# APPLIED LIFE DATA ANALYSIS

This Page Intentionally Left Blank

# APPLIED LIFE DATA ANALYSIS

This Page Intentionally Left Blank

# APPLIED LIFE DATA ANALYSIS

WAYNE NELSON
*General Electric Co.*
*Corporate Research and Development*
*Schenectady, New York*

JOHN WILEY & SONS
New York • Chichester • Brisbane • Toronto • Singapore

A NOTE TO THE READER
This book has been electronically reproduced from digital information stored at John Wiley & Sons, Inc. We are pleased that the use of this new technology will enable us to keep works of enduring scholarly value in print as long as there is a reasonable demand for them. The content of this book is identical to previous printings.

Copyright © 1982 by Wayne Nelson
Published by John Wiley & Sons, Inc.

All rights reserved. Published simultaneously in Canada.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except as permitted under Sections 107 or 108 of the 1976 United States Copyright Act, without either the prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, 222 Rosewood Drive, Danvers, MA 01923, (978) 750-8400, fax (978) 750-4470. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, (201) 748-6011, fax (201) 748-6008, E-Mail: PERMREQ@WILEY.COM.

To order books or for customer service please, call 1(800)-CALL-WILEY (225-5945).

***Library of Congress Cataloging in Publication Data***:

Nelson, Wayne, 1936-
Applied life data analysis.

(Wiley series in probability and mathematical statistics. Applied probability and statistics section, ISSN 0271-6356) (A Wiley publication in applied statistics)
Includes bibliographical references and index.
1. Reliability (Engineering) I. Title.
II. Series. III. Series: Wiley publication in applied statistics.

TS173.N44 620'.00452 81-14779
ISBN 0-471-09458-7 AACR2

Printed in the United States of America

20 19 18 17

**In Grateful Memory**
**of my Mother and Father**

This Page Intentionally Left Blank

# Preface

Many major companies spend millions of dollars each year on product reliability. Much management and engineering effort goes into evaluating risks and liabilities, predicting warranty costs, evaluating replacement policies, assessing design changes, identifying causes of failure, and comparing alternate designs, vendors, materials, manufacturing methods, and the like. Major decisions are based on product life data, often from a few units. This book presents modern methods for extracting from life test and field data the information needed to make sound decisions. Such methods are successfully used in industry on a great variety of products by many who have modest statistical backgrounds.

This book is directed to engineers and industrial statisticians working on product life data. It will also aid workers in other fields where survival is studied, for example, in medicine, biology, actuarial science, economics, business, and criminology. Also, this book may supplement texts for many statistics and engineering courses, since it gives a wealth of practical examples with real data, emphasizes applied data analysis, employs computer programs, and systematically presents graphical methods, the method of maximum likelihood, censored data analysis, prediction methods, and linear estimation.

Life data generally contain running times on unfailed units, which require special statistical methods. In the past, these rapidly developing methods were associated with aerospace applications, but they are more widely used for consumer and industrial products. This book presents many applications to diverse products ranging from simple dielectrics and small appliances to locomotives and nuclear reactors.

This book draws from my experience teaching courses on life data analysis throughout the General Electric Company and at Rensselaer Polytechnic Institute and Union College. These courses have been popular with practicing engineers and graduate students in engineering, statistics, and operations research.

This book is organized to serve practitioners. The simplest and most widely useful material appears first. The book starts with basic models and simple graphical analyses of data, and it progresses through advanced analytic methods. All preliminary material for a topic is stated, and each topic is self-contained for easy reference, although this results in some repetition. Thus this book serves as a reference as well as a textbook. Derivations are generally omitted unless they help one understand the material. Such derivations appear in advanced sections for those who seek a fundamental understanding and wish to develop new statistical models and data analyses.

Readers of this book need a previous course in statistics and, for some advanced material, facility in calculus or matrix algebra. While many methods employ new and advanced statistical theory, the book emphasizes how to apply them. Certain methods (particularly those in Chapters 8 and 12), while important, are difficult to use unless one has special computer programs, which are now available.

There is much literature on life data analysis. So I have selected topics useful in my consulting. However, I briefly survey other topics in the final chapter.

Chapter 1 describes life data analysis, provides background material, and gives an overview of the book in detail. Chapter 2 presents basic concepts and statistical distributions for product life. Chapters 3 and 4 present graphical methods for estimating a life distribution from complete and censored life data. Chapter 5 explains statistical models and analyses for data on competing failure modes and on series systems. Chapters 6, 7, and 8 provide analytic methods, mainly linear and maximum likelihood methods, for estimating life distributions from complete and censored data. Chapter 9 provides methods for analyzing inspection data (quantal-response and interval data). Chapters 10, 11, and 12 provide methods for comparing samples (hypothesis tests) and for pooling estimates from a number of samples. Chapter 13 surveys other topics.

The real data in all examples come mostly from my consulting for the General Electric Company and other companies. Many of these real data sets are messy. Proprietary data were protected by vaguely naming a product and by multiplying the data by a factor. So engineers are advised not to use examples as typical of any product.

