Dynamics for Engineers

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Contents

Preface

The undergraduate curricula of almost all disciplines of engineering include some courses on modelling and analysis of dynamical systems. They include modelling approaches based on block diagram, signal flow graph, and so on, which obtain the system models in the form of transfer functions. Analysis of stability and other properties is then carried out with the help of root locus, Nyquist criterion and similar tools. At base, the Laplace transform remains the tool of analysis of the engineer. This restricts the exposure of the students to the behaviour of linear systems only.

Over the past few years, there has been an increasing realization that most of the physical systems are nonlinear, and linearity is a very special case. Most of the systems that an engineer has to deal with are nonlinear, and nonlinear dynamics pervades the engineer's workplace. But the training of the engineer often renders him/her hopelessly short of the challenges. This book is aimed at addressing this problem, first by introducing those methodologies of system modelling that make no reference to linearity, and then by developing an understanding of dynamics where linear system description is put in proper perspective – as local linear approximation in the neighbourhood of an equilibrium point.

The book is divided into two parts. In the first part, the methods and techniques for translating a physical problem into mathematical language by formulating differential equations are introduced. Some part of it draws from classical mechanics, but instead of working with particles and groups of particles as in classical mechanics textbooks, I have dealt mainly with electrical and mechanical systems so that the ideas developed can easily be applied in engineering.

The basic methodology of deriving differential equations follows from Newton's laws for mechanical systems and from Kirchoff's laws for electrical circuits. However, the Newtonian method in its pure form is not suitable for handling practical systems. In Chapter 2, the Newtonian formalism is introduced, and the practical problems of this method are illustrated. In Chapter 3, the application of Kirchoff's laws in derivation of dynamical equations for electrical circuits is considered. The mesh current method, the node voltage method and the more general graph theoretic methods are introduced. This chapter is particularly useful for students of the electrical sciences, and may be skipped by those of the other disciplines without breaking the continuity of exposition.

The Lagrangian method is introduced in Chapter 4, and its application in handling electrical, mechanical and electromechanical systems is illustrated. To show the basic unity of dynamical systems, an equivalence of the mechanical and the electrical systems is shown at relevant places. In order to make the model amenable to the solution techniques to be introduced in the latter part of the book, first-order differential equations must be obtained. Chapter 5 shows how the Lagrangian equations can be used to obtain the equations in first-order through the definition of conjugate momenta. The Hamiltonian formalism – which allows one to obtain the first-order equations directly – is generally applied to conservative systems. I have shown how this approach, with the inclusion of the dissipative term, can be made useful in handling engineering systems also.

Then I have introduced the Bond Graph methodology – a powerful technique for obtaining first-order differential equations for a wide variety of physical and engineering systems. This method algorithmizes the process of obtaining equations, so that once the bond graph of a system is formulated, computer programs can handle the job of obtaining equations and simulating the model. This method is widely applicable to engineering problems, but has not yet entered the mainstream engineering curriculum. Engineering students are often found to put off learning this powerful technique for some day "when there is time at hand". In this book, the basic elements of this method are introduced in a span of only 40 pages without getting into the nitty-gritty of modelling complicated systems, so that the reader can get a feel of bond graph–based system modelling without spending too much time on it.

The main advantage of the methods introduced in Part I is that they are equally applicable to both linear and nonlinear systems. I believe, these will constitute the basic modelling tools in the hands of the scientist and the engineer in future.

After the differential equations are obtained, one has to *solve them*, and from the solutions, one has to understand how a given system is going to behave in specific circumstances. The second part of the book is aimed at developing an intuitive understanding of the dynamics of physical systems. For this, the concepts of *state space* and *vector field* are introduced, and a geometric view of the dynamics in the state space is provided – since with the availability of computers and computer graphics that viewpoint has become visualizable and intuitively appealing.

The method of locally linearizing a nonlinear system through the Jacobian matrix is introduced, and the dynamics of linear systems are then analysed. There are many approaches to solving linear differential equations, and I have chosen the one that allows the relationship of the eigenvalues and eigenvectors with system dynamics to be highlighted. This gives a geometric view of the dynamics, and facilitates a smooth transition to the understanding of the dynamics of nonlinear systems. The special features of the dynamics of nonlinear systems, like limit cycles, high-period orbits and chaotic orbits, are then discussed.

