observation of children at play and at lessons were as follows:

1. A group of nine children from 21 to 45 months was found to have used the word *two* 158 times, the word *three* 47 times, the word *four* 18 times, and the word *five* 4 times.
2. The children seldom had to count "one–two" in order to use the word *two* correctly; for the word *three* counting was necessary about half the time; for the word *four* it was necessary most of the time; for higher numbers it was necessary all the time.

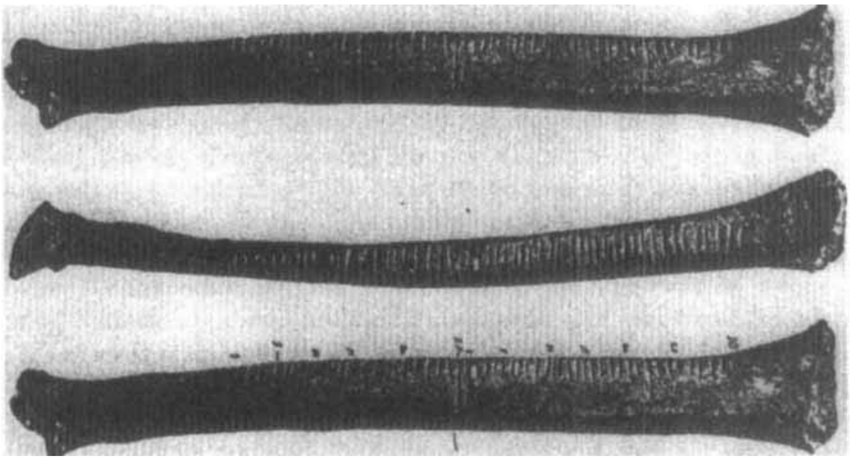
One can thus observe in children the capacity to recognize groups of two or three without performing any conscious numerical process. This observation suggests that these numbers are primitive, while larger numbers are a conscious creation. It also illustrates what was said above about the need for arranging a collection in some linear order so as to be able to find its cardinal number.

1.3. Archaeological evidence of counting. Very ancient animal bones containing notches have been found in Africa and Europe, suggesting that some sort of counting procedure was being carried on at a very early date, although what exactly was being counted remains unknown. One such bone, the radius bone of a wolf, was discovered at Veronice (Czech Republic) in 1937. This bone was marked with two series of notches, grouped by fives, the first series containing five groups and the second six. Its discoverer, Karel Absolon (1887–1960), believed the bone to be about 30,000 years old, although other archaeologists thought it considerably younger. The people who produced this bone were clearly a step above mere survival, since a human portrait carved in ivory was found in the same settlement, along with a variety of sophisticated tools. Because of the grouping by fives, it seems likely that this bone was being used to count something. Even if the groupings are meant to be purely decorative, they point to a use of numbers and counting for a practical or artistic purpose.

Another bone, named after the fishing village of Ishango on the shore of Lake Edward in Zaire where it was discovered in 1960 by the Belgian archaeologist Jean de Heinzelin de Braucourt (1920–1998), is believed to be between 8500 and 11,000 years old. The Ishango Bone, which is now in the Musée d'Histoire Naturelle in Brussels, contains three columns of notches. One column consists of four series of notches containing 11, 21, 19, and 9 notches. Another consists of four series containing 11, 13, 17, and 19 notches. The third consists of eight series containing 3, 6, 4, 8, 10, 5, 5, and 7 notches, with larger gaps between the second and third series and between the fourth and fifth series. These columns present us with a mystery. Why were they put there? What activity was being engaged in by the person who carved them? Conjectures range from abstract experimentation with numbers to keeping score in a game. The bone could have been merely decorative, or it could have been a decorated tool. Whatever its original use, it comes down to the present generation as a reminder that human beings were engaging in abstract thought and creating mathematics a very, very long time ago.

2. Continuous magnitudes

In addition to the ability to count, a second important human faculty is the ability to perceive spatial and temporal relations. These perceptions differ from the discrete objects that elicit counting behavior in that the objects involved are perceived as being divisible into arbitrarily small parts. Given any length, one can always imagine cutting it in half, for example, to get still smaller lengths. In contrast, a penny cut in half does not produce two coins each having a value of one-half cent. Just as human beings are endowed with the ability to reason numerically and understand the concept of equal distribution of money or getting the correct change with a purchase, it appears that we also have an innate ability to reason spatially, for example, to understand that two areas are equal even when they have different shapes, provided that they can be dissected into congruent pieces, or that



The Veronice wolf bone, from the *Illustrated London News*, October 2, 1937.

two vessels of different shape have the same volume if one each holds exactly enough water to fill the other.