For help on this book I am overwhelmed with a great feeling of gratitude to many. Dr. Gerald J. Hahn, my co-worker, above all others, encouraged me, helped me to obtain support from General Electric, generously contributed much personal time reading the manuscript, and offered many useful suggestions. Gerry is the godfather of this book. I am much indebted for

support from management at General Electric Corporate Research and Development—Dr. Art Bueche, Mr. Stu Miller, Mr. Virg Lucke, Dr. Dick Shuey, Mr. E. Lloyd Rivest, Dr. Dave Oliver, Dr. Hal Chestnut, and Mr. Bill Chu. Professor Al Thimm, encouraged by Professor Josef Schmee, both of Union College, kindly provided me with an office, where I worked on this book, and a class that I taught from my manuscript during a leave from GE. Professor John Wilkinson of Rensselaer Polytechnic Institute gave me the original opportunity to teach courses and develop preliminary material for this book.

Colleagues have generously given much time reading the manuscript and offering their suggestions. I am particularly grateful to Paul Feder, Gerry Hahn, Joseph Kuzawinski, Bill Meeker, John McCool, Ron Regal, Josef Schmee, Bob Miller, Bill MacFarland, Leo Aroian, Jim King, Bill Tucker, and Carolyn Morgan.

Many clients generously let me use their data. They also inspired methods (such as hazard plotting) that I developed for their problems. Many students contributed suggestions. There are too many to name, unfortunately.

The illustrations are mostly the superb work of Mr. Dave Miller. The manuscript benefited much from the skillful technical typing of Jean Badalucco, Ceil Crandall, Edith White, and Ruth Dodd.

WAYNE NELSON

*Schenectady, New York*
*November 1981*

## About the Author

Wayne Nelson privately teaches and consults for companies, technical societies, and universities in the United States and abroad. Formerly with General Electric Co. Corp. Research and Development in Schenectady, NY over 20 years, he consults on engineering and scientific applications of statistical methods for reliability data, accelerated testing, quality control, planned experiments, sampling, tolerancing, measurement errors, and data analysis. He is an adjunct Professor at Union College. He can be reached at 739 Huntingdon Dr., Schenectady, NY 12309.

For his contributions to Reliability, Accelerated Testing, and Reliability Education, he was elected a Fellow of the American Statistical Assoc. (ASA), the American Society for Quality Control (ASQC), and the Institute of Electrical and Electronics Engineers (IEEE). He has a B.S. in Physics from Caltech and an M.S. in Physics (NSF Fellow) and a Ph.D. in Statistics (NSF Fellow) from the University of Illinois. He has published over 90 articles on statistical methods for engineering applications. For his publications, he received the 1969 Brumbaugh Award, the 1970 Jack Youden Prize, and the 1972 Frank Wilcoxon Prize of the ASQC. He received Outstanding Presentation Awards of the ASA for presentations at the 1977, 1979, 1987, 1988, and 1989 Joint Statistical Meetings. General Electric presented him the 1981 Dushman Award for his contributions to research and applications of reliability data analysis and accelerated testing.

He also authored the book *Accelerated Testing: Statistical Models, Test Plans, and Data Analyses* (John Wiley & Sons, 1990). He contributed to *Handbook of Reliability Engineering and Management* (McGraw-Hill, 1988) and to *Practical Machinery Management* (Gulf Publ., 1983). He wrote booklets *How to Analyze Reliability Data*, *How to Analyze Data with Simple Plots*, and *How to Plan and Analyze Accelerated Tests* for the ASQC and contributed to several engineering standards on data analysis.

# Contents


**1. Overview and Background** **1**

1. Introduction, 1
2. Overview of the Book, 3
3. Background Material, 5

**2. Basic Concepts and Distributions for Product Life** **15**

1. General Theory on Continuous Distributions (Illustrated by the Exponential Distribution), 16
2. Normal Distribution, 28
3. Lognormal Distribution, 32
4. Weibull Distribution, 36
5. Smallest Extreme Value Distribution, 39
6. Gamma and Chi-Square Distributions, 44
7. Other Distributions, 47
8. Shifted Distributions, 54
9. Conditional and Truncated Distributions, 56
10. Joint Distributions and the Joint Normal Distribution, 71
11. General Theory on Discrete Distributions (Geometric Distribution), 78
12. Poisson Distribution, 84
13. Binomial Distribution, 91
14. Hypergeometric Distribution, 96
15. Multinomial Distribution, 98

Problems, 100

**3. Probability Plotting of Complete and Singly Censored Data** **103**

1. Motivation, 104
2. How to Make a Probability Plot of Complete Data, 108
3. Percentiles and Probabilities, 110

4. Exponential Probability Plot, 110
5. Normal Probability Plot, 111
6. Lognormal Probability Plot, 113
7. Weibull Probability Plot, 115
8. Extreme Value Probability Plot, 116
9. Aids for Data Plotting, 117
10. Probability Plots of Singly Censored Data, 120
11. Theory for Probability Papers, 125
Problems, 128

**4. Graphical Analysis of Multiply Censored Data** **131**

1. Hazard Plotting, 132
2. Probability Plotting of Multiply Censored Data, 146
3. Theory for Hazard Papers, 154
Problems, 158

**5. Series Systems and Competing Risks** **162**

1. Series Systems of Different Components, 163
2. Series Systems of Identical Parts, 168
3. Hazard Plots for Data with Competing Failure Modes, 173
Problems, 187

**6. Analysis of Complete Data** **193**

1. Basic Ideas for Data Analysis, 194
2. Poisson Data, 200
3. Binomial Data, 204
4. Exponential Data, 209
5. Normal and Lognormal Data, 216
6. Weibull and Extreme Value Data, 226
7. Distribution-Free Methods, 235
Problems, 243

**7. Linear Methods for Singly Censored Data** **247**

1. Basic Concepts, 248
2. Exponential Data, 251
3. Normal and Lognormal Data, 261
4. Weibull and Extreme Value Data, 276
5. Order Statistics, 294
Problems, 309