Even though limit cycles find wide application in engineering wherever some oscillatory behaviour is desired (as in oscillators, power electronics, etc.), treatment of the stability of limit cycles is rarely found in dynamics or control textbooks. An approach to this problem, developed in nonlinear dynamics, has now reached sufficient maturity to deserve being taught at the undergraduate level. In this book, I have illustrated the powerful method of obtaining discrete-time models or “maps”, developed by the French mathematician Henri Poincaré, and have discussed the various ways a limit cycle can lose stability. In that process, I have included an exposition on the dynamics of discrete-time systems, which is finding increasing application in engineering.

The content of this book is suitable for teaching undergraduate students of all branches of engineering at the second- or third-year level. Though the scope of the topic is much wider, the material presented in this book is limited to the extent that can be taught in a one-semester course – which, I feel is a proper supplement of a control systems course. Some parts of it can also be integrated in an existing course on control theory.

The book addresses dynamics problems coming from a wide range of engineering disciplines, and can be used by mechanical engineers, electrical engineers, aeronautical engineers, civil engineers, and so on. For example, an electrical engineer who is not interested in mechanical systems may read Chapters 1 and 3 to pick up the methods of obtaining differential equations specific to electrical systems, and then may proceed on to Chapter 7 onward to develop ideas of dynamics. For those who have to deal with electromechanical systems like relays, motors and other electrically actuated mechanical systems, Chapters 4, 5 and 6 will be particularly useful. The students of mechanical, aeronautical and allied engineering disciplines, on the other hand, may skip Chapter 3.

This book can also be used by those who are primarily interested in linear systems, because the methods of obtaining differential equations are equally applicable to linear systems. Moreover, Chapters 9 and 10 specifically deal with solving linear differential equations and developing a visual impression of linear dynamics. Courses with such leaning may leave Chapters 11 and 12 as options or for a later study.

It should be kept in mind that this book is not meant to be an exhaustive treatise on system modelling. The purpose of Part I is to acquaint the engineering students with the general modelling approaches, which they can later pursue to any level of detail following the lead provided at the end of each chapter. This is also not a typical nonlinear dynamics book, and includes only those aspects of nonlinear dynamics which a twenty-first century engineer should be exposed to. I have deliberately chosen not to burden the students with too much material.

The subject matter of this book was developed through the teaching of the subject "Dynamics of Physical Systems" at the Indian Institute of Technology, Kharagpur, India. I am indebted to the vision of Prof. Y. P. Singh who was instrumental in introducing this subject as an important component in the curriculum of the Bachelor's degrees in Electrical Engineering and Energy Engineering at the IIT, Kharagpur. In the course of writing the book, valuable feedback was received from the students to whom this subject was taught. They contributed in many ways in giving it the shape of a concrete course material. Among them, Mr. Manish Agarwal deserves special mention, for he helped in formulating many of the exercise problems.

I am particularly indebted to Prof. Amalendu Mukherjee of the Mechanical Engineering Department of our institute, who went through the whole manuscript and made many valuable suggestions. Mr. Ashoke Mukherjee and Prof. G. P. Rao helped me in improving the language. I am indebted to the Continuing Education Cell of IIT, Kharagpur, for providing the financial support for the preparation of the manuscript. I also thank Ms. Wendy Hunter of John Wiley & Sons for her continuous support at times of difficulty. Last, but not the least, I thank my daughter Anita, my son Kiran and my wife Manua for being patient when I could not spare time for my family while preparing the manuscript.

I invite the students and teachers who will use the book to send me comments and suggestions for its further improvement.

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Part I

Obtaining differential equations for physical systems

Everything in nature is continuously changing. However static and unchanging some of the things may look, they are all changing – some fast and some slowly. And any system whose status changes with time is called a *dynamical system*.

The study of dynamical systems has an intrinsic value for scientists who attempt to understand how nature functions. For engineers, it is the bread and butter. Everything he has to deal with is a dynamical system. He has to design them, operate them and he has to predict how a given system is going to behave in a particular circumstance.

Dynamical systems are described by *differential equations* – whose solutions show how the variables of the system depend on the independent variable time. Hence the thrust of the following chapters will be to formulate the differential equations for different types of systems.

Dynamics for Engineers S. Banerjee

1

Introduction to System Elements

1.1 Introduction

Any electrical, mechanical or electromechanical system is composed of some *elements* that interact with one another to produce the dynamics of the total system. To model the dynamics of the system, therefore, it is necessary to understand the dynamical properties of the individual elements.

