

# Bayesian Networks

## A Practical Guide to Applications

**Edited by**

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# Foreword

When we, in the late 1980s, worked in a European ESPRIT project on what later became the MUNIN network, expert systems and neural networks were the predominant new artificial intelligence techniques. However, we felt that the most important ingredient of medical diagnosis, *causality with inherent uncertainty*, could not be captured by these techniques.

Rather than trying to model the experts we felt that we should go back to the ancient scientific tradition of modeling the domain, and the new challenge was to incorporate causal uncertainty. We called our models *causal probabilistic networks* (CPNs). They are now called Bayesian networks. The task, we thought, is quite simple: determine a CPN through dialogues with the experts. The rest is just mathematics and computer power.

We were wrong in two ways. It is not ‘just’ mathematics and computer power. But even worse, to determine a CPN through dialogues with the experts is much more intriguing than we anticipated. Over the two decades since the revival of Bayesian networks, several books have addressed the first problem. Although the need is widely recognized, no book has so far focused on the second problem.

This book meets the demand for an aid in developing Bayesian network models in practice. The authors have done a great job in collecting a large sample of Bayesian network applications from a wide range of domains.

Each chapter tells a story about a particular application. However, they do more than that. By studying the various chapters, the reader can learn very much about how to collaborate with domain experts and how to combine domain knowledge with learning from databases. Furthermore, the reader will be presented to a long list of advantages, problems and shortcomings of Bayesian network modeling and inference.

The sample also reflects the two sides of Bayesian network. On the one hand, a Bayesian network is a causal probabilistic network. On the other hand, a Bayesian network is a way of decomposing a large joint probability distribution. In some of the applications, causality is an important part of the model construction, and in other applications, causality is not an issue.

I hope that this book will be studied by everyone who is about to model a domain containing causality with inherent uncertainty: this book will teach him/her if and how to use Bayesian networks.

Finn V. Jensen  
Aalborg University



# Preface

The spectacular improvements of the technologies to produce, transmit, store and retrieve information are leading to a paradox: in many circumstances, making the best use of the available information is much more difficult today than a few decades ago. Information is certainly abundant and easily accessible, but at the same time (and to a large extent, consequently) often inconsistent, contradictory, and of uncertain traceability and reliability. The process of interpreting information remains an essential one, because uninterpreted information is nothing else than noise, but becomes more and more delicate. To mention only one domain covered in this book, striking examples of this phenomenon are the famous criminal cases which remain unsolved, despite the accumulation over years of evidences, proofs and expert opinions.

Given this challenge of optimally using information, it is not surprising that a gain of interest for statistical approaches has appeared in many fields in recent years: the purpose of statistics is precisely to convert information into a usable form.

Bayesian networks, named after the works of Thomas Bayes (ca. 1702–1761) on the theory of probability, have emerged as the result of mathematical research carried out in the 1980s, notably by Judea Pearl at UCLA, and from that time on, have proved successful in a large variety of applications.

This book is intended for users, and also *potential* users of Bayesian networks: engineers, analysts, researchers, computer scientists, students and users of other modeling or statistical techniques. It has been written with a dual purpose in mind:

- highlight the versatility and modeling power of Bayesian networks, and also discuss their limitations and failures, in order to help potential users to assess the adequacy of Bayesian networks to their needs;
- provide practical guidance on constructing and using of Bayesian networks.

We felt that these goals would be better achieved by presenting real-world applications, i.e., models actually in use or that have been at least calibrated, tested, validated, and possibly updated from real-world data – rather than demonstration models, prototypes, or hypothetical models. Anyone who has constructed and used models to solve practical problems has learned that the process is never as straightforward as in textbook cases, due to some ever-present difficulties: inability of the model to capture some features of the problem, missing input data, untractability

(model size/computing time), and non-validable results. Our aim in the book is, also, to identify and document the techniques invented by practitioners to overcome or reduce the impact of these difficulties.

Besides a brief theoretical introduction to Bayesian networks, based on some basic, easily reproducible, examples (Chapter 1), the substance of this book is 20 application chapters (Chapters 2–21), written by invited authors.