**8. Maximum Likelihood Analysis of Multiply Censored Data 313**

Introduction, 313
1. Exponential Distribution, 316
2. Normal and Lognormal Distributions, 323
3. Weibull and Extreme Value Distributions, 333
4. Data with Competing Failure Modes—Maximum Likelihood Analysis, 347
5. General Maximum Likelihood Methods and Theory, 356
6. Numerical Methods, 386
Problems, 395

**9. Analyses of Inspection Data (Quantal-Response and Interval Data) 405**

Introduction, 405
1. Graphical Analyses of Quantal-Response Data, 407
2. Maximum Likelihood Fitting for Quantal-Response Data, 411
3. Graphical Analyses of Interval Data, 414
4. Maximum Likelihood Analyses for Interval Data, 418
5. ML Theory for Interval and Quantal-Response Data, 421
Problems, 427

**10. Comparisons (Hypothesis Tests) for Complete Data 433**

Introduction, 433
1. Comparison of Poisson Samples, 438
2. Comparison of Binomial Samples, 447
3. Comparison of Multinomial Samples, 453
4. Comparison of Exponential Samples, 459
5. Comparison of Normal and Lognormal Samples, 469
6. Nonparametric, Weibull, and Other Distributions, 490
7. Theoretical Concepts for Hypothesis Tests, 491
Problems, 494

**11. Comparisons With Linear Estimators (Singly Censored and Complete Data) 504**

1. One-Sample Comparisons, 505
2. Pooled Estimate of a Common Parameter Value, 509
3. Two-Sample Comparisons, 510
4. *K*-Sample Comparisons, 515
Problems, 519

**12. Maximum Likelihood Comparisons (Multiply Censored and Other Data) 522**

1. One-Sample Comparisons, 523
2. Pooled Estimate of a Common Parameter Value, 525
3. Two-Sample Comparisons, 527
4. *K*-Sample Comparisons, 531
5. Likelihood Ratio Tests, 534
6. General Theory for Likelihood Ratio and Related Tests, 539

Problems, 553

**13. Survey of Other Topics 558**

1. System Reliability Models and Their Analyses, 559
2. Maintainability Models and Data Analyses, 561
3. Availability Models and Data Analyses, 562
4. Estimation of System Reliability From Component Data, 562
5. Bayesian Methods in Reliability, 562
6. Reliability Demonstration and Acceptance Tests, 563
7. Reliability Growth (Duane Plots), 564
8. Renewal Theory and Data Analysis, 564
9. Life as a Function of Other Variables, 565
10. Accelerated Life Testing, 566
11. Depreciation, Replacement, and Maintenance Policies, 566
12. Books on Reliability and Life Data Analysis, 567
13. Abstracts, Bibliographies, and Indices on Reliability and Life Data Analysis, 569
14. Journals with Papers on Reliability and Life Data Analysis, 570
15. Computerized Literature Searches, 571

Problems, 572

**Appendix A Tables 575**

**References 603**

**Index 617**

# APPLIED LIFE DATA ANALYSIS

This Page Intentionally Left Blank

# 1

# Overview And Background

## 1. INTRODUCTION

This chapter presents (1) an overview of this book's contents and (2) background information for the rest of the book. To read this chapter and the rest of the book, one needs a basic statistics course. Although addressed mostly to engineering applications, this book applies to many other fields. A key characteristic of life data analysis distinguishes it from other areas in statistics: namely, data are usually censored or incomplete in some way. Like other areas, it is concerned with estimates and confidence limits for population parameters and with predictions and prediction limits for future samples. The following paragraphs describe applications and the history of life data analysis.

**Applications.** This book presents methods for analysis of product life data and gives many engineering applications. In this book, examples of applications to products include diesel engine fans, transformers, locomotive controls, generator field windings, material strength, generator retaining rings, locomotive reliability, electrical insulating fluids and oils, the strength of electrical connections, Class-B and Class-H motor insulations, appliance cords, fuses, turbine disks, shave dies, alarm clocks, batteries, toasters, capacitors, cryogenic cables, motor coils, engine cylinders and pistons, power lines, large motor warranty costs, turbine wheels, and distribution transformers.

The methods apply to other fields and types of data as the following examples show. Economists and demographers study the length of time people are in the work force (Kpedekpo, 1969). Employers are concerned with the length of time employees work before changing jobs. Mental health officials use tables of length of stay in facilities to predict patient load. Businessmen wish to know the shelf life of goods and the time it takes inventory to turn over; for example, one manufacturer wanted to know the distribution of time from manufacture to installation of a major appliance. Wildlife managers use mortality tables to predict wildlife population sizes and determine hunting seasons. Hoadley (1970) studied the length of time telephones remain disconnected in vacant quarters in order to determine which telephones to remove for use elsewhere and which to leave in for the next customer. Kosambi (1966) proposed that knowledge of the distribution of the time that coins remain in circulation can help the mint plan production. The success of medical treatments for certain diseases is measured by the length of patient survival (Gross and Clark, 1975). The distribution of time from prison release to committing a crime measures the success of prison programs. A trading stamp company estimated the proportion of stamps that would be redeemed; this was used to determine needed cash reserves to cover outstanding stamps. Potency of some pesticides (and chemicals) is bioassayed by observing the times to death (or other reaction) of a sample of insects or animals. Life insurance companies determine premiums from mortality tables. The life of TV programs has been evaluated (Prince, 1967). Jaeger and Pennock (1957) estimated service life of household goods. The Association for the Advancement of Medical Instrumentation (1975) has a proposed standard with methods for estimating the life of heart pacemakers. Zahn (1975) described a psychological experiment on the time a (planted) "stranded" motorist must wait for aid from a passerby. The durability of garments is studied by manufacturers (Goldsmith, 1968). Wagner and Altman (1973) studied the time in the morning when baboons come down from the trees.

