

Investment Mathematics

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Preface

This book is intended for practitioners in the investment world, particularly those studying for professional examinations. It is also suitable for students of finance and investment in higher education, either as a main text or as an additional reference book.

The book is divided into three parts. Part I looks at the fundamental analysis of investments from a mathematical viewpoint, relying heavily on compound interest techniques which are developed in the first chapter. There is particular emphasis on the valuation of investments and the calculation of rates of return. Mathematical developments are illustrated with practical examples. The material is presented in such a way that those without formal training in mathematics will be able to follow the text without difficulty.

Part II provides the necessary statistical background for investment specialists. Modern financial economics and developments in the actuarial field have emphasised the importance of probability distributions and have made investors more aware of the concept that expected investment returns are the expected values of random variables. The statistical topics in Part II complete the foundation which allows the reader to tackle topics in Part III. Like Part I, the approach in Part II assumes little formal mathematical training.

Part III deals with a number of specialist topics which are applications of the material covered in Parts I and II. It is expected that readers will only study those chapters of Part III which are relevant to their particular work or course. Topics covered are modern portfolio theory and asset pricing, market indices, portfolio performance measurement, bond analysis, option pricing models, and stochastic investment models.

An important feature of this international text is the way in which chapters are self-contained, and yet follow logically one from another. This will enable readers to choose a reading path appropriate to their own specific needs without a loss of continuity; it also enables the text to be used as a reference manual. *Investment Mathematics* is an accessible text which will provide readers with a sound analytical framework within which the valuation of investments and investment in a wider context may be studied.

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Part I

Security Analysis

The first part of this book develops the foundations of security analysis, drawing heavily upon the concept of compound interest. Consequently, the first chapter is devoted to explaining the basics of compound interest. Chapters 2 and 3 use these techniques in the analysis of fixed-interest bonds, equities and real estate. Chapter 4 deals with the mathematical problems of investment in inflationary conditions and Chapter 5 deals with the analysis of index-linked bonds. Chapter 6 makes use of the material presented in Chapters 4 and 5 to consider the mathematics associated with foreign currency investments. There has been dramatic growth in the market for derivative securities in recent times and the basics of these instruments are outlined in Chapter 7.

1.1 INTRODUCTION

The purpose of this chapter is to provide an introduction to *compound interest* which is the foundation of investment mathematics. In particular, methods are developed for calculating the accumulated value and present value of an investment. Although we concentrate on financial investment in securities in this book, for completeness, the second half of this chapter is devoted to discussion of real investment in projects.

Readers unfamiliar with the concepts of *exponents* and *geometric series*, which occur frequently in compound interest calculations, should first read the Annex to this chapter.

1.2 ACCUMULATED VALUES

Suppose that a woman deposits £100 with a bank which pays a rate of interest of 6% per annum to its depositors and that she leaves any interest earned to accumulate within the account. At the start of the second year she would have $£100 \times 1.06 = £106$ in the account, which would earn 6% interest during the second year. So in the second year the £106 grows at 6% to $£106 \times 1.06$, or alternatively $£100(1.06)^2$, which equals £112.36. The important point to grasp here is that interest is itself earning interest. This is the essence of what is known as *compound interest*. If interest were spent rather than left in the account, total interest received at the end of the second year would amount to only £12 rather than £12.36.

By the end of the third year, the £112.36 will grow to $£112.36 \times 1.06$, or alternatively $£100(1.06)^3$, which equals £119.10. Continuing the process further, the £100 grows to $£100(1.06)^4$ after four years, $£100(1.06)^5$ after 5 years and so on. In the general case, £100 grows to $£100(1.06)^n$ after n years.

We may generalise the above *compounding* process further. Suppose that an amount $A(0)$ is invested at time 0 at a compound interest rate of i per interval, where i is written as a decimal rather than a percentage. Then the accumulated amount $A(n)$ after n intervals of time is given by the formula

$$A(n) = A(0)(1 + i)^n \quad (1.1)$$

This may be represented diagrammatically as shown in Figure 1.1. Note that the rate at which interest accrues has not been expressed as a rate *per annum* but as a rate *per interval*. A unit interval of time is not restricted to one year. It could be a half-year, a quarter of a year, an hour or any other interval of time.