In selecting the applications, we strove to achieve the following objectives:

1. cover the major types of applications of Bayesian networks: diagnosis, explanation, forecasting, classification, data mining, sensor validation, and risk analysis;
2. cover as many domains of applications as possible: industry (energy, defense, robotics), computing, natural and social sciences (medicine, biology, ecology, geology, geography), services (banking, business, law), while ensuring that each application is accessible and attractive for nonspecialists of the domain;
3. invite ‘famous names’ of the field of Bayesian networks, but also authors who are primarily known as experts of their field, rather than as Bayesian networks practitioners; find a balance between corporate and academic applications;
4. describe the main features of the most common Bayesian network software packages.

Chapter 22 is a synthesis of the application chapters, highlighting the most promising fields and types of applications, suggesting ways that useful lessons or applications in one field might be used in another field, and analysing, in the perspective of artificial intelligence, where the field of Bayesian networks as a whole is heading.

A companion website for this book can be found at: [www.wiley.com/go/pourret](http://www.wiley.com/go/pourret)

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# 1

# Introduction to Bayesian networks

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## 1.1 Models

### 1.1.1 Definition

The primary objective of this book is to discuss the power and limits of Bayesian networks for the purpose of constructing real-world models. The idea of the authors is not to extensively and formally expound on the formalism of mathematical models, and then explain that these models have been – or may be – applied in various fields; the point of view is, conversely, to explain why and how some recent, real-world problems have been modeled using Bayesian networks, and to analyse what worked and what did not.

Real-world problems – thus the starting point of this chapter – are often described as *complex*. This term is however seldom defined. It probably makes more sense to say that human cognitive abilities, memory, and reason are limited and that reality is therefore difficult to understand and manage. Furthermore, in addition to the biological limitations of human capabilities, a variety of factors, either cultural (education, ideology), psychological (emotions, instincts), and even physical (fatigue, stress) tend to distort our judgement of a situation.

One way of trying to better handle reality – in spite of these limitations and biases – is to use representations of reality called *models*. Let us introduce a basic example.

**Example 1.** Consider an everyday life object, such as a DVD recorder. The life cycle of the device includes its phases of design, manufacture, marketing, sale, use, possibly break down/repair, and disposal. The owner of a DVD recorder is involved in a temporally delimited part of its life cycle (i.e., when the object is in his/her living-room) and has a specific need: being able to use the device. The user's instruction manual of a DVD recorder is a description of the device, written in natural language, which exclusively aims at explaining how the device is to be operated, and is expressly dedicated to the owner: the manual does not include any internal description of the device.

In this example, the user's instruction manual is a *model* of the DVD recorder.

The 20 application chapters of this book provide numerous examples of models: models of organizations (Japanese electrical companies), of facts (criminal cases), of individuals (students in a robotics course, patients suffering from liver disorders), of devices (a sprinkler system), of places (potentially 'mineralized' geographic areas in India), of documents (texts of the parliament of Andalusia), of commodities (Chilean wines), or of phenomena (crime in the city of Bangkok, terrorism threats against US military assets). These parts of reality are material or immaterial: we will use the word 'objects' to refer to them.

These objects, which are delimited in time and space, have only one common point: at some stage of their life cycle (either before they actually occurred in reality, or in 'real-time' when they occurred, or after their occurrence) they have been modeled, and Bayesian networks have been employed to construct the model.

Example 1 suggests that the purpose of a model is to satisfy a need of some person or organization having a particular interest in one or several aspects of the object, but not in a comprehensive understanding of its properties. Using the terminology of corporate finance, we will refer to these individuals or groups of individuals with the word *stakeholders*. Examples of stakeholders include users, owners, operators, investors, authorities, managers, clients, suppliers, competitors. Depending on the role of the stakeholder, the need can be to:

- document, evaluate, operate, maintain the object;
- explain, simulate, predict, or diagnose its behavior;
- or – more generally – make decisions or prepare action regarding the object.

The very first benefit of the model is often to help the stakeholder to explicitly state his need: once a problem is explicitly and clearly expressed, it is sometimes not far from being solved.