This book presents engineering applications and uses mostly engineering and reliability terminology. Biomedical, actuarial, and other fields have their own terminology for many concepts; some of their terminology is mentioned. Differences in terminology may cause initial difficulties to those who read publications in other fields.

**History.** Todhunter (1949) describes the early human life table of Halley (Chapter 2) and Bernoulli's work on the effect of smallpox innoculation on the distribution of life. Insurance companies have long used actuarial methods for constructing human life tables. Early in this century, actuarial methods were used to estimate survivorship of (1) medical patients under

different treatments and of (2) equipment, particularly on railroads. In the 1950s and 1960s, reliability engineering blossomed; this resulted from demands for more reliable equipment from military and space programs. In this period, engineering design methods for reliable equipment made great strides. However, reliability data analysis mostly employed the oversimple exponential and Poisson distributions. In the 1950s and 1960s, most advances in life data analysis came from biomedical applications. Now methods are widely being developed for engineering applications to many consumer and industrial products. This book brings together recent methods for life data analysis. This field continues to grow, although many important problems remain unsolved, as this book shows.

## 2. OVERVIEW OF THE BOOK

This section describes this book's contents, organization, and how to use the book. The types of data mentioned here are described in Section 3.

Chapter 1 gives an overview of the book and presents needed background. Chapter 2 describes distributions for life and failure data. Chapter 3 presents simple probability plots for analyzing complete and singly censored data. Chapter 4 presents hazard plots for multiply censored data. Chapter 5 describes models for and graphical analyses of data with a mix of failure modes. Chapters 6 through 9 give analytic methods for (1) estimates and confidence limits for distribution parameters, percentiles, reliabilities, and other quantities and for (2) predictions and prediction limits for future samples. Chapter 6 treats analysis of complete data. Chapter 7 gives linear methods for singly censored data. Chapter 8 gives maximum likelihood methods for multiply censored data. Chapter 9 gives maximum likelihood methods for quantal-response and interval data. Chapter 10 presents various methods for comparing samples of complete data by confidence intervals and hypothesis tests; such methods include 1-, 2-, and $K$-sample comparisons and estimation by pooling a number of samples. Chapter 11 presents such comparisons based on linear estimates from singly censored samples. Chapter 12 presents such comparisons based on maximum likelihood methods for multiply censored and other types of data. Chapter 13 surveys topics in reliability and life data analysis that are not presented in the book.

Figure 2.1 shows this book's chapters. They are organized by type of data (complete, singly censored, multiply censored, etc.) and by statistical method (elementary, linear, and maximum likelihood). The chapters are in order of difficulty. Early chapters present simple graphical methods, and later ones present advanced analytic methods. The arrows in Figure 2.1 show which chapters are background for later chapters. Also, each chapter introduction

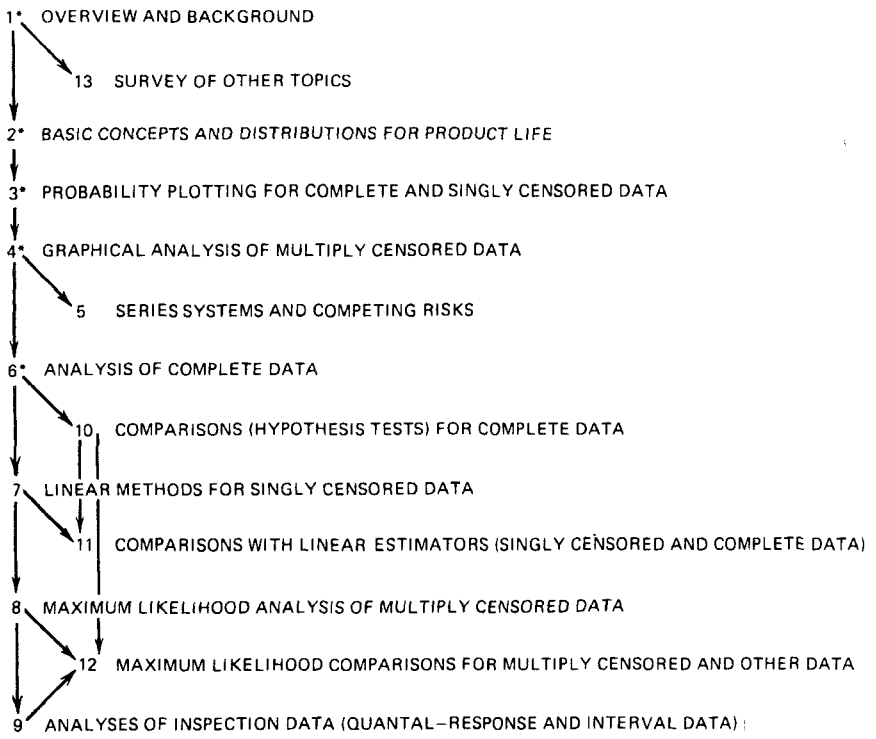


**Figure 2.1.** Book organization (asterisk denotes basic material).

refers to needed background and describes the difficulty of the chapter. Some section introductions do the same, and they state if a section is theoretical and can be skipped. Most sections are written for those who just wish to apply the methods. The first four chapters are simple and basic reading for all. The more advanced Chapters 5 through 9 are in order of difficulty. Chapters 10, 11, and 12 can be read after the corresponding Chapters 6 through 9.