Though, in general, the elements of a practical engineering system may be quite complex, with complicated (often nonlinear) dynamical properties, we can take the first step by considering a few discrete elements often found in practical systems, and idealizing them into linear components. Often, the component parts of a system may be spatially extended, and if one looks into what goes on inside that component, the problem may become unmanageable. What one does, instead, is to take "lumped representation" of the component where one focuses attention on the way it interacts with other elements through the "end points", and ignores the behaviour at the points inside it. In the following section, we shall discuss some such system elements.

The behaviour of certain electrical quantities is mathematically identical to that of certain mechanical quantities. This enables us to establish an equivalence between electrical and mechanical systems. These equivalences are also discussed in the following sections.

1.1.1 The inertial element

The property of inertia is to resist change in velocity. A moving point mass or a rotating rigid body are examples of the inertial element in mechanical systems. If a force $\mathbf{f}$ is applied on a translational mass that moves with momentum $\mathbf{p}$, then the

Dynamics for Engineers S. Banerjee

fundamental property of the inertial element is given by the relation

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}.$$

With $\mathbf{q}$, $\mathbf{v}$ and $\mathbf{a}$ representing the position, velocity and acceleration respectively, this relation can be written as

$$\mathbf{f} = m\frac{\mathrm{d}^2\mathbf{q}}{\mathrm{d}t^2} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a},$$

which defines the mass m as the slope of the (linear) graph of the magnitude $f = |\mathbf{f}|$ versus $a = |\mathbf{a}|$ (Fig. 1.1(a)). In SI units, the unit of mass is kg, and that of force is Newton (or kg m/s^2).

For the rotation of a rigid body about a fixed axis, the same relationship obtains between the applied torque $\mathbf{f}$ and angular momentum, given by

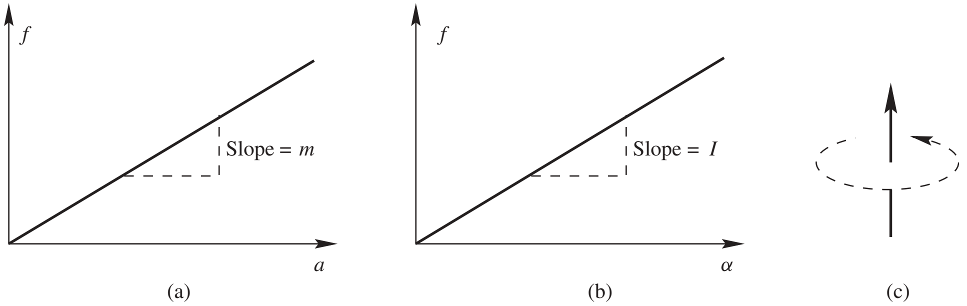
$$\mathbf{f} = I\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = I\boldsymbol{\alpha},$$

where $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the angular velocity and acceleration respectively. The unit of moment of inertia is kg m^2 and that of torque is N m (Newton meters).

For translational motion in rectangular coordinate system, the directions of the vectors are obtained in a straightforward manner. But what represents the direction of the vectors in rotational motion? The vectors in the relation above are obtained by the right-hand rule, with the vector pointing in the direction of the thumb (Fig. 1.1(c)).

Inertial element in the electrical domain is the inductance, whose dynamical property is given by

$$E = L\frac{\mathrm{d}i}{\mathrm{d}t} = L\frac{\mathrm{d}^2q}{\mathrm{d}t^2}.$$



(a) (b) (c)

Figure 1.1 The law of inertia in (a) translational and (b) rotational motion, where the vectors point in the direction shown in (c) by the right-hand corkscrew rule.

Thus the electrical equivalent of mechanical force is voltage or electromotive force, and the electrical equivalent of position is charge. The analogy also indicates that the energy stored in an inductor, $\frac{1}{2}L\dot{q}^2$, is the electrical equivalent of kinetic energy. The unit of inductance is Henry (or V s/A). The property of inertia is to resist change in speed and the property of inductance is to resist change in current.