The construction of a model involves the intervention of at least one human *expert* (i.e., someone with a practical or theoretical experience of the object), and is sometimes also based upon direct, uninterpreted observations of the object. Figure 1.1 illustrates this process: in Example 1, the object is the DVD recorder; the stakeholder is the owner of the device, who needs to be able to perform the installation, connections, setup and operation of the DVD recorder; the model is the user's instruction manual, which is based on the knowledge of some expert

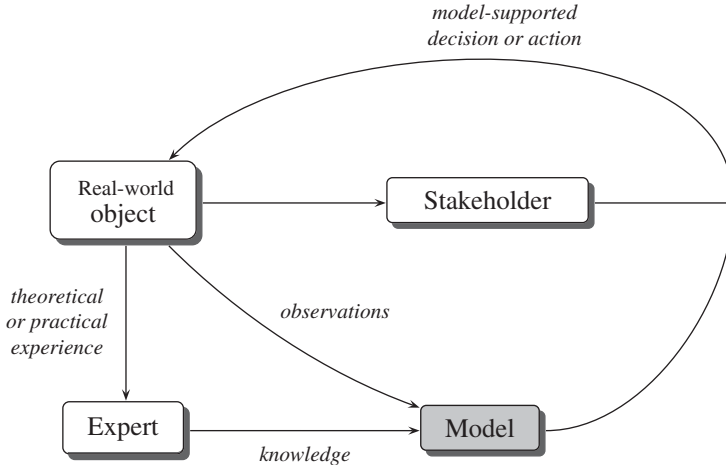


Figure 1.1 Construction and use of a model.

(i.e., the designer), and describes the object from the stakeholder’s point of view, in an understandable language; the model-supported action is the use of the device.

Based on the experience of the models discussed in this book, we may agree on the following definition – also fully consistent with Example 1.

**Definition 2 (Model)** *A model is a representation of an object, expressed in a specific language and in a usable form, and intended to satisfy one or several need(s) of some stakeholder(s) of the object.*

### 1.1.2 Use of a model: the inference process

Definition 2 specifies that models are written in a *usable* form. Let us analyse how models are used, i.e., explicitly state the *model-supported decision or action* arrow shown in Figure 1.1.

When the model includes an evaluation of the stakeholder’s situation, or a recommendation of decision or action, then the stakeholder makes his decision on the basis of the evaluation or follows the recommendation.

However, most models require – prior to their use – to be adapted to the specific situation of the stakeholder, by the integration of input data.

In Example 1, the input data are the type of the device, the information displayed by the device, and the actions already carried out by the user. The output information is the next action to be done.

Models are thus used to produce information (evaluations, appropriate decisions or actions) on the basis of some input information, considered as valid. This process is called *inference*.

Table 1.1 The inference process: given some input data, what can be inferred from the knowledge of the melting point of gold?

| Input data                                       | Inferred information                  |
|--|---------------------------------------|
| The ring is of solid gold. Temperature is 1000°C | The ring won't melt.                  |
| Temperature is 1000°C. The ring melts.           | The ring is not of gold.              |
| Temperature is 1100°C. The ring does not melt.   | The ring is not of gold.              |
| Temperature is 1100°C. The ring melts.           | The ring is possibly of gold.         |
| The ring is of solid gold. It does not melt.     | The temperature is lower than $T_m$ . |

For example, if we assume that the statement

$$\text{The melting point of gold is } T_m = 1064.18^\circ\text{C} \tag{1.1}$$

is true, then it constitutes a model which can be used in numerous ways, depending on the available input data: Table 1.1 shows examples of information that can be inferred using the model, on the basis of some input data.

The use of real-world models is not always as straightforward as in the example of Table 1.1. For example, the model user may have some prior belief regarding whether the ring is of gold or not. Also, whether the rings melts or does not melt might be difficult to tell; finally, the temperature might not be known with a high level of precision. In such cases, the use of the model will not produce definitive ‘true’ statement, but just modify one’s assessment of the situation. For instance, if the ring is *believed* not to be of gold, the temperature *estimated* at 1100°C, and the ring *seems* not to melt, then the model output is that the ring is most unlikely of gold. If the uncertainties in the input data can be quantified with probabilities, then the use of the model increases the probability that the ring is not of gold. This is an example of *probabilistic* inference.