Maximum likelihood methods (Chapters 8, 9, and 12) are versatile and apply to most distributions and types of data. Also, they have good statistical properties. If time is limited, one might skip the linear methods (Chapter 7) in favor of maximum likelihood methods.

The book employs the following scheme for numbering sections, equations, figures, and tables. Within each chapter, the sections are numbered simply 1, 2, 3, etc.; subsections are numbered 4.1, 4.2, etc. Equation numbers give the (sub)section number and equation number; for example,

(2.3) is the third numbered equation in Section 2. Figure and table numbers include the section number; Figure 2.3 is the third figure in Section 2. Unless another chapter is stated, any reference to an equation, figure, or table is to one in the same chapter.

There are two types of problems at the end of a chapter. One type involves an analysis of data with the methods in that chapter; the other involves extending the results of the chapter to other problems. An asterisk (*) marks more laborious or difficult problems.

The book cites references by means of the Harvard system. A citation includes the author's name, year of publication, and his publications in that year. For example, "Nelson (1972b)" refers to Nelson's second referenced publication in 1972. All references are listed near the end of the book.

Basic statistical tables are in an appendix near the end of the book. Other tables must be obtained from the literature and are referenced.

The index of the book is detailed. It will be an aid to those who wish to use the book as a reference for selected methods. Also, to aid users, each section is written to be self-contained, thus repeating some material.

The book omits many derivations. Reasons for this are the following: (1) users can properly apply most methods, not knowing derivations, (2) many derivations are easy for a reader or instructor to supply, and (3) more time can be spent on methods useful in practice. Many derivations appear in Mann, Schafer, and Singpurwalla (1974), Gross and Clark (1975), Bain (1978), and Lawless (1982).

## 3. BACKGROUND MATERIAL

Background material useful for the rest of this book is briefly presented here. The topics are (1) statistical models, (2) population and sample, (3) valid data, (4) failure and exposure, (5) types of data, (6) nature of data analysis, (7) estimates and confidence intervals, (8) hypothesis tests, (9) predictions, (10) practical and statistical significance, (11) numerical calculations, (12) notation.

**Statistical models.** Supposedly identical units made and used under the same conditions usually have different values of performance, dimensions, life, etc. Variability of such a performance variable is inherent in all products, and it is described by a statistical model or distribution.

**Population and sample.** A statistical model describes some **population**. A manufacturer of fluorescent lamps is concerned with the future production of a certain lamp—an essentially infinite population. A manufacturer of

locomotives is concerned with a small population of locomotives. A metallurgist is concerned with the future production of a new alloy—an essentially infinite population. A generator manufacturer is concerned with the performance of a small population of units to be manufactured next year. To obtain information, we use a **sample** (a set of units) from the population. We analyze the sample data to get information on the underlying population distribution or to predict future data from the population.

**Valid data.** There are many practical aspects to the collection of valid and meaningful data. Some are described below. Throughout, this book assumes that such aspects are properly handled.

Most statistical work assumes that the sample is from the population of interest. A sample from another population or a subset of the population can give misleading information. For example, failure data from appliances on a service contract may overestimate failure rates for appliances not on contract. Also, laboratory test data may differ greatly from field data. Data on units made last year may not adequately predict this year's units. In practice, it is often necessary to use such data. Then engineering judgment must determine how well such data represent the population of interest and how much one can rely on the information.

Most statistical work assumes that the data are obtained by simple random sampling from the population of interest. Such sampling gives each possible set of $n$ units the same chance of being the chosen sample; random numbers should be used to ensure random selection. In practice, other statistical sampling methods are sometimes used, the most common methods being stratified sampling and two-stage sampling. Data analyses must take into account the sampling method. This book assumes throughout that simple random sampling is used. Some samples are taken haphazardly, that is, without probability sampling. Such samples may be quite misleading.

In practice, measurements must be meaningful and correct. Also, one needs to avoid blunders in handling data. Bad data can be unknowingly processed by computers and by hand.

**Failure and exposure.** Failure must be precisely defined in practice. For dealings between producers and consumers, it is essential that the definition of a failure be agreed upon in advance to minimize disputes. For many products, failure is catastrophic, and it is clear when failure occurs. For some products, performance slowly degrades, and there is no clear end of life. One can then define that a failure occurs when performance degrades below a specified value. Of course, one can analyze data according to each of a number of definitions of failure. One must decide whether time is calendar time or operating hours or some other measure of exposure, for

example, the number of start-ups, miles traveled, energy output, cycles of operation, etc. Also, one must decide whether to measure time of exposure starting at time of manufacture, time of installation, or whatever. Engineers define failure and exposure.

**Types of data.** The proper analysis of data depends on the type of data. The following paragraphs describe the common types of life data from life tests and actual service.

Most nonlife data are **complete**; that is, the value of each sample unit is observed. Such life data consist of the time to failure of each sample unit. Figure 3.1*a* depicts a complete sample. Chapters 3, 6, and 10 treat such data. Much life data are incomplete. That is, the exact failure times of some units are unknown, and there is only partial information on their failure times. Examples follow.