One important feature of counting as opposed to measuring—arithmetic as opposed to geometry—is its exactitude. Two sets having the same number of members are numerically *exactly equal*. In contrast, one cannot assert that two sticks, for example, are *exactly* the same length. This difference arises in countless contexts important to human society. Two people may have exactly the same amount of money in the bank, and one can make such an assertion with complete confidence after examining the balance of each of them. But it is only within some limit of error that one could assert that two people are of the same height. The word *exact* would be inappropriate in this context. The notion of absolute equality in relation to continuous objects means *infinite precision* and can be expressed only through the concept of a *real number*, which took centuries to distill. That process is one important thread in the tapestry of mathematical history.

Very often, a spatial perception is purely geometrical or topological, involving similarity (having the same shape), connectivity (having holes or being solid), boundedness or infinitude, and the like. We can see the origins of these concepts in many aspects of everyday life that do not involve what one would call formal geometry. The perception of continuous magnitudes such as lengths, areas, volumes, weights, and time is different from the perception of multiple copies of a discrete object. The two kinds of perception work both independently and together to help a human being or animal cope with the physical world. Getting these two "draft horses" harnessed together as parts of a common subject called mathematics has led to a number of interesting problems to be solved.

2.1. Perception of shape by animals. Obviously, the ability to perceive shape is of value to an animal in determining what is or is not food, what is a predator, and so forth; and in fact the ability of animals to perceive space has been very well documented. One of the most fascinating examples is the ability of certain species of bees to communicate the direction and distance of sources of plant nectar by performing a dance inside the beehive. The pioneer in this work was Karl von

Frisch (1886–1982), and his work has been continued by James L. Gould and Carol Grant Gould (*1995*). The experiments of von Frisch left many interpretations open and were challenged by other specialists. The Goulds performed more delicately designed experiments which confirmed the bee language by deliberately misleading the bees about the food source. The bee will traverse a circle alternately clockwise and counterclockwise if the source is nearby. If it is farther away, the alternate traversals will spread out, resulting in a figure 8, and the dance will incorporate sounds and waggling. By moving food sources, the Goulds were able to determine the precision with which this communication takes place (about 25%). Still more intriguing is the fact that the direction of the food source is indicated by the direction of the axis of the figure 8, oriented relative to the sun if there is light and relative to the vertical if there is no light.

As another example, in his famous experiments on conditioned reflexes using dogs as subjects the Russian scientist Pavlov (1849–1936) taught dogs to distinguish ellipses of very small eccentricity from circles. He began by projecting a circle of light on the wall each time he fed the dog. Eventually the dog came to expect food (as shown by salivation) every time it saw the circle. When the dog was conditioned, Pavlov began to show the dog an ellipse in which one axis was twice as long as the other. The dog soon learned not to expect food when shown the ellipse. At this point the malicious scientist began making the ellipse less eccentric, and found, with fiendish precision, that when the axes were nearly equal (in a ratio of 8 : 9, to be exact) the poor dog had a nervous breakdown (Pavlov, *1928*, p. 122).

2.2. Children's concepts of space. The most famous work on the development of mathematical concepts in children is due to Jean Piaget (1896–1980) of the University of Geneva, who wrote many books on the subject, some of which have been translated into English. Piaget divided the development of the child's ability to perceive space into three periods: a first period (up to about 4 months of age) consisting of pure reflexes and culminating in the development of primary habits, a second period (up to about one year) beginning with the manipulation of objects and culminating in purposeful manipulation, and a third period in which the child conducts experiments and becomes able to comprehend new situations. He categorized the primitive spatial properties of objects as proximity, separation, order, enclosure, and continuity. These elements are present in greater or less degree in any spatial perception. In the baby they come together at the age of about 2 months to provide recognition of faces. The human brain seems to have some special "wiring" for recognizing faces.

The interesting thing about these concepts is that mathematicians recognize them as belonging to the subject of topology, an advanced branch of geometry that developed in the late nineteenth and early twentieth centuries. It is an interesting paradox that the human ability to perceive shape depends on synthesizing various topological concepts; this progression reverses the pedagogical and historical ordering between geometry and topology. Piaget pointed out that children can make topological distinctions (often by running their hands over models) before they can make geometric distinctions. Discussing the perceptions of a group of 3-to-5-year-olds, Piaget and Inhelder (*1967*) stated that the children had no trouble distinguishing between open and closed figures, surfaces with and without holes, intertwined rings and separate rings, and so forth, whereas the seemingly simpler relationships of geometry—distinguishing a square from an ellipse, for example were not mastered until later.

2.3. Geometry in arts and crafts. Weaving and knitting are two excellent examples of activities in which the spatial and numerical aspects of the world are combined. Even the sophisticated idea of a rectangular coordinate system is implicit in the placing of different-colored threads at intervals when weaving a carpet or blanket so that a pattern appears in the finished result. One might even go so far as to say that curvilinear coordinates occur in the case of sweaters.