### 1.1.3 Construction

Definition 2 is extremely general: the way a model is constructed obviously depends on several factors, such as the nature of the object, the stakeholder’s need(s), the available knowledge and information, the time and resources devoted to the model elaboration, etc. Nevertheless, we may identify two invariants in the process of constructing a model.

#### 1.1.3.1 Splitting the object into elements

One of the precepts of Descartes in his famous *Discourse on the Method* is ‘to divide each of the difficulties under examination into as many parts as possible and as might be necessary for its adequate solution’ [126].

Indeed, modeling an object implies splitting it into elements and identifying a number of aspects or attributes that characterise the elements.

Deriving a collection of attributes from one single object could at first glance appear as a poor strategy, but this really helps to simplify the problem of satisfying

the stakeholder's need: on one hand, each of the attributes is easier to analyze than the whole object; on the other hand, only the attributes which are relevant for the need of the stakeholder are taken into consideration.

### 1.1.3.2 Saying how it works: the modeling language

To allow inference, the model must include a description of how the elements interact and influence each other. As said in Definition 2, this involves the use of a specific language, which is either textual (natural language, formal rules), mathematical (equations, theorems), or graphical (plans, maps, diagrams).

The well-known consideration 'A good drawing is better than a long speech' also applies to models. Figures are more easily and quickly interpreted, understood and memorized than words. Models which are represented or at least illustrated in a graphical form tend to be more popular and commonly used. It is possible to admit that, throughout history, most successful or unsuccessful attempts of mankind to overcome the complexity of reality have involved, at some stage, a form a graphical representation. Human paleolithic cave paintings – although their interpretation in terms of hunting magic is not beyond dispute – may be considered as the first historical models, in the sense of Definition 2.

## 1.2 Probabilistic vs. deterministic models

### 1.2.1 Variables

During the modeling process, the exact circumstances in which the model is going to be used (especially, what input data the model will process) are, to a large extent, unknown. Also, some of the attributes remain unknown when the model is used: the attributes which are at some stage unknown are more conveniently described by *variables*.

In the rest of the chapter, we therefore consider an object which is characterized by a collection of numerical or symbolic variables, denoted  $X_1, X_2, \dots, X_n$ . To simplify the formalism, we suppose that the domain of each of the  $X_j$  variables, denoted  $\mathcal{E}_j$ , is discrete.

One may basically distinguish two kinds of variables. The first category is the variables whose values are specified (typically by the stakeholder) at some stage of the model use. Such variables typically describe:

- some aspects of the context: the values of such variables are defined prior to the use of the model and do not change afterwards (in Example 1: which version of the DVD recorder is being installed?);
- some aspects of the object which are directly 'controlled' by the stakeholder:
  - attributes of the object the stakeholder can observe (in Example 1: what is displayed on the control screen of the device?);

- decisions or actions he or she could carry out (in Example 1: what button should be pressed?).

The second category of variables are those which are not directly or not completely controlled – although possibly influenced – by the stakeholder’s will (in Example 1: Is the device well setup/out of order?).

At any stage of the practical use of a model, the variables under control have their value fixed, and do not explicitly behave as variables anymore. We may therefore suppose without any loss of generality that the model only comprises variables which are not under control.

### 1.2.2 Definitions

A deterministic model is a collection of statements, or rules regarding the  $X_i$  variables. A sentence (in natural language) such as

$$\text{Elephants are grey} \quad (1.2)$$

is a deterministic model which can be used to identify the race of an African mammal, on the basis of its colour. This model can be considered as a more elegant and intuitive expression of an equation such as

$$\text{colour}(\text{elephant}) = \text{grey},$$

or of the following formal rule:

$$\text{if animal=elephant then colour=grey.}$$

Also, if  $X_1$  and  $X_2$  are variables that correspond to the race and colour of a set of mammals, then the model can be converted in the formalism of this chapter:

$$\text{if } X_1 = \text{'elephant'}, \text{ then } X_2 = \text{'grey'}. \quad (1.3)$$

If the number of variables and the number of possible values of each of them are large, then the object can theoretically reside in a considerable number of states. Let us suppose however that all of these configurations can be enumerated and analyzed. Then the probabilistic modeling of the object consists in associating to any object state (or set of states), a *probability*, i.e., a number between 0 and 1, quantifying how plausible the object state (or set of states) is. We thus define the *joint probability distribution* of the set of variables  $X_1, X_2, \dots, X_n$ , denoted

$$\mathbb{P}(X_1, X_2, \dots, X_n).$$

The domain of this function is the Cartesian product  $\mathcal{E}_1 \times \dots \times \mathcal{E}_n$  and its range is the interval  $[0;1]$ .