Sometimes when life data are analyzed, some units are unfailed, and their failure times are known only to be beyond their present running times. Such data are said to be **censored on the right.** Unfailed units are called run-outs, survivors, removals, and suspended units. Similarly, a failure time known only to be before a certain time is said to be **censored on the left.** If all unfailed units have a common running time and all failure times are earlier, the data are said to be **singly censored** on the right. Singly censored data arise when units are started on test together and the data are analyzed before all units fail. Such data are singly **time censored** if the censoring time is fixed; then the number of failures in that fixed time is random. Figure 3.1*b* depicts such a sample. Time censored data are also called **Type I censored.** Data are singly **failure censored** if the test is stopped when a specified number of failures occurs, the time to that fixed number of failures being random. Figure 3.1*c* depicts such a sample. Time censoring is more common in practice; failure censoring is more common in the literature, as it is mathematically more tractable. Chapters 3, 7, and 11 treat singly censored data.

Much data censored on the right have differing running times intermixed with the failure times. Such data are called **multiply censored** (also progressively, hyper-, and arbitrarily censored). Figure 3.1*d* depicts such a sample. Multiply censored data usually come from the field, because units go into service at different times and have different running times when the data are recorded. Such data may be time censored (running times differ from failure times, as shown in Figure 3.1*d*) or failure censored (running times equal failure times, as shown in Figure 3.1*e*). Chapters 4, 8, and 12 treat such data.

A mix of **competing failure modes** occurs when sample units fail from different causes. Figure 3.1*f* depicts such a sample. Data on a particular

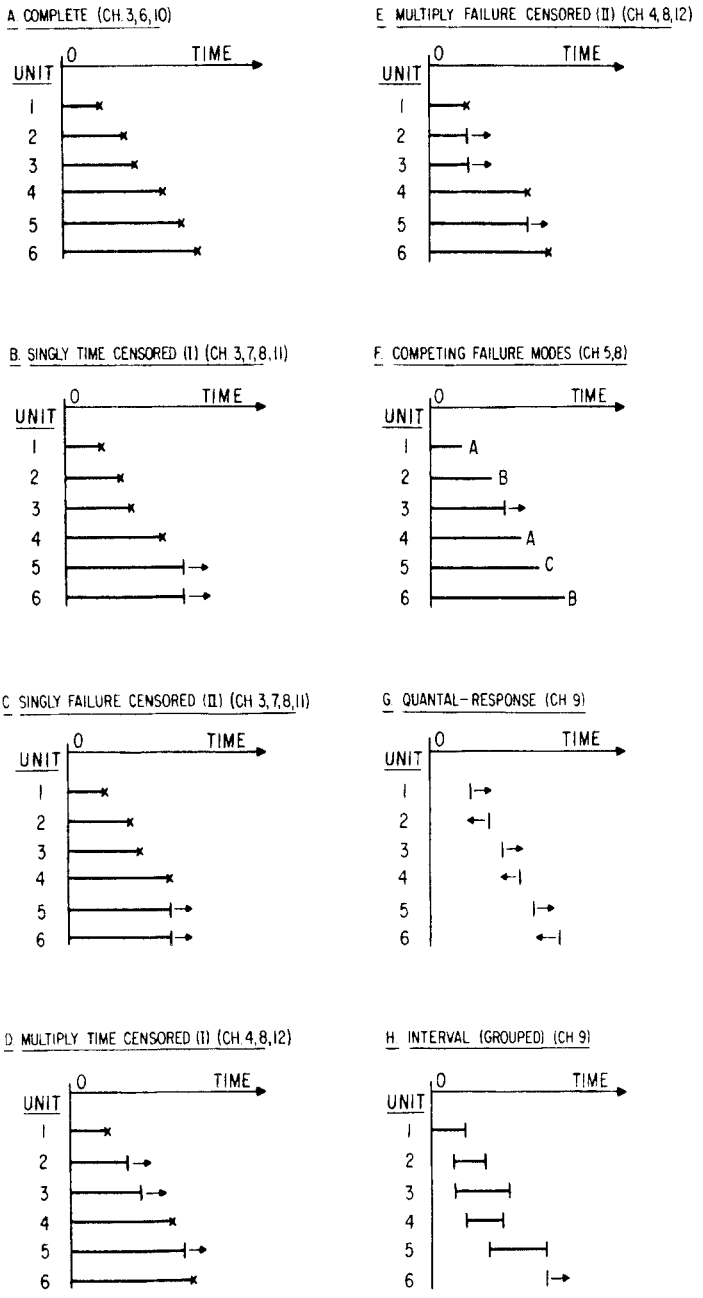


**Figure 3.1.** Types of data (failure time $\times$, running time $\mapsto$, failure occurred earlier $\leftarrow$)

discrete, 82
exponential, 24
exponential ML estimate, 367
extreme value, 42
factor, 250
hypergeometric, 98
*local* estimate, 318, 344
lognormal, 34
order statistic, 265, 297
normal, 31
Poisson, 88
prediction error, 199
unequal, 479
Weibull, 38
Weibull ML estimates, 343
*see also* Asymptotic variance *and* Covariance matrix