1.1.2 The compliant element

The property of a compliant element is to resist change in the separation between its end points, that is, to resist compression or stretching. If a translational spring is given a relative displacement of q between the two ends, it produces a force f given by

$$f = kq = k\int v\,\mathrm{d}t,$$

where k is the spring constant (or stiffness). Thus k represents the slope of the graph, assumed to be linear, between the force and the relative displacement, and its unit is N/m. Fig. 1.2(a) shows the characteristics of the hard and soft springs, as well as that of the linear spring represented by the equation above. For torsional springs, the same relationship applies, with f representing the torque, q representing the relative angular displacement and k representing the torsional stiffness.

The corresponding electrical element is the capacitance, whose dynamical relation between the voltage across the capacitor e and the charge in the capacitor q (Fig. 1.2(b)) is given by

$$e = \frac{1}{C}q = \frac{1}{C}\int i\,\mathrm{d}t.$$

The unit of charge is Coulomb (or A s) and that of capacitance is Farad (or A s/V).

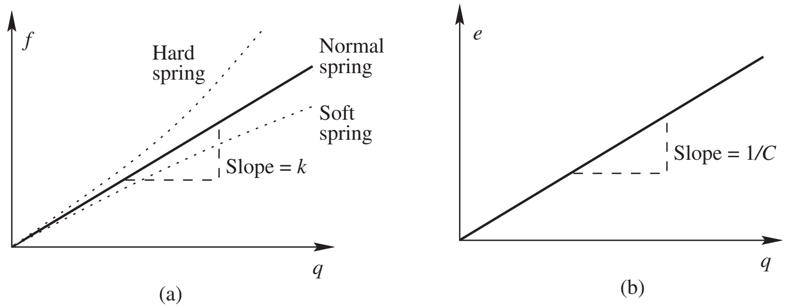


Figure 1.2 (a) The spring element in mechanical translational motion and (b) the capacitance in electrical circuit.

The potential energy stored in a spring is given by $\frac{1}{2}kq^2$, and the electrical equivalent of potential energy is the energy stored in a capacitor, given by $\frac{1}{2C}q^2$.

1.1.3 The resistive element

The electrical resistance is represented by the Ohm's law, which is a linear approximation of the relationship between the voltage across it and the current flowing through it:

$$e = Ri = R\frac{\mathrm{d}q}{\mathrm{d}t}.$$

There are, in the main, two types of mechanical friction or damping:

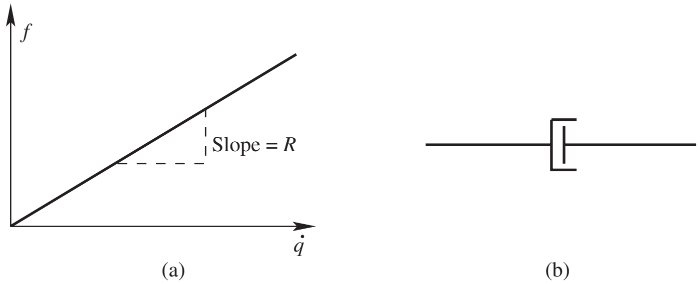
1. Viscous friction
2. Coulomb friction.

Viscous friction is generated when two surfaces separated by a liquid slide against each other. The damping force due to friction opposes the motion, and depends on the nature of fluid flow between the surfaces. The relationship between the relative velocity of sliding and the generated damping force is quite complex, but can be approximated to a linear relationship (Fig. 1.3) given by

$$f = R.v = R\frac{\mathrm{d}q}{\mathrm{d}t}.$$

Therefore viscous friction is similar in nature to electrical resistance. In rotational motion, the same relation applies between the relative angular velocity of the two surfaces and the torque created by friction.

Coulomb friction is generated when two dry surfaces slide against each other. Suppose a solid body is resting on a dry surface and a force is applied to move it.



(a) (b)

Figure 1.3 (a) The characteristics of viscous friction, and (b) the representation of a viscous damper.

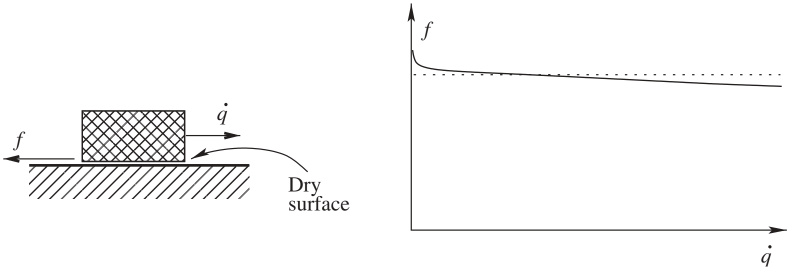


Figure 1.4 The characteristics of Coulomb friction.