Equation (1.1) is valid not only for an integral number of time intervals but also for fractions of time intervals, provided the rate of growth is constant. Consider the amount by which the investment has grown after $1/m$ of a time interval (where m is an integer). If the sum has grown by a factor $1 + y$, we can say that

$$(1 + y)^m = 1 + i$$

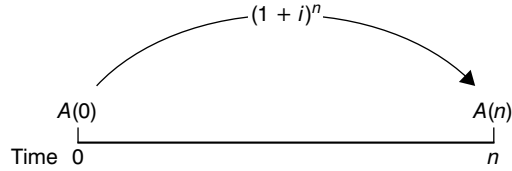


Figure 1.1

or

$$(1 + y) = (1 + i)^{1/m}$$

Thus, after n time intervals and r periods of $1/m$ time intervals (r and m integers), the sum $A(0)$ would grow to

$$A(0) \cdot (1 + i)^{n + (r/m)}$$

In other words, after any time period t , $A(0)$ would grow to $A(0) \cdot (1 + i)^t$.

Example 1.1

What is the accumulated value of £300 invested at 5% per annum for (a) 6 years?
(b) $12\frac{1}{2}$ years?

Answer

- (a) $A(6) = £300(1.05)^6 = \underline{£402.03}$.
(b) $A(12.5) = £300(1.05)^{12.5} = \underline{£552.06}$.

Example 1.2

What is the accumulated value of \$200 invested for $8\frac{1}{4}$ years where the rate of interest is (a) 3% per half-year? (b) 2% per quarter?

Answer

- (a) The unit interval of time is a half-year so that n is $8.25 \times 2 = 16.5$

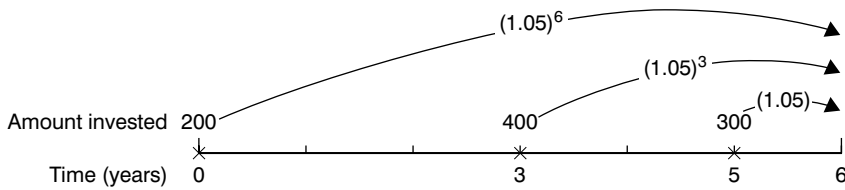
$$A(16.5) = \$200(1.03)^{16.5} = \underline{\$325.72}$$

- (b) The unit interval of time is a quarter so that n is $8.25 \times 4 = 33$

$$A(33) = \$200(1.02)^{33} = \underline{\$384.45}$$

Example 1.3

What is the accumulated value in 6 years' time of £200 invested now, £400 invested at the end of year 3 and £300 invested at the end of year 5, if the rate of interest is 5% per annum?

Answer**Figure 1.2**

In this case there are a number of payments at different points in time. The solution is straightforward when it is realised that the accumulated values of the payments can be considered separately and then simply added together (see Figure 1.2).

£200 is invested for 6 years, £400 for 3 years and £300 for 1 year, so

$$\begin{aligned}\text{Accumulated value} &= £200(1.05)^6 + £400(1.05)^3 + £300(1.05) \\ &= \underline{\underline{£1046.07}}\end{aligned}$$

1.3 EFFECTIVE AND NOMINAL RATES OF INTEREST

Rates of interest only have meaning when they are related to a time interval. Thus, in Examples 1.1 and 1.3 above, the rate of interest was 5% *per annum* whereas in Example 1.2(a) the rate of interest was 3% *per half-year* and in Example 1.2(b) the rate of interest was 2% *per quarter*. Rates of interest expressed in this way give the actual rate of increase over the stated interval of time. They are known as *effective* rates of interest.

Example 1.4

If the effective annual rate of interest is 7%, what is the effective monthly rate of interest?

Answer

Let the effective monthly rate of interest be j . Then

$$\begin{aligned}(1 + j)^{12} &= 1.07 \\ 1 + j &= 1.00565 \\ j &= 0.565\%\end{aligned}$$

Where the effective rate of interest is expressed in terms of $1/p$ of a year, it is often converted to an annual rate by simply multiplying by p . Thus, 3% per half-year would be quoted as being “6% per annum, convertible half-yearly”. Similarly, 2% per

quarter would be quoted as being “8% per annum, convertible quarterly”. Interest rates quoted in this way are known as *nominal* rates of interest. Quoted interest rates on savings products offered by banks are often nominal rates, e.g. convertible half-yearly.