Wald's test, 551, 555
Warranty period, 276, 294
Weakest link, 36, 40
Wear-out, 26
WEIB, *see* Computer program
Weibull analysis, approximate BLUEs, 339
BLIEs, 339
BLUEs, 277-279
compare $K$ parameters, 517
comparisons, 490, 507-517
competing modes, 348
complete data, 226-235, 337, 491
conditional confidence limits, 338, 339
conditional prediction limits, 339
confidence limits, 282-291
confidence limits for parameters, 230, 231, 283, 285, 337, 338, 344
confidence limits for percentiles, 233, 287-290, 337, 339, 345
confidence limits for percentiles simultaneous, 534
confidence limits for ratio of two percentiles, 531
confidence limits for ratio of two scale parameters, 531
confidence limits for ratio of two shape parameters, 529
confidence limits for reliability, 234, 290, 337, 338, 347
confidence limits for scale, 231, 285, 340
confidence limits for shape, 230, 283, 340
distribution, *see* Distribution, Weibull
estimate for percentile, 232, 280, 345
estimate for reliability, 233, 280, 346
estimate for scale, 229, 277
estimate for shape, 227, 279
estimators from selected order statistics, 281
interval data, 418-420, 426
known shape, 235, 294, 399
ML estimate calculation, 340
ML estimates, 337, 340
multiply censored data, 333-347
other methods, 235, 339
prediction, 291-294
prediction limits for first order statistic, 235, 293
prediction limits for order statistic $j$, 292
predictor for first order statistic, 234, 293
predictor for order statistic $j$, 292
quantal-response data, 411-413, 426
sample size, 231
shape parameter known, 235, 294, 399
singly censored data, 276-294, 338
*see also* Data, Extreme value analysis
Weibull distribution, 36-39, 43
relation to extreme value distribution, 43
Wholly significant, 483, 487, 533, 534
Wholly significant difference, 517
Wilson-Hilferty approximation, 46, 267, 285, 506

WILEY SERIES IN PROBABILITY
AND MATHEMATICAL STATISTICS

ESTABLISHED BY WALTER A. SHEWHART AND SAMUEL S. WILKS
Editors
*Ralph A. Bradley, J. Stuart Hunter, David G. Kendall*
*Rupert G. Miller, Jr., Geoffrey S. Watson*

*Probability and Mathematical Statistics*

ADLER • The Geometry of Random Fields
ANDERSON • The Statistical Analysis of Time Series
ANDERSON • An Introduction to Multivariate Statistical Analysis
ARAUJO and GINE • The Central Limit Theorem for Real and Banach Valued Random Variables
ARNOLD • The Theory of Linear Models and Multivariate Analysis
BARLOW, BARTHOLOMEW, BREMNER, and BRUNK • Statistical Inference Under Order Restrictions
BARNETT • Comparative Statistical Inference
BHATTACHARYYA and JOHNSON • Statistical Concepts and Methods
BILLINGSLEY • Probability and Measure
CASSEL, SARNDAL, and WRETMAN • Foundations of Inference in Survey Sampling
COCHRAN • Contributions to Statistics
DE FINETTI • Theory of Probability, Volumes I and II
DOOB • Stochastic Processes
FELLER • An Introduction to Probability Theory and Its Applications, Volume I, *Third Edition,* Revised; Volume II, Second Edition
FULLER • Introduction to Statistical Time Series
GRENANDER • Abstract Inference
HANNAN • Multiple Time Series
HANSEN, HURWITZ, and MADOW • Sample Survey Methods and Theory, Volumes I and II
HARDING and KENDALL • Stochastic Geometry
HOEL • Introduction to Mathematical Statistics, *Fourth Edition*
HUBER • Robust Statistics
IOSIFESCU • Finite Markov Processes and Applications
ISAACSON and MADSEN • Markov Chains
KAGAN, LINNIK, and RAO • Characterization Problems in Mathematical Statistics
KENDALL and HARDING • Stochastic Analysis
LAHA and ROHATGI • Probability Theory
LARSON • Introduction to Probability Theory and Statistical Inference, *Second Edition*
LARSON • Introduction to the Theory of Statistics
LEHMANN • Testing Statistical Hypotheses
MATTHES, KERSTAN, and MECKE • Infinitely Divisible Point Processes
PARZEN • Modern Probability Theory and Its Applications
PURI and SEN • Nonparametric Methods in Multivariate Analysis
RANDLES and WOLFE • Introduction to the Theory of Nonparametric Statistics
RAO • Linear Statistical Inference and Its Applications, *Second Edition*
ROHATGI • An Introduction to Probability Theory and Mathematical Statistics
RUBINSTEIN • Simulation and The Monte Carlo Method
SCHEFFE • The Analysis of Variance
SEBER • Linear Regression Analysis
SEN • Sequential Nonparametrics: Invariance Principles and Statistical Inference
SERFLING • Approximation Theorems of Mathematical Statistics
TJUR • Probability Based on Radon Measures

*Probability and Mathematical Statistics (Continued)*

WILLIAMS • Diffusions, Markov Processes, and Martingales, Volume I: Foundations

ZACKS • Theory of Statistical Inference

*Applied Probability and Statistics*

ANDERSON, AUQUIER, HAUCK, OAKES, VANDAELE, ar WEISBERG • Statistical Methods for Comparative Studies

ARTHANARI and DODGE • Mathematical Programming in Statistics

BAILEY • The Elements of Stochastic Processes with Applications to the Natural Sciences

BAILEY • Mathematics, Statistics and Systems for Health

BARNETT • Interpreting Multivariate Data

BARNETT and LEWIS • Outliers in Statistical Data

BARTHOLOMEW • Stochastic Models for Social Processes, *Third Edition*

BARTHOLOMEW and FORBES • Statistical Techniques for Manpower Planning

BECK and ARNOLD • Parameter Estimation in Engineering and Science

BELSLEY, KUH, and WELSCH • Regression Diagnostics: Identifying Influential Data and Sources of Collinearity