As the force is increased gradually, the body does not move till a critical value of force is reached. Upto that point, the applied force equals the *static friction force*. At the critical juncture, just before the beginning of the sliding motion, the static friction force attains a maximum value. As soon as sliding occurs, the nature of the friction force changes, and now it is the *kinetic friction force*. At the beginning of motion, it has a value slightly less than the maximum value of static friction, and decreases with the increase of the relative velocity (Fig. 1.4).

For the purpose of this book, we will mainly assume friction elements to be of the viscous type. The viscous damper will be denoted by the symbol shown in Fig. 1.3(b). Friction elements with nonlinear characteristics can be modelled by considering R to be a variable parameter, and by giving it a suitable functional form.

1.1.4 The voltage source and externally impressed force

The voltage source and externally impressed force represent the *source of effort* in electrical and mechanical system respectively. What they apply on a system – mechanical force in case of mechanical system and electromotive force in case of electrical system – are independent variables, not affected by the rest of the system. The notation of the voltage source is shown in Fig. 1.5(a).

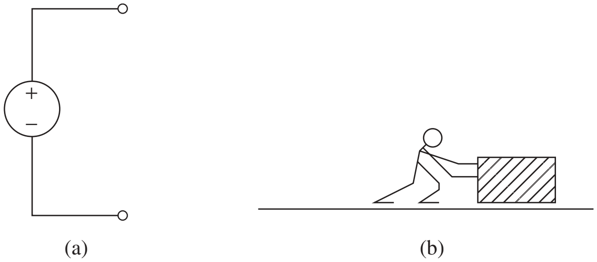


Figure 1.5 The voltage source (a) and the mechanically impressed force (b).

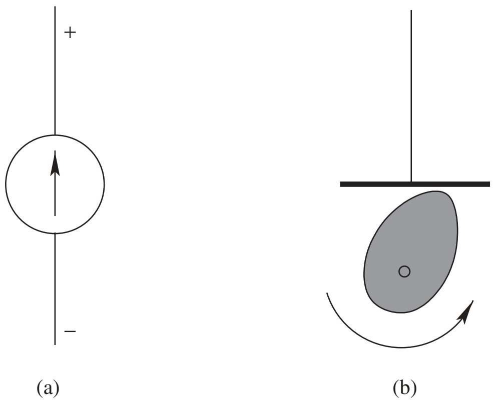


Figure 1.6 (a) The current source in an electrical circuit, and (b) the cam in a mechanical system.

However, the current through a voltage source may vary depending upon the rest of the circuit. Since the power delivered by the voltage source is the product of the voltage and the current, it follows that the power flowing from a source to the rest of the system depends not only on the characteristics of the source but also on the system configuration.

Likewise, the agent applying an external force in a mechanical system has to move along with the mass on which the force is applied (Fig. 1.5(b)), and hence the power delivered depends both on the character of the applied force and the rest of the system.

1.1.5 The current source and externally impressed sources of flow

The correct source in an electrical circuit has the property that the current through it is an independent variable and does not depend on the rest of the circuit (this immediately implies that a current source cannot be open-circuited). But the voltage across the current source and hence the power delivered depends on the rest of the circuit. The notation of the current source is shown in Fig. 1.6(a).

In a mechanical system the equivalent of a current source is a *cam* which imposes a given (externally determined) velocity on some part of a system (Fig. 1.6(b)). The force of interaction between the cam and the rest of the system is variable, and depends on the system dynamics.

1.2 Chapter Summary

Translational or rotational motion in a mechanical system is characterized by three types of quantities:

- inertia (or the tendency to resist acceleration),
- compliance (or the tendency to resist compression and stretching),
- friction (or the tendency to resist motion).

In an electrical system, the dynamics or the change in the variables like voltage, current, and so on, are governed by three types of quantities:

- inductance (or the tendency to resist change in current),
- capacitance (or the tendency to resist change in voltage),
- resistance (or the tendency to resist the flow of charge).

These are quantities that are internal to a system. There are also quantities that are given by things external to the system – the applied force and cam in a mechanical system, and the voltage source and the current source in an electrical system.

There is an equivalence between some mechanical quantities and some electrical quantities, in the sense that their dynamical character is the same. For the sake of convenience, the equivalence between mechanical quantities and electrical quantities is summarized in Table 1.1.