Corresponding to a nominal rate of interest there always exists an effective annual rate of interest. For example, suppose £100 is invested for a year at a rate of interest of 8% per annum, convertible half-yearly (i.e. an effective rate of interest of 4% per half-year). This is not the same as an effective rate of interest of 8% per annum. The amount at the end of the year would be $£100(1 + 0.08/2)^2$. If the corresponding annual rate of interest is i per annum, the amount at the end of the year may also be written as $£100(1 + i)$. We therefore have

$$£100(1 + 0.08/2)^2 = £100(1 + i)$$

or

$$\begin{aligned} (1 + 0.08/2)^2 &= 1 + i \\ 1.0816 &= 1 + i \\ i &= 0.0816 \quad \text{or} \quad 8.16\% \end{aligned} \tag{1.2}$$

That is, a rate of interest expressed as 8% per annum convertible half-yearly is the same as an effective rate of interest of 8.16% per annum.

Equation (1.2) above may be generalised. Suppose we are given a rate of interest per annum, convertible p times a year (denoted by the symbol $i^{(p)}$). Then the equivalent rate of interest per annum (denoted by the symbol i) is given by

$$\left(1 + \frac{i^{(p)}}{p}\right)^p = 1 + i \tag{1.3}$$

If p is greater than 1 so that interest is convertible more frequently than once a year, interest itself earns interest within the year, and the effective rate of interest i exceeds the nominal rate of interest, $i^{(p)}$. As p increases so that intervals between additional interest become smaller and smaller, the margin between i and $i^{(p)}$ widens at a decreasing rate, tending to a limit which corresponds to interest being *continuously compounded*. In this case (for which p is infinite), the resultant annual convertible rate of interest is referred to as the *force of interest* and is given the symbol δ . The equivalent rate of interest per annum is given by

$$e^\delta = 1 + i$$

where

$$e = \lim_{p \rightarrow \infty} \left(1 + \frac{1}{p}\right)^p = 2.718$$

If the force of interest is quoted, then we have:

$$A(t) = A(0)e^{\delta t} \tag{1.4}$$

Example 1.5

If the effective annual rate of interest is 6%, what is (a) the annual rate of interest convertible half-yearly? (b) the force of interest?

Answer

(a) Let the annual rate of interest convertible half-yearly be $i^{(2)}$. Then

$$\begin{aligned} \left(1 + \frac{i^{(2)}}{2}\right)^2 &= 1.06 \\ 1 + \frac{i^{(2)}}{2} &= 1.029\ 56 \\ \frac{i^{(2)}}{2} &= 0.029\ 56 \\ i^{(2)} &= \underline{5.912\%} \end{aligned}$$

(b) Let the force of interest be δ . Then

$$\begin{aligned} e^\delta &= 1.06 \\ \delta &= \log_e 1.06 \\ \delta &= 0.0583 \\ \delta &= \underline{5.83\%} \end{aligned}$$

1.4 THE ACCUMULATED VALUE OF AN ANNUITY-CERTAIN

An annuity-certain is a series of payments at fixed intervals of time for a fixed period of time. The payments may be of a constant amount or they may vary.

(1) *A series of payments of one per interval payable in arrears for n intervals*

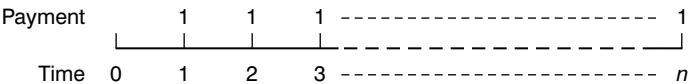


Figure 1.3

The accumulated value of this series of payments is often denoted by the symbol $s_{\overline{n}|}$. Occasionally it is necessary to make clear the ruling rate of interest by placing it to the right, e.g. $s_{\overline{n}|10\%}$.

Suppose that the rate of interest is i per interval, Then

$$s_{\overline{n}|} = (1 + i)^{n-1} + (1 + i)^{n-2} + \cdots + (1 + i) + 1$$

or, reversing the order of the terms on the right-hand side,

$$s_{\overline{n}|} = 1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1}$$

The right-hand side of this equation is the sum of a geometric series with first term equal to 1 and common ratio equal to $1+i$. There are n terms in the series. The sum of this geometric series is

$$\frac{1 \cdot (1 - (1+i)^n)}{1 - (1+i)} \quad \text{using equation (1.22)}$$

We therefore have the result:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \quad (1.5)$$

It is not usually necessary to calculate $s_{\overline{n}|}$ as values can be obtained from compound interest tables (see back of this book).