### 1.2.3 Benefits of probabilistic modeling

#### 1.2.3.1 Modeling power

As far as modeling capability is concerned, probabilistic models are undeniably more powerful than deterministic ones. Indeed, a deterministic model may always be considered as a particular or simplified case of probabilistic model. For example, the model of sentence (1.2) above is a particular case of a slightly more complicated, probabilistic one:

$$x\% \text{ of elephants are grey} \quad (1.4)$$

where  $x = 100$ . This model can also be written using a conditional probability:

$$\mathbb{P}(X_2 = \text{'grey'} \mid X_1 = \text{'elephant'}) = x. \quad (1.5)$$

Incidentally, the probabilistic model is a more appropriate representation of reality in this example, since, for instance, a rare kind of elephant is white.

#### 1.2.3.2 The power of doubt – exhaustiveness

Doubt is a typically human faculty which can be considered as the basis of any scientific process. This was also pointed out by Descartes, who recommended ‘never to accept anything for true which is not clearly known to be such; that is to say, carefully to avoid precipitancy and prejudice, and to comprise nothing more in one’s judgement than what was presented to one’s mind so clearly and distinctly as to exclude all ground of doubt.’ The construction of a probabilistic model requires the systematic examination of all possible values of each variable (each subset  $\mathcal{E}_j$ ), and of each configuration of the object (i.e., each element of  $\mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ ). This reduces the impact of cultural and psychological biases and the risk to forget any important aspect of the object. Furthermore, it is hard to imagine a more precise representation of an object: each of the theoretically possible configurations of the object is considered, and to each of them is associated one element of the infinite set  $[0;1]$ .

#### 1.2.3.3 Usability in a context of partial information

In many circumstances, probabilistic models are actually much easier to use than deterministic ones. Let us illustrate this with an example.

**Example 3.** A hiker has gone for a walk in a forest, and brings back home some flashy coloured mushrooms. He wants to decide whether he will have them for dinner, or not. Instead of consulting an encyclopedia of mushrooms, he phones a friend, with some knowledge of the domain. His friend tells him that:

75% of mushrooms with flashy colours are poisonous.

In this example, a deterministic model, such as an encyclopedia of mushrooms, would certainly help identify the exact nature of the mushrooms, but this requires an extensive examination, and takes some time. The probabilistic model provided by the hiker's friend is more suitable to satisfy his need, i.e., make a quick decision for his dinner, than the deterministic one. In fact, if the hiker wants to use the only available information 'the mushroom is flashy-coloured', then a form of probabilistic reasoning – possibly a very basic one – is absolutely necessary.

### 1.2.4 Drawbacks of probabilistic modeling

In spite of its benefits listed in the previous paragraph, the joint probability distribution  $\mathbb{P}(X_1, X_2, \dots, X_n)$  is rarely employed *per se*. The reason is that this mathematical concept is rather unintuitive and difficult to handle.

Firstly, it can be graphically represented only if  $n = 1$  or 2. Even in the simplest nontrivial case  $n = p = 2$  (illustrated in Figure 1.2), the graphical model is rather difficult to interpret. When  $n \geq 3$ , no graphical representation is possible, which, as mentioned above, restrains the model usability.

Secondly, the joint probability distribution gives rise to a phenomenon of combinatorial explosion. For instance, if each variable takes on  $p$  different values ( $p \geq 1$ ), then the joint probability distribution has to be described by the probabilities of  $p^n$  potential configurations of the object, i.e., ten billion values if  $n = p = 10$ .