Table 1.1 Equivalence of variables and parameters in mechanical and electrical systems

Mechanical quantity	Electrical quantity
Displacement	Charge
Velocity	Current
Force	Voltage
Mass	Inductance
Spring constant	1/capacitance
Damping coefficient	Resistance
Potential energy	Energy in capacitor
Kinetic energy	Energy in inductor

Further Reading

C. Nelson Dorny, *Understanding Dynamic Systems: Approaches to Modeling, Analysis and Design*, Prentice Hall, Englewood Cliffs, New Jersey, 1993.

R. L. Woods and K. L. Lawrence, *Modeling and Simulation of Dynamic Systems*, Prentice Hall, Upper Saddle River, New Jersey, 1997.

2

Obtaining Differential Equations for Mechanical Systems by the Newtonian Method

The basic methodology for formulation of differential equations was provided by Isaac Newton (1642–1727). He discovered, following Galileo's thoughts, that the application of a force changes the velocity of a body. The rate of change of velocity, or acceleration, is proportional to the applied force. And the mass of the body appears in the constant of proportionality as $F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t}$.

Following Newton, D'Alembert (1717–1783) proposed an equivalent but slightly different viewpoint. He saw the entity called *mass* as having the property of actively trying to maintain status quo. If left to itself, a mass maintains its own velocity with respect to any inertial frame of reference. But if it is not allowed to maintain its inertial status by the application of an external force, an opposing tendency or "inertial force" develops in it, whose magnitude is given by the rate of change of momentum (or mass × acceleration), which acts in opposition to the applied force. These two are equal in magnitude, and the body exists in the unity of the two opposing tendencies.

That gives us a way of writing an equation. And since acceleration is the double derivative of position, the resulting equation would be a differential equation. Thus, following Newton, a simple methodology of understanding the dynamics of any system emerged; just look for the opposing tendencies in a system since motion or change in any system is the result of these opposing tendencies.

▶ **Example 2.1** Consider the mass-spring system in Fig. 2.1. The externally applied force is F. Let the position of the mass m be measured such that it has value zero at the unstretched position of the spring, and is positive in the direction of the applied force.

Dynamics for Engineers S. Banerjee

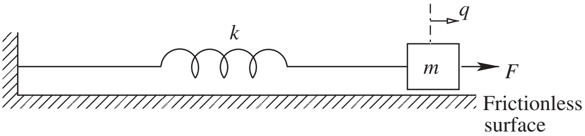


Figure 2.1 The mass-spring system of Example 2.1.

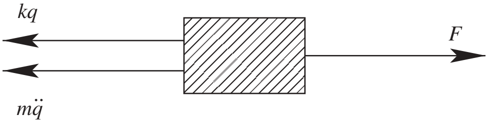


Figure 2.2 The free body diagram of the system of Example 2.1 following D'Alembert's principle.

For understanding the balance of forces, one draws what is known as the free body diagram (FBD) – where the forces acting on each mass point are shown separately. In this system, there is only one mass point, and its FBD is shown in Fig. 2.2. When the elongation of the spring is q, the force on the mass exerted by the spring is kq, where k is the spring constant. Therefore the total force on the mass is $F - kq$. Owing to this force the mass moves, and the rate of change of momentum acts in opposition to the applied force. Equating these two opposing tendencies, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}) = F - kq$$

or

$$m\ddot{q} + kq - F = 0.$$

This is the differential equation that describes the dynamics of the mass. ◄

2.1 The Configuration Space

The positional status of any system can be uniquely specified with the help of a few real numbers. If we have one particle (or a solid body), its position can be specified with a vector $\mathbf{r}$ consisting of three real numbers representing the three coordinates. If we have two solid bodies making up a system, we would need two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ consisting of six position coordinates. Likewise, for a system with N mass points, one requires N vectors $\mathbf{r}_j$ (j varying from 1 to N) consisting of $3N$ coordinates.

An interesting feature is that there is no fundamental distinction between the x-coordinate of one body and the z-coordinate of another. All coordinates have equal value in defining the configuration of the whole system. It is therefore convenient to express the positional status or configuration of a system as a collection of $3N$ real numbers $(x_1, x_2, x_3, \ldots, x_{3N})$, each number representing a position coordinate of one of the participating bodies.