(2) *A series of payments of one per interval payable in advance for n intervals*

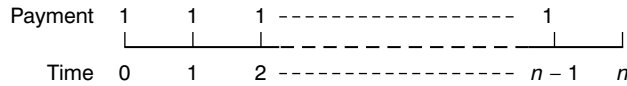


Figure 1.4

The accumulated value of this series of payments is often denoted by the symbol $\ddot{s}_{\overline{n}|}$. Note that this is the same as the series of payments in (1) above except that each payment accumulates for one additional interval. It therefore follows that

$$\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|} \quad (1.6)$$

1.5 PRESENT VALUES

We have considered how to determine the accumulated value of a payment (or a number of payments) at some point in the future. We now consider the reverse problem of determining the amount which must be invested now to provide for a payment (or a number of payments) at some point (or points) in the future. In other words, we wish to determine the present value of amounts received at specified future points in time. As with accumulated values, present values depend on the ruling rate of interest but instead of *accumulating* we are *discounting*.

Suppose that we wish to make a payment of £600 in exactly 7 years from now and that the rate of interest is 9% per annum. How much must be invested at the present time to provide this amount? If the initial amount invested is $A(0)$, then using equation (1.1) derived earlier, we obtain

$$£600 = A(0)(1.09)^7$$

so that

$$\begin{aligned} A(0) &= \frac{£600}{(1.09)^7} \\ &= £328.22 \end{aligned}$$

In general, the amount $A(0)$ which must be invested now to provide $A(t)$ after t intervals of time is given by

$$A(0) = \frac{A(t)}{(1+i)^t} \quad (1.7)$$

where i is the rate of interest per interval, and t need not be an integer. Alternatively, if δ is the force of interest,

$$A(0) = A(t)e^{-\delta t}$$

$A(0)$ is known as the *present value* of $A(t)$.

Equation (1.7) may be represented diagrammatically as in Figure 1.5.

$1/(1+i)$ is often given the symbol v and so equation (1.7) would then become

$$A(0) = A(t) \cdot v^t \quad (1.8)$$

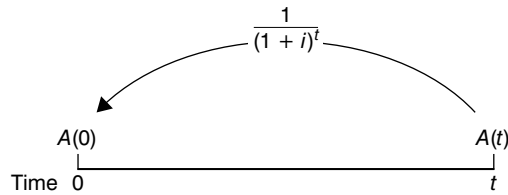


Figure 1.5

Example 1.6

What is the present value of £100 due in exactly 8 years from now if the rate of interest is 7% per annum?

Answer

Here we have

$$t = 8 \quad A(t) = £100 \quad i = 0.07$$

So

$$\begin{aligned} \text{Present value} &= \frac{£100}{(1.07)^8} \\ &= \underline{£58.20} \end{aligned}$$

Note that the present value is less than the amount due in the future.

Example 1.7

Assuming a rate of interest of 4% per half-year, how much must be invested now to provide the following payments?

£200 after 2 years
plus £600 after $3\frac{1}{2}$ years
plus £500 after 5 years

Answer

The unit interval of time is a half-year in this case.

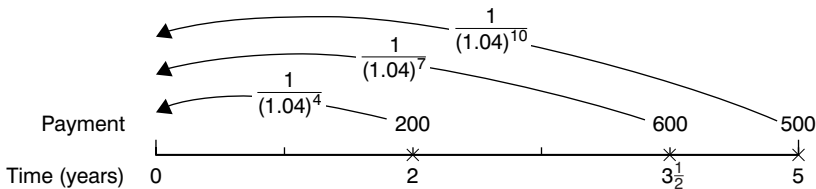


Figure 1.6

The amount which must be invested now is

$$\frac{£200}{(1.04)^4} + \frac{£600}{(1.04)^7} + \frac{£500}{(1.04)^{10}} = \underline{£964.69}$$

1.6 THE PRESENT VALUE OF AN ANNUITY-CERTAIN

The following types of annuity-certain are particularly important in investment and it is therefore useful to know the general formulae for their present values.