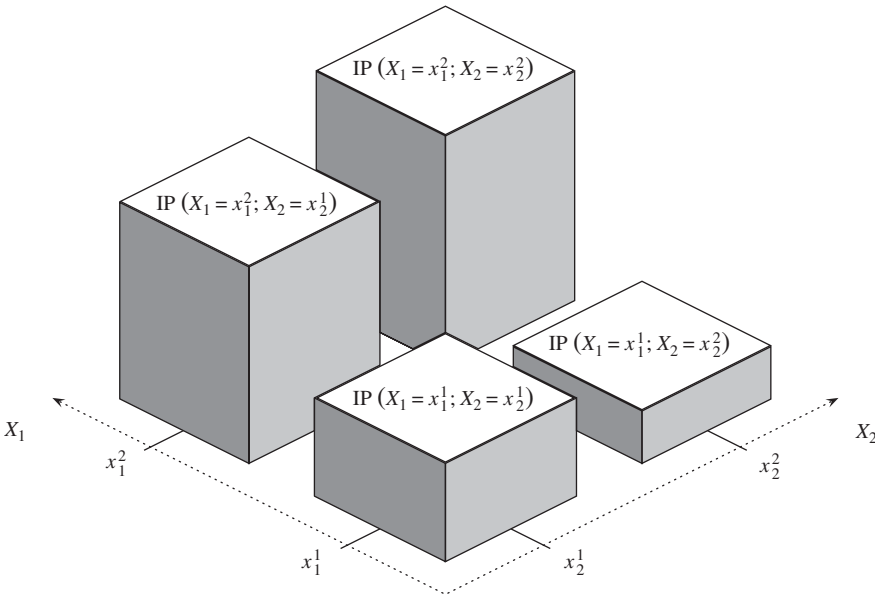


Figure 1.2 Representation of the joint probability distribution of a pair of random variables  $(X_1, X_2)$ .

## 1.3 Unconditional and conditional independence

### 1.3.1 Unconditional independence

Following Descartes's precept of dividing the difficulties, one may try to split the set of  $n$  variables into several subsets of smaller sizes which can relevantly be analyzed separately.

Suppose for example that the set of  $n$  variables may be divided into two subsets of sizes  $j$  and  $n - j$  such as:

$$\mathbb{P}(X_1, X_2, \dots, X_n) = \mathbb{P}(X_1, \dots, X_j) \mathbb{P}(X_{j+1}, \dots, X_n). \quad (1.6)$$

Then the modeling problem can be transformed into two simpler ones. One can derive the joint probability of subset  $X_1, \dots, X_j$ , then that of subset  $X_{j+1}, \dots, X_n$ , and use Equation (1.6) to obtain the complete model.

The equality of two functions expressed by Equation (1.6) means that the subsets of variables  $(X_1, \dots, X_j)$  and  $(X_{j+1}, \dots, X_n)$  are independent, or – to avoid confusion with a concept which is defined below – *unconditionally* independent. This means that any information regarding the  $(X_1, \dots, X_j)$  subset (for instance, ' $X_1 = 7$ ' or ' $X_1 + X_2 > 3$ ') does not change the probability distribution of the second subset  $(X_{j+1}, \dots, X_n)$ .

However, unconditional independence between two subsets of variables is very unlikely to happen in real-world models. If it does happen, the initial definition of the object is not relevant: in such a case, it makes more sense to construct two separate models.

### 1.3.2 Conditional independence

A more common – or at least much more reasonably acceptable in real-world models – phenomenon is the so-called 'conditional independence'. Let us introduce this concept by two examples.

#### 1.3.2.1 The lorry driver example

**Example 4.** A lorry driver is due to make a 600-mile trip. To analyze the risk of his falling asleep while driving, let us consider whether (1) he sleeps 'well' (more than seven hours) on the night before and (2) he feels tired at the beginning of the trip.

In this example, there are obvious causal relationships between the driver's sleep, his perceived fatigue, and the risk of falling asleep: the three variables are dependent. Let us suppose however that we know that the lorry driver feels tired at the beginning of the trip. Then knowing whether this is due to a bad sleep the previous night, or to any other reason is of no use to evaluate the risk. Similarly, if the lorry driver does *not* feel tired at the beginning of the trip, one may then consider that the quality of his sleep on the night before has no influence on the risk. Given these

considerations, the risk of falling asleep is said to be *conditionally independent* of the quality of sleep, *given* the lorry driver's fatigue.

