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Stochastic simulation and applications in finance with Matlab programs

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Stochastic Simulation and Applications in Finance with MATLAB® Programs

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Huu Tue Huynh: To my late parents, my wife Carole, and all members of my family.

Van Son Lai: To my parents, my wife Quynh-Trang, my son Laurent Lam, and my brothers and sisters.

Issouf Soumaré: To my wife Fatou, my son Moussa, and my daughter Candia.

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Preface

Since the seminal works of Black-Scholes-Merton in 1973, the world of finance has been revolutionized by the emergence of a new field known as financial engineering. On the one hand, markets (foreign exchange, interest rate, commodities, etc.) have become more volatile, which creates an increase in the demand for derivatives products (options, forwards, futures, swaps, hybrids and exotics, and credit derivatives to name a few) used to measure, control, and manage risks, as well as to speculate and take advantage of arbitrage opportunities.

On the other hand, technological advances have enabled financial institutions and other markets players to create, price and launch new products and services to not only hedge against risks, but also to generate revenues from these risks. In addition to a deep grasp of advanced financial theories, the design, analysis and development of these complex products and financial services, or financial engineering, necessitate a mastering of sophisticated mathematics, statistics and numerical computations.

By way of an integrated approach, the object of this book is to teach the reader:

- to apply stochastic calculus and simulation techniques to solve financial problems;
- to develop and/or adapt the existing contingent claims models to support financial engineering platforms and applications.

There are several books in the market covering stochastic calculus and Monte Carlo simulations in finance. These books can be roughly grouped into two categories: introductory or advanced. Unfortunately, the books at the introductory level do not answer the needs of upper-level undergraduate and graduate students and finance professionals and practitioners. Advanced books, being very sophisticated and specialized, are tailored for researchers and users with solid and esoteric scientific backgrounds in mathematics and statistics. Furthermore, these books are often biased towards the research interests of the authors, hence their scope is narrowed and their applications in finance limited. By and large, the existing books are less suitable for day-to-day use which is why there is a need for a book that can be used equally by beginners and established researchers wishing to acquire an adequate knowledge of stochastic processes and simulation techniques and to learn how to formulate and solve problems in finance.

This book, which has developed from the master programme in financial engineering at Laval University in Canada first offered in 1999, aims to reinforce several aspects of simulation techniques and their applications in finance. Building on an integrated approach, the book provides a pedagogical treatment of the material for senior undergrad and graduate students as well as professionals working in risk management and financial engineering. While initiating students into basic concepts, it covers current up-to-date problems. It is written in a clear, concise and rigorous pedagogical language, which widens accessibility to a larger audience without sacrificing mathematical rigor. By way of a gradual learning of existing theories and new developments, our goal is also to provide an approach to help the reader follow the relevant literature which continually expands at a rapid pace.

This book is intended for students in business, economics, actuarial sciences, computer sciences, general sciences, and engineering, programmers and practitioners in financial, investment/asset and risk management industries. The prerequisites for the book are some familiarity in linear algebra, differential calculus and programming.

The book introduces and trains users in the formulation and resolution of financial problems. As exercises, it provides computer programs for use with the practical examples, exercises and case studies, which give the reader specific recipes for solving problems involving stochastic processes in finance. The programming language is the MATLAB®[1] software which is easy to learn and popular among professionals and practitioners. Moreover, the programs could be readily converted for use with the platform C++. Note that, unlike the MATLAB financial toolboxes which are still limited in scope, our proposed exercises and case studies tackle the complex problems encountered routinely in finance.

Overall, the general philosophy of the book can be summarized as follows:

- keep mathematical rigor by minimizing abstracts and unnecessary jargon;
- each concept, either in finance or in computation, leads to algorithms and is illustrated by concrete examples in finance.

Therefore, after they are discussed, the topics are presented in algorithmic forms. Furthermore, some of the examples which treat current financial problems are expounded in case studies, enabling students to better comprehend the underlying financial theory and related quantitative methods.

Every effort has been made to structure the chapters in a logical and coherent manner, with a clear thread and linkage between the chapters which is not apparent in most existing books. Each chapter has been written with regard to the following four principles: pedagogy, rigor, relevance and application. Advanced readers can skip the chapters they are familiar with and go straight to those of interest.

The book starts with a refresher of basic probability and statistics which underpin random processes and computer simulation techniques introduced later. Most of the developed tools are used later to study computational problems of derivative products and risk management. The text is divided into the following four major parts. The first part (Chapters 1 to 3) reviews basic probability and statistics principles. The second part (Chapters 4 to 6) introduces the Monte Carlo and Quasi Monte Carlo simulations topics and techniques. In addition to the other commonly used variance reduction techniques, we introduce the quadratic resampling technique of Barraquand (1995) to obtain the prescribed distribution characteristics of the simulated samples, which is important to improve the quality of the simulations. We also present the Markov Chain Monte Carlo (MCMC) and important sampling methods. The third part (Chapters 7 and 8) treats random processes, stochastic calculus, Brownian bridges, jump

[1] MATLAB is a registered trademark of The MathWorks, Inc. For more information, see htt://www.mathworks.com.

processes and stochastic differential equations. Finally, the fourth part (Chapters 9 to 15) develops the applications in finance.

To price contingent claims, two equivalent approaches are used in finance: the state variables approach consisting of solving partial differential equations and the probabilistic or equivalent martingale approach. The equivalence between the two approaches is established via the Feynman-Kac theorem. Our purpose is to teach how to solve numerically stochastic differential equations using Monte Carlo simulations, which essentially constitutes the pedagogical contribution of our book.

The fourth part of the book presents different applications of stochastic processes and simulation techniques to solve problems frequently encountered in finance. This part is structured as follows. Chapter 9 lays the foundation to price and replicate contingent claims. Chapter 10 prices European, American and other complex and exotic options using Monte Carlo simulations. Chapter 11 presents modern continuous-time models of the term structure of interest rates and the pricing of interest rate derivatives. Chapters 12 and 13 develop valuation models of corporate securities and credit risk. Chapters 14 and 15 overview risk management and develop estimations of Value at Risk (VaR) by combining Monte Carlo and Quasi Monte Carlo simulations with Principal Components Analysis.

Although this is an introductory and pedagogical book, nonetheless, in Chapter 10 we explain many useful and modern simulation techniques such as the Least-Squares Method (LSM) of Longstaff and Schwartz (2001) and the dynamic programming with Stratified State Aggregation of Barraquand and Martineau (1995) to price American options, the extreme value simulation technique proposed by El Babsiri and Noel (1998) to price exotic options and the Retrieval of Volatility Method proposed by Cvitanic, Goukassian and Zapatero (2002) to estimate the option sensitivity coefficients or hedge ratios (the Greeks). Note that, to our knowledge, with the exception of LSM, this is the first book to bring to the fore these important techniques. In Chapter 11 on term structure of interest rates modeling and pricing of interest rate derivatives, we present the interest rate model of Heath, Jarrow and Morton (1992) and the industry-standard Market Model of Brace, Gatarek and Musiela (2001). An extensive treatment of corporate securities valuation and credit risk based on the structural approach of Merton (1974) is presented in chapter 12. Chapter 13 gives case studies on financial guarantees to show how the simulations techniques can be implemented, and this chapter is inspired from the research publications of the authors. As such, Chapters 12 and 13 provide indispensable fundamentals for a reader to embark on the study of structured products design and credit derivatives.

To perform a sound simulation experiment, one has to undertake roughly the following three steps: (1) modeling of the problem to be studied, (2) calibration/estimation of the model parameters, and (3) backtesting using real data and recalibration. This book focuses on the use of Monte Carlo and Quasi Monte Carlo simulations in finance for the sake of pricing and risk management assuming the dynamics of the underlying variables are known.

We do not pretend that the book provides complete coverage of all topics and issues; future editions would include application examples of the Markov Chain Monte Carlo (MCMC) simulation technique, estimation techniques of the parameters of the diffusion processes and the determination of the assets variance-covariance matrix, the spectral analysis, real options, volatility derivatives, etc.

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1
Introduction to Probability

Since financial markets are very volatile, in order to model financial variables we need to characterize randomness. Therefore, to study financial phenomena, we have to use probabilities.

Once defined, we will see how to use probabilities to describe the evolution of random parameters that we later call random processes. The key step here is the quantitative construction of the events' probabilities. First, one must define the events and then the probabilities associated to these events. This is the objective of this first chapter.

1.1 INTUITIVE EXPLANATION

1.1.1 Frequencies

Here is an example to illustrate the notion of relative frequency. We toss a dice N times and observe the outcomes. We suppose that the 6 faces are identified by letters A, B, C, D, E and F. We are interested in the probability of obtaining face A. For that purpose, we count the number of times that face A appears and denote it by $n(A)$. This number represents the frequency of appearance of face A.

Intuitively, we see that the division of the number of times that face A appears, $n(A)$, by the total number N of throws, $\frac{n(A)}{N}$, is a fraction that represents the probability of obtaining face A each time that we toss the dice. In the first series of experiments when we toss the dice N times we get $n_1(A)$ and if we repeat this series of experiments another time by tossing it again N times, we obtain $n_2(A)$ of outcomes A.

It is likely that $n_1(A)$ and $n_2(A)$ are different. The fractions $\frac{n_1(A)}{N}$ and $\frac{n_2(A)}{N}$ are then different. Therefore, how can we say that this fraction quantifies the probability of obtaining face A? To find an answer, we need to continue the experiment. Even if the fractions are different, when the number N of throws becomes very large, we observe that these two fractions converge to the same value of $\frac{1}{6}$.

Intuitively, this fraction measures the probability of obtaining face A, and when N is large, this fraction goes to $\frac{1}{6}$. Thus, each time we toss the dice, it is natural to take $\frac{1}{6}$ as the probability of obtaining face A.

Later, we will see that from the law of large numbers these fractions converge to this limit. This limit, $\frac{1}{6}$, corresponds to the concept of the ratio of the number of favorable cases over the total number of cases.

1.1.2 Number of Favorable Cases Over The Total Number of Cases

When we toss a dice, there is a total of 6 possible outcomes, $\{1, 2, 3, 4, 5, 6\}$, corresponding to the letters on faces $\{A, B, C, D, E, F\}$. If we wish to obtain face A and we have only one such case, then the probability of getting face A is quantified by the fraction $\frac{1}{6}$. However, we may be interested in the event {"the observed face is even"}. What does this mean? The even face can be 2, 4 or 6. Each time that one of these three faces appears, we have a realization of the event {"the observed face is even"}. This means that when we toss a dice, the total number of possible cases is always 6 and the number of favorable cases associated to even events is 3. Therefore, the probability of obtaining an even face is simply $\frac{3}{6}$ and intuitively this appears to be correct.

From this consideration, in the following section we construct in an axiomatic way the mechanics of what is happening. However, we must first establish what is an event, and then we must define the probabilities associated with an event.

1.2 AXIOMATIC DEFINITION

Let's define an universe in which we can embed all these intuitive considerations in an axiomatic way.

1.2.1 Random Experiment

A random experiment is an experiment in which we cannot precisely predict the outcome. Each result obtained from this experiment is random *a priori* (before the realization of the experiment). Each of these results is called a simple event. This means that each time that we realize this experiment we can obtain only one simple event. Further we say that all simple events are exclusive.

Example 2.1 Tossing a dice is a random experiment because before the toss, we cannot exactly predict the future result. The face that is shown can be 1, 2, 3, 4, 5 or 6. Each of these results is thus a simple event. All these 6 simple events are mutually exclusive.

We denote by Ω the set of all simple events. The number of elements in Ω can be finite, countably infinite, uncountably infinite, etc. The example with the dice corresponds to the first case (the case of a "finite number of results", $\Omega = \{1, 2, 3, 4, 5, 6\}$).

Example 2.2 We count the number of phone calls to one center during one hour. The number of calls can be 0, 1, 2, 3, etc. up to infinity. An infinite number of calls is evidently an event that will never occur. However, to consider it in the theoretical development allows us to build useful models in a relatively simple fashion. This phone calls example corresponds to the countably infinite case ($\Omega = \{ 0, 1, 2, 3, \ldots, \infty\}$).

Example 2.3 When we throw a marble on the floor of a room, the position on which the marble will stop is a simple event of the experiment. However, the number of simple events is infinite and uncountable. It corresponds to the set of all points on the floor.

Building a probability theory for the case of finite experiments is relatively easy, the generalization to the countably infinite case is straightforward. However, the uncountably infinite case is different. We will point out these differences and technicalities but we will not dwell on the complex mathematical aspects.

1.2.2 Event

We consider the experiment of a dice toss. We want to study the "even face" event. This event happens when the face shown is even, that is, one of 2, 4, or 6.

Thus, we can say that this event "even face" contains three simple events $\{2, 4, 6\}$. This brings us to the definition:

Definition 2.4 *Let Ω be the set of simple events of a given random experiment. Ω is called the sample space or the universe. An event is simply a sub-set of Ω.*

Is any subset of Ω an event? This question will be answered below. We must not forget that an event occurs if the realized simple event belongs to this event.

1.2.3 Algebra of Events

We saw that an event is a subset of Ω. We would like to construct events from Ω. Let Ω be the universe and let ξ be the set of events we are interested in. We consider the set of all events. ξ is an algebra of events if the following axioms are satisfied:

A1: $\Omega \in \xi$,
A2: $\forall A \in \xi$, $A^c = \Omega \backslash A \in \xi$ (where $\Omega \backslash A$, called the complementary of A, is the set of all elements of Ω which do not belong to A),
A3: $\forall A_1, A_2, \ldots, A_n \in \xi, A_1 \cup A_2 \cup \ldots \cup A_n \in \xi$.

Axiom A1 says that the universe is an event. This event is certain since it happens each time that we undertake the experiment. Axiom A1 and axiom A2 imply that the empty set, denoted by $\emptyset$, is also an event but it is impossible since it never happens. Axiom A3 says that the union of a finite number of events is also an event. To be able to build an algebra of events associated with a random experiment encompassing a countable infinity of simple events, axiom A3 will be replaced by:

A3': $\cup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots \in \xi$.

This algebra of events plays a very important role in the construction of the probability of events. The probabilities that we derive should follow the intuition developed previously.

1.2.4 Probability Axioms

Let Ω be the universe associated with a given random experiment on which we build the algebra of events ξ. We associate to each event $A \in \xi$ a probability noted Prob(A), representing the probability of event A occurring when we realize the experiment. From our intuitive setup, this probability must satisfy the following axioms:

P1: $\text{Prob}(\Omega) = 1$,
P2: $\forall A \in \xi, 0 \leq \text{Prob}(A) \leq 1$,
P3: if $A_1, A_2, \ldots, A_n, \ldots$ is a series of mutually exclusive events, that is: $\forall i \neq j, A_i \cap A_j = \emptyset$, then

$$\text{Prob}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \text{Prob}(A_n). \tag{1.1}$$

Axiom P3 is called $\sigma-$additivity of probabilities. This axiom allows us to consider random experiments with an infinity of possible outcomes. From these axioms, we can see that

$$\text{Prob}(\emptyset) = 0 \quad \text{and} \quad \text{Prob}(A^c) = 1 - \text{Prob}(A) \tag{1.2}$$

which are intuitively true.

A very important property easy to derive is presented below.

Property 2.5 *Consider two events A and B, then*

$$\text{Prob}(A \cup B) = \text{Prob}(A) + \text{Prob}(B) - \text{Prob}(A \cap B). \tag{1.3}$$

The mathematical proof is immediate.

Proof: Let $A \setminus C$ be the event built from elements of A that do not belong to C.

$$A = (A \setminus C) \cup C \quad \text{where} \quad C = A \cap B. \tag{1.4}$$

Since $A \setminus C$ and C are disjoint, from axiom P3,

$$\text{Prob}(A) = \text{Prob}(A \setminus C) + \text{Prob}(C). \tag{1.5}$$

Similarly

$$\text{Prob}(B) = \text{Prob}(B \setminus C) + \text{Prob}(C). \tag{1.6}$$

Adding these two equations yields:

$$\text{Prob}(A \setminus C) + \text{Prob}(B \setminus C) + \text{Prob}(C) = \text{Prob}(A) + \text{Prob}(B) - \text{Prob}(C). \tag{1.7}$$

Moreover,

$$A \cup B = (A \setminus C) \cup (B \setminus C) \cup C, \tag{1.8}$$

and since $A \setminus C$, $B \setminus C$ and C are disjoint, we have

$$\text{Prob}(A \cup B) = \text{Prob}(A \setminus C) + \text{Prob}(B \setminus C) + \text{Prob}(C), \tag{1.9}$$

thus,

$$\text{Prob}(A \cup B) = \text{Prob}(A) + \text{Prob}(B) - \text{Prob}(C). \tag{1.10}$$

Example 2.6 Let's go back to the dice toss experiment with

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

and consider the events:

(a) $A = \{\text{“face smaller than 5”}\} = \{1, 2, 3, 4\}$.

Since events $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$ are mutually exclusive, we know from axiom P3 that:

$$\text{Prob}(A) = \text{Prob}(\{1\}) + \text{Prob}(\{2\}) + \text{Prob}(\{3\}) + \text{Prob}(\{4\}) = \frac{4}{6}.$$

(b) $B = \{\text{“even faces”}\} = \{2, 4, 6\}$.

Thus, $A \cup B = \{1, 2, 3, 4, 6\}$ and $A \cap B = \{2, 4\}$. We also have, $\text{Prob}(A) = \frac{4}{6}$, $\text{Prob}(B) = \frac{3}{6}$, $\text{Prob}(A \cap B) = \text{Prob}(\{2, 4\}) = \frac{2}{6}$, which implies

$$\text{Prob}(A \cup B) = \text{Prob}(A) + \text{Prob}(B) - \text{Prob}(A \cap B) = \frac{4}{6} + \frac{3}{6} - \frac{2}{6} = \frac{5}{6}.$$

Next, we discuss events that may be considered as independent. To present this, we must first discuss the concept of conditional probability, i.e., the probability of an event occurring given that another event already happened.

1.2.5 Conditional Probabilities

Let A and B be any two events belonging to the same algebra of events. We suppose that B has occurred. We are interested in the probability of getting event A. To define it, we must look back to the construction of the algebras of events.

Within the universe Ω in which A and B are two well-defined events, if B has already happened, the elementary event associated with the result of this random experiment must be an element belonging to event B. This means that given B has already happened, the result of the experiment is an element of event B.

Intuitively, the probability of A occurring is simply the probability that this result is also an event of B. If B has already happened, the probability of getting A knowing B is the probability of $A \cap B$ divided by the probability of B. Therefore, we obtain

$$\text{Prob}(A|B) = \frac{\text{Prob}(A \cap B)}{\text{Prob}(B)}. \tag{1.11}$$

This definition of the conditional probability is called Bayes' rule.

This probability satisfies the set of axioms for probabilities introduced at the beginning of the section:

$$\text{Prob}(\Omega|B) = 1, \tag{1.12}$$

$$0 \leq \text{Prob}(A|B) \leq 1, \tag{1.13}$$

$$\text{Prob}(A^c|B) = 1 - \text{Prob}(A|B), \tag{1.14}$$

and

$$\text{Prob}(\cup_{n=1}^{\infty} A_n|B) = \sum_{n=1}^{\infty} \text{Prob}(A_n|B), \quad \forall i \neq j, \quad A_i \cap A_j = \emptyset. \tag{1.15}$$

This definition is illustrated next by way of examples.

Example 2.7 Consider the dice toss experiment with event

$$A = \{\text{“face smaller than 5"}\} = \{1, 2, 3, 4\}$$

and event

$$B = \{\text{“even face"}\} = \{2, 4, 6\}.$$

We know that

$$\text{Prob}(B) = \text{Prob}(\{2, 4, 6\}) = \frac{3}{6}$$

and

$$\text{Prob}(A) = \text{Prob}(\{1, 2, 3, 4\}) = \frac{4}{6}.$$

However, we want to know what is the probability of obtaining an even face knowing that the face is smaller than 5 (in other words, A has already happened). From Bayes' rule:

$$\begin{aligned}\text{Prob}(B|A) &= \frac{\text{Prob}(A \cap B)}{\text{Prob}(A)} \\ &= \frac{\text{Prob}(\{2, 4\})}{\text{Prob}(\{1, 2, 3, 4\})} \\ &= \frac{2/6}{4/6} \\ &= \frac{1}{2}.\end{aligned}$$

Example 2.8 From a population of N persons, we observe n_s smokers and n_c people with cancer. From these n_s smokers we observe $n_{s,c}$ individuals suffering from cancer. For this population, we can say that the probability that a person is a smoker is $\frac{n_s}{N}$ and the probability that a person has cancer is $\frac{n_c}{N}$. The probability that a person has cancer given that he is already a smoker is:

$$\text{Prob}(\text{cancer}|\text{smoker}) = \frac{\text{Prob(smoker and cancer)}}{\text{Prob(smoker)}} = \frac{n_{s,c}}{n_s}.$$

From this experiment, we note that the conditional probability can be smaller or greater than the probability considered *a priori*. Following this definition of the conditional probability, we examine next the independence of two events.

1.2.6 Independent Events

Two events are said to be statistically independent when the occurrence of one of them doesn't affect the probability of getting the other. A and B are said to be statistically independent if

$$\text{Prob}(A|B) = \text{Prob}(A). \tag{1.16}$$

From Bayes' rule, if A and B are two independent events then

$$\text{Prob}(A \cap B) = \text{Prob}(A)\text{Prob}(B). \tag{1.17}$$

Example 2.9 Consider the experiment of tossing a dice twice. Intuitively, we hope that the result of the first toss would be independent of the second one. From our preceding exposition, we can establish this independence as follows. Indeed, the universe of this experiment contains 36 simple events denoted by $(R1, R2)$ where $R1$ and $R2$ are respectively the results of the first and second tosses, with $(R1, R2)$ taking values (n, m) in

$$\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}.$$

The probability the first element $R1$ equals n is

$$\text{Prob}(R1 = n) = \frac{1}{6}, \quad \forall n \in \{1, 2, 3, 4, 5, 6\}$$

and the probability the second element $R2$ equals m is

$$\text{Prob}(R2 = m) = \frac{1}{6}, \quad \forall m \in \{1, 2, 3, 4, 5, 6\}.$$

Since $\text{Prob}(R1 = n, R2 = m) = \frac{1}{36}$, then the conditional probability

$$\begin{aligned} \text{Prob}(R2 = m|R1 = n) &= \frac{\text{Prob}(R1 = n, R2 = m)}{\text{Prob}(R1 = n)} \\ &= \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}, \end{aligned}$$

which gives us $\text{Prob}(R2 = m|R1 = n) = \text{Prob}(R2 = m) = \frac{1}{6}$. Hence, we conclude that $R2$ and $R1$ are independent.

Notes and Complementary Readings

The concepts presented in this chapter are fundamentals of the theory of probabilities. The reader could refer to the books written by Ross (2002 a and b) for example.

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2
Introduction to Random Variables

In the previous chapter, we introduced some concepts of events and defined probabilities on sets of events. In this chapter, we will focus on the representation of realized events on the real axis and probability space in order to provide a quantification to be used in financial problems.

A random variable is a function mapping the sample space Ω to the real axis. Afterwards, a complete characterization of such random variables will be given by introducing the probability density function, the cumulative distribution function and the characteristic function. We will show examples of the most frequently-encountered random variables in finance. The characteristic function will be presented in order to give the reader a better understanding of random variables. We will not use it extensively later in the book, but it is useful to be familiar with it to enable us to follow some proofs.

We will also introduce the concept of transformation of random variables. This concept is the basis of random variables simulation under known distributions and will be used in subsequent chapters.

2.1 RANDOM VARIABLES

We have defined random events and the probabilities associated with these events. In finance, as in the sciences, random events are always associated with quantities such as indices, costs and interest rates which vary in a random way. This means that we could link these experiments' random effects to the real axis. In other words, we associate a real number with the result given by the experiment.

Before realizing the experiment, this number is not known – it behaves as a random result from a random experiment. This approach means that we are looking to create a random experiment on the real axis the results of which are what we will call a random variable.

Mathematically, the random experiment on the real axis is created by using a function (denoted by X) from the universe of events Ω on the real axis. The random results observed on the axis under this function are used as the basis to define the random events on the real axis. This representation on the real axis obeys the same rules or is subject to the same constraints as the original events.

This function, or transformation X, must satisfy the following condition: let x be any real value, the set of all elementary events ω such that $\{X(\omega) \leq x\}$ is an event associated with the original random experiment

$$A = \{\omega \text{ such that } X(\omega) \leq x\}. \tag{2.1}$$

In mathematical terms, this function is said to be measurable.

Now, using this random variable, we only need to look at the universe of events as the real axis and the events as a subset of the real line. The most simple events are open or closed intervals and open or closed half axes. The constructed algebra of events based on these natural events is known as the Borel Algebra of the real axis. One simple way to describe the Borel Algebra of the real axis is to construct events by combining the simple open and closed intervals and open and closed half axis of the real axis. Since Borel Algebra is not really necessary to follow the text, we will not dwell on it further.

2.1.1 Cumulative Distribution Function

Let X be a real-valued random variable, by definition its cumulative distribution function, noted $F_X(x)$, is:

$$F_X(x) = \text{Prob}(A) \quad \text{where} \quad A = \{\omega \text{ such that } X(\omega) \leq x\}. \tag{2.2}$$

Based on the previous definition of probability, we deduce the following properties:

(a) $0 \leq F_X(x) \leq 1$,
(b) $F_X(x)$ is monotone, non decreasing, i.e., if $x_1 < x_2$ then we have $F_X(x_1) \leq F_X(x_2)$,
(c) $F_X(-\infty) = 0$ and $F_X(+\infty) = 1$.

Properties (a), (b) and (c) follow from the probability axioms. When $F_X