(1) *A series of payments of one per interval payable in arrears for n intervals*

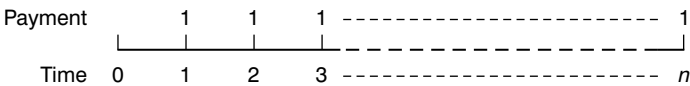


Figure 1.7

The present value of this series of payments is often denoted by the symbol $a_{\overline{n}|}$. Suppose that the rate of interest is i per interval. Then

$$a_{\overline{n}|} = \frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n}$$

The right-hand side of this equation is the sum of a geometric series with first term equal to $1/(1+i)$ and common ratio equal to $1/(1+i)$.

The sum of this geometric series is

$$\frac{\frac{1}{(1+i)} \left(1 - \frac{1}{(1+i)^n} \right)}{1 - \frac{1}{(1+i)}}$$

Multiplying the numerator and denominator by $1+i$, we obtain

$$\frac{1 - \frac{1}{(1+i)^n}}{1+i-1} = \frac{1 - \frac{1}{(1+i)^n}}{i}$$

We therefore have the result:

$$a_{\overline{n}|} = \frac{1 - \frac{1}{(1+i)^n}}{i} \tag{1.9}$$

Values of $a_{\overline{n}|}$ are commonly found in compound interest tables (see back of book).

(2) *A series of payments of one per interval payable in advance for n intervals*

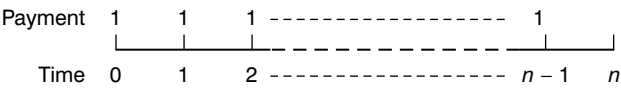


Figure 1.8

The present value of this series of payments is often denoted by the symbol $\ddot{a}_{\overline{n}|}$. This is the same as the series of payments in (1) above except that each payment is discounted for one interval fewer. It therefore follows that

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|} \tag{1.10}$$

(3) *An infinite series of payments made at the end of each interval; the first payment is d_1 and each subsequent payment is $(1+g)$ times the previous payment*

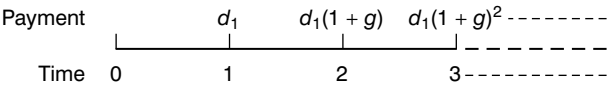


Figure 1.9

Suppose again that the rate of interest is i per interval. Then

$$\text{Present value} = \frac{d_1}{(1+i)} + \frac{d_1(1+g)}{(1+i)^2} + \frac{d_1(1+g)^2}{(1+i)^3} + \dots$$

The right-hand side of this equation is an infinite geometric series with first term equal to $d_1/(1+i)$ and common ratio equal to $(1+g)/(1+i)$. If i is greater than g , the series is convergent and the sum of the series is

$$\frac{\frac{d_1}{(1+i)}}{1 - \frac{(1+g)}{(1+i)}}$$

Multiplying the numerator and denominator by $1+i$ we obtain

$$\frac{d_1}{(1+i) - (1+g)}$$

so that the present value equals

$$\frac{d_1}{i - g} \quad (1.11)$$

Note that when there is zero growth, i.e. $g = 0$, we have an infinite series of payments (known as a *perpetuity*) of d_1 per interval, payable in arrears with present value equal to d_1/i . The same result can be obtained by multiplying equation (1.9) by d_1 and setting $n = \infty$.

(4) An increasing annuity in which the first payment is one after one interval, the second payment is two after two intervals, and so on, with a final payment of n after n intervals

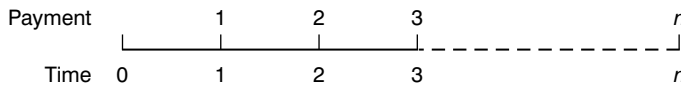


Figure 1.10

The present value of this series of payments is often denoted by the symbol $(Ia)_{\overline{n}|}$. It is convenient to use v for $1/(1+i)$. Then

$$(Ia)_{\overline{n}|} = v + 2v^2 + \dots + nv^n \quad (1.12)$$

$$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \dots + nv^{n-1} \quad (1.13)$$

Subtracting equation (1.12) from equation (1.13),

$$\begin{aligned} i(Ia)_{\overline{n}|} &= 1 + v + v^2 + \dots + v^{n-1} - nv^n \\ (Ia)_{\overline{n}|} &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \end{aligned} \quad (1.14)$$