To express it formally, let  $X_1$ ,  $X_2$  and  $X_3$  be binary variables telling whether the lorry driver sleeps well the night before, whether he feels tired at the beginning of the trip, and whether he will fall asleep while driving. Then  $X_3$  is independent of  $X_1$ , for any given value of  $X_2$ . In terms of probabilities, we have:

$$\mathbb{P}(X_3 | X_1 \text{ and } X_2) = \mathbb{P}(X_3 | X_2). \quad (1.7)$$

In such a case, knowing the values of  $X_1$  and  $X_2$  is not better than knowing only the value of  $X_2$ , and it is useless to describe the behavior of  $X_1, X_2, X_3$  by a function of three variables; indeed, we may deduce from Equation (1.7):

$$\mathbb{P}(X_1, X_2, X_3) = \mathbb{P}(X_1) \mathbb{P}(X_2 | X_1) \mathbb{P}(X_3 | X_2), \quad (1.8)$$

which shows that the risk model can be constructed by successively studying the quality of sleep, then its influence on the state of fatigue, and then the influence of the state of fatigue on the risk of falling asleep.

### 1.3.2.2 The doped athlete example

**Example 5.** In a sports competition, each athlete undergoes two doping tests, aimed at detecting if he/she has taken a given prohibited substance: test  $A$  is a blood test and test  $B$  a urine test. The two tests are carried out in two different laboratories, without any form of consultation.

It is quite obvious in Example 5 that the results of the two tests are not independent variables. If test  $A$  is positive, then the participant is likely to have used the banned product; then test  $B$  will probably be also positive.

Now consider a participant who has taken the banned substance. Then tests  $A$  and  $B$  can be considered independent, since the two laboratories use different detection methods. Similarly, tests  $A$  and  $B$  can be considered independent when the participant has *not* taken the banned substance: the results of both tests are conditionally independent, given the status of the tested athlete. Formally, if  $X_1$  is a binary variable telling whether the athlete is 'clean' or not,  $X_2$  is the result of test  $A$ , and  $X_3$  the result of test  $B$ , we can write:

$$\mathbb{P}(X_3 | X_1 \text{ and } X_2) = \mathbb{P}(X_3 | X_1). \quad (1.9)$$

Equation (1.9) can exactly be translated into 'knowing whether the athlete has taken the substance is enough information to estimate the chances of test  $B$  being positive'. A symmetrical equation holds regarding test  $A$ :

$$\mathbb{P}(X_2 | X_1 \text{ and } X_3) = \mathbb{P}(X_2 | X_1). \quad (1.10)$$

Here again, it is useless to describe the behavior of  $X_1, X_2, X_3$  by a function of three variables. Equations (1.9) and (1.10) yield:

$$\mathbb{P}(X_1, X_2, X_3) = \mathbb{P}(X_1) \mathbb{P}(X_2 | X_1) \mathbb{P}(X_3 | X_1), \quad (1.11)$$

which means that considerations on the proportion of doped athletes  $\mathbb{P}(X_1)$ , and on the reliabilities of each tests are sufficient to construct the model.

## 1.4 Bayesian networks

### 1.4.1 Examples

In the lorry driver and doped athlete examples, we have identified the most direct and significant influences between the variables, and simplified the derivation of the joint probability distribution. By representing these influences in a graphical form, we now introduce the notion of Bayesian network.

In Example 4, our analysis has shown that there is an influence of variable  $X_1$  on variable  $X_2$ , and another influence of variable  $X_2$  on variable  $X_3$ ; we have assumed that there is no direct relation between  $X_1$  and  $X_3$ . The usual way of representing such influences is a diagram of nodes and arrows, connecting influencing variables (parent variables) to influenced variables (child variables). The structure corresponding to Example 4 is shown in Figure 1.3.

Similarly, the influences analyzed in Example 5 may be represented as shown in Figure 1.4.