BENNETT and FRANKLIN • Statistical Analysis in Chemistry and the Chemical Industry

BHAT • Elements of Applied Stochastic Processes

BLOOMFIELD • Fourier Analysis of Time Series: An Introduction

BOX • R. A. Fisher, The Life of a Scientist

BOX and DRAPER • Evolutionary Operation: A Statistical Method for Process Improvement

BOX, HUNTER, and HUNTER • Statistics for Experimenters: An Introduction to Design, Data Analysis, and Model Building

BROWN and HOLLANDER • Statistics: A Biomedical Introduction

BROWNLEE • Statistical Theory and Methodology in Science and Engineering, *Second Edition*

BURY • Statistical Models in Applied Science

CHAMBERS • Computational Methods for Data Analysis

CHATTERJEE and PRICE • Regression Analysis by Example

CHERNOFF and MOSES • Elementary Decision Theory

CHOW • Analysis and Control of Dynamic Economic Systems

CHOW • Econometric Analysis by Control Methods

CLELLAND, BROWN, and deCANI • Basic Statistics with Business Applications, *Second Edition*

COCHRAN • Sampling Techniques, *Third Edition*

COCHRAN and COX • Experimental Designs, *Second Edition*

CONOVER • Practical Nonparametric Statistics, *Second Edition*

CORNELL • Experiments with Mixtures: Designs, Models and The Analysis of Mixture Data

COX • Planning of Experiments

DANIEL • Biostatistics: A Foundation for Analysis in the Health Sciences, *Second Edition*

DANIEL • Applications of Statistics to Industrial Experimentation

DANIEL and WOOD • Fitting Equations to Data: Computer Analysis of Multifactor Data, *Second Edition*

DAVID • Order Statistics, *Second Edition*

DEMING • Sample Design in Business Research

DODGE and ROMIG • Sampling Inspection Tables, *Second Edition*

DRAPER and SMITH • Applied Regression Analysis, *Second Edition*

DUNN • Basic Statistics: A Primer for the Biomedical Sciences, *Second Edition*

DUNN and CLARK • Applied Statistics: Analysis of Variance and Regression

ELANDT-JOHNSON • Probability Models and Statistical Methods in Genetics

*continued on back*

*Applied Probability and Statistics (Continued)*

ELANDT-JOHNSON and JOHNSON • Survival Models and Data Analysis

FLEISS • Statistical Methods for Rates and Proportions, *Second Edition*

FRANKEN • Queues and Point Processes

GALAMBOS • The Asymptotic Theory of Extreme Order Statistics

GIBBONS, OLKIN, and SOBEL • Selecting and Ordering Populations: A New Statistical Methodology

GNANADESIKAN • Methods for Statistical Data Analysis of Multivariate Observations

GOLDBERGER • Econometric Theory

GOLDSTEIN and DILLON • Discrete Discriminant Analysis

GROSS and CLARK • Survival Distributions: Reliability Applications in the Biomedical Sciences

GROSS and HARRIS • Fundamentals of Queueing Theory

GUPTA and PANCHAPAKESAN • Multiple Decision Procedures: Theory and Methodology of Selecting and Ranking Populations

GUTTMAN, WILKS, and HUNTER • Introductory Engineering Statistics, *Third Edition*

HAHN and SHAPIRO • Statistical Models in Engineering

HALD • Statistical Tables and Formulas

HALD • Statistical Theory with Engineering Applications

HAND • Discrimination and Classification

HARTIGAN • Clustering Algorithms

HILDEBRAND, LAING, and ROSENTHAL • Prediction Analysis of Cross Classifications

HOEL • Elementary Statistics, *Fourth Edition*

HOEL and JESSEN • Basic Statistics for Business and Economics, *Third Edition*

HOLLANDER and WOLFE • Nonparametric Statistical Methods

JAGERS • Branching Processes with Biological Applications

JESSEN • Statistical Survey Techniques

JOHNSON and KOTZ • Distributions in Statistics
Discrete Distributions
Continuous Univariate Distributions—1
Continuous Univariate Distributions—2
Continuous Multivariate Distributions

JOHNSON and KOTZ • Urn Models and Their Application: An Approach to Modern Discrete Probability Theory

JOHNSON and LEONE • Statistics and Experimental Design in Engineering and the Physical Sciences, Volumes I and II, *Second Edition*

JUDGE, HILL, GRIFFITHS, LUTKEPOHL and LEE • Introduction to the Theory and Practice of Econometrics

JUDGE, GRIFFITHS, HILL and LEE • The Theory and Practice of Econometrics

KALBFLEISCH and PRENTICE • The Statistical Analysis of Failure Time Data

KEENEY and RAIFFA • Decisions with Multiple Objectives

LAWLESS • Statistical Models and Methods for Lifetime Data

LEAMER • Specification Searches: Ad Hoc Inference with Nonexperimental Data

McNEIL • Interactive Data Analysis

MANN, SCHAFER and SINGPURWALLA • Methods for Statistical Analysis of Reliability and Life Data

MEYER • Data Analysis for Scientists and Engineers

MILLER • Survival Analysis

MILLER, EFRON, BROWN, and MOSES • Biostatistics Casebook

MONTGOMERY and PECK • Introduction to Linear Regression

NELSON • Applied Life Data Analysis

OTNES and ENOCHSON • Applied Time Series Analysis: Volume I, Basic Techniques