Not only do arts and crafts *involve* the kind of abstract and algorithmic thinking needed in mathematics, their themes have often been inspired by mathematical topics. We shall give several examples of this inspiration in different parts of this book. At this point, we note just one example, which the author happened to see at a display of quilts in 2003. The quilt, shown on the cover of this book, embodies several interesting properties of the *Golden Ratio* $\Phi = (1+\sqrt{5})/2$, which is the ratio of the diagonal of a pentagon to its side. This ratio is known to be involved in the way many trees and flowers grow, in the spiral shell of the chambered nautilus, and other places. The quilt, titled "A Number Called Phi," was made by Mary Knapp of Watertown, New York. Observe how the quilter has incorporated the spiral connection in the sequence of nested circles and the rotation of each successive inscribed pentagon, as well as the phyllotaxic connection suggested by the vine.

Marcia Ascher (*1991*) has assembled many examples of rather sophisticated mathematics inspired by arts and crafts. The Bushoong people of Zaire make part of their living by supplying embroidered cloth, articles of clothing, and works of art to others in the economy of the Kuba chiefdom. As a consequence of this work, perhaps as preparation for it, Bushoong children amuse themselves by tracing figures on the ground. The rule of the game is that a figure must be traced without repeating any strokes and without lifting the finger from the sand. In graph theory this problem is known as the *unicursal tracing problem.* It was analyzed by the Swiss mathematician Leonhard Euler (1707–1783) in the eighteenth century in connection with the famous Königsberg bridge problem. According to Ascher, in 1905 some Bushoong children challenged the ethnologist Emil Torday (1875–1931) to trace a complicated figure without lifting his finger from the sand. Torday did not know how to do this, but he did collect several examples of such figures. The Bushoong children seem to learn intuitively what Euler proved mathematically: A unicursal tracing of a connected graph is possible if there are at most two vertices where an odd number of edges meet. The Bushoong children become very adept at finding such a tracing, even for figures as complicated as that shown in Fig. 1.

3. Symbols

We tend to think of symbolism as arising in algebra, since that is the subject in which we first become aware of it as a concept. The thing itself, however, is implanted in our minds much earlier, when we learn to talk. Human languages, in which sounds correspond to concepts and the temporal order or inflection of those sounds maps some relation between the concepts they signify, exemplify the process of abstraction and analogy, essential elements in mathematical reasoning. Language is, all by itself, ample proof that the symbolic ability of human beings is highly developed. That symbolic ability lies at the heart of mathematics.

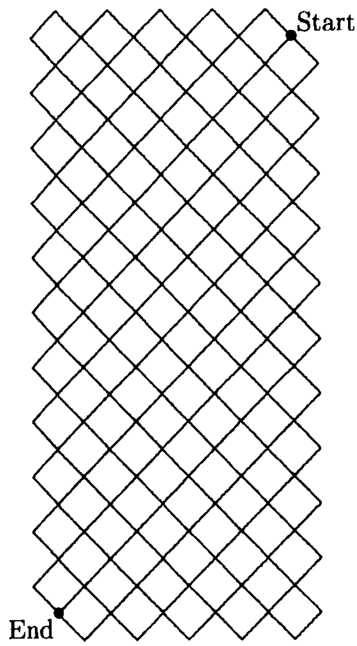


FIGURE 1. A graph for which a unicursal tracing is possible.

Once numbers have been represented symbolically, the next logical step would seem to be to introduce symbols for arithmetic operations or for combining the number symbols in other ways. However, this step may not be necessary for rapid computation, since mechanical devices such as counting rods, pebbles, counting boards, and the like can be used as analog computers. The operations performed using these methods can rise to a high level of sophistication without the need for any written computations. An example of the use of an automatic counting device is given by Ascher (*1997*) in a discussion of a system of divination used by the Malagasy of Madagascar, in which four piles of seeds are arranged in a column and the seeds removed from each pile two at a time until only one or two seeds remain. Each set of seeds in the resulting column can be interpreted as "odd" or "even." After this procedure is performed four times, the four columns and four rows that result are combined in different pairs using the ordinary rules for adding odds and evens to generate eight more columns of four numbers. The accuracy of the generation is checked by certain mathematical consequences of the method used. If the results are satisfactory, the 16 sets of four odds and evens are used as an oracle for making decisions and ascribing causes to such events as illnesses.

The Malagasy system of divination bears a resemblance to the procedures described in the Chinese classic *I Ching* (*Permutation Classic*). In the latter, a set of 50 yarrow sticks is used, the first stick being laid down to begin the ceremony. One stick is then placed between the ring and small fingers of the left hand to represent the human race. The remaining 48 sticks are then divided without counting into two piles, and one pile held in each hand. Those in the right hand are then discarded four at a time until four or fewer remain. These are then transferred to the left hand, and the same reduction is applied to the other pile, so that at