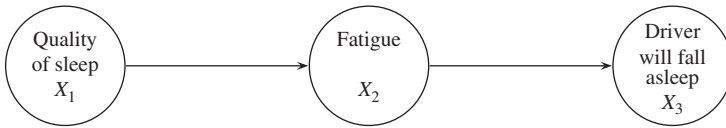


Figure 1.3 A representation of the influences between variables in Example 4. Variable  $X_3$  is *conditionally independent* of  $X_1$  given  $X_2$ .

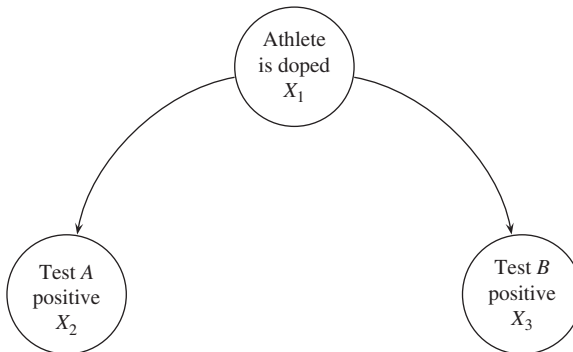


Figure 1.4 A representation of the influences between variables in Example 5. Variables  $X_2$  and  $X_3$  are *conditionally independent* given  $X_1$ .

Considering the graphical structures of Figures 1.3 and 1.4, and more precisely the parents of each variable, we observe that both Equations (1.8) and (1.11) can be written in the following form:

$$\mathbb{P}(X_1, X_2, X_3) = \mathbb{P}(X_1 | \text{parents}(X_1)) \mathbb{P}(X_2 | \text{parents}(X_2)) \mathbb{P}(X_3 | \text{parents}(X_3)). \quad (1.12)$$

Equation (1.12) is the formal definition of a Bayesian network, in the three-variable case: through a process of analyzing and sorting out the unconditional independences between the three variables, we have been able to convert  $\mathbb{P}(X_1, X_2, X_3)$  into a product of three conditional probabilities. This definition is generalized in the next paragraph.

### 1.4.2 Definition

**Definition 6 (Bayesian network)** *Let us consider  $n$  random variables  $X_1, X_2, \dots, X_n$ , a directed acyclic graph with  $n$  numbered nodes, and suppose node  $j$  ( $1 \leq j \leq n$ ) of the graph is associated to the  $X_j$  variable. Then the graph is a Bayesian network, representing the variables  $X_1, X_2, \dots, X_n$ , if:*

$$\mathbb{P}(X_1, X_2, \dots, X_n) = \prod_{j=1}^n \mathbb{P}(X_j | \text{parents}(X_j)), \quad (1.13)$$

where:  $\text{parents}(X_j)$  denotes the set of all variables  $X_i$ , such that there is an arc from node  $i$  to node  $j$  in the graph.

As shown in the examples, Equation (1.13) simplifies the calculation of the joint probability distribution. Let us suppose for instance that each variable has  $p$  possible values, and less than three parent variables. Then the number of probabilities in the model is lower than  $n \cdot p^4$ , although the object can reside in  $p^n$  configurations. If  $n = p = 10$ , the reduction factor is greater than one hundred thousands.

A crucial point is that this simplification is based on an graphical, intuitive representation, and not on some highly technical considerations. A diagram of boxes and arrows can be easily interpreted, discussed and validated on a step-by-step basis by the stakeholders: there is no ‘black box’ effect in the modeling process.

Another important remark can be deduced from Definition 6.

**Proposition 7.** *Any joint probability distribution may be represented by a Bayesian network.*

Indeed, we may formally express  $\mathbb{P}(X_1, X_2, \dots, X_n)$  as follows:

$$\begin{aligned} \mathbb{P}(X_1, X_2, \dots, X_n) &= \mathbb{P}(X_1) \mathbb{P}(X_2, \dots, X_n | X_1) \\ &= \mathbb{P}(X_1) \mathbb{P}(X_2 | X_1) \cdots \mathbb{P}(X_3, \dots, X_n | X_1, X_2) \\ &= \dots \\ &= \mathbb{P}(X_1) \mathbb{P}(X_2 | X_1) \cdots \mathbb{P}(X_n | X_1, \dots, X_{n-1}). \end{aligned} \quad (1.14)$$