Geometrically, one can visualize a $3N$-dimensional space with the configuration coordinates x_i as the axes. The configuration of a system at any instant would be represented by a *point* in this space. This is the *configuration space*. Dynamics of a system would then be represented by movement of the point in the configuration space.

2.2 Constraints

Most physical systems have *constraints* that restrict the movement of the point in the configuration space. The bob of a pendulum is constrained to move on the surface of a sphere. A ball sliding down a plane is constrained to remain on the plane. People riding a roller coaster are constrained to move along given trajectories.

In these cases, the constraints can be expressed by algebraic equations of the form

$$f(x_1, x_2, x_3, \ldots, x_{3N}, t) = 0$$

or, equivalently,

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_N, t) = 0. \tag{2.1}$$

A constraint of the form (2.1) or reducible to that form, is called a *holonomic* constraint.

▶ **Example 2.2** For a spherical pendulum, the equation of constraint is

$$x_1^2 + x_2^2 + x_3^2 - l^2 = 0.$$

If it is a planar pendulum, there would be another constraint given by $x_3 = 0$. These equations are of the form (2.1) without time dependence. If the point of suspension is moved as a function of time in x_1 direction, the constraint equation becomes

$$[x_1 + f(t)]^2 + x_2^2 + x_3^2 - l^2 = 0.$$

Here the constraint equation is a function of both x_j and time. ◀

Holonomic constraints can be further subdivided into two types. A constraint that does not depend explicitly on time is called *scleronomic*. The equation of a scleronomic constraint would be of the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_N) = 0.$$

The constraints that depend on time and hence are not of the above form are called *rheonomic*. They have to be expressed in the form (2.1).

In a three-dimensional configuration space an algebraic equation of the form (2.1) represents a surface. The configuration point would be constrained to remain on this surface throughout the dynamical motion. If two such constraints exist, the configuration point would be constrained to remain on the line obtained by the intersection of the two surfaces. This argument applies to higher dimensional configuration space also, though the surfaces would then be difficult to visualize. It turns out that every holonomic constraint reduces the dimension of the system by one. The existence of holonomic constraints, therefore, simplifies system modelling problems a great deal.

There are constraints that cannot be described by algebraic equations of the form (2.1). These are called *non-holonomic* systems. Billiard balls are constrained to move within the boundaries of the table. Here the coordinates representing the location of the balls are bounded within some range of real numbers. The constraint is therefore expressible as an *inequality*, which is a signature of non-holonomic systems. Non-holonomic systems also include cases where the *velocities* of the elements are related by an algebraic equation, as[1]

$$f(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \dot{\mathbf{r}}_3, \ldots, \dot{\mathbf{r}}_N, t) = 0. \tag{2.2}$$

► **Example 2.3** (Rosenberg 1977, p. 29) Two objects cannot occupy the same position at any time. To illustrate that this obvious fact imposes a non-holonomic constraint, consider two particles moving along a straight line, and let their positions at any instant of time be given by their distance from some fixed point on that line (Fig. 2.3). Since their positions

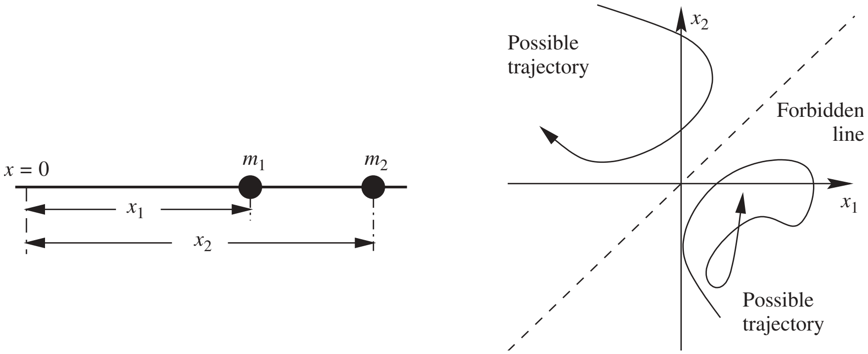


Figure 2.3 Two particles moving on a straight line. Two possible trajectories are shown in the configuration space.

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[1] It is to be noted that a holonomic constraint equation of the form (2.1) can be differentiated term-wise to obtain a relationship between the velocities. Such an equation relating the velocities would not imply that the system is non-holonomic, because it can be integrated to the form (2.1). A differential constraint of the form (2.2) is non-holonomic if it is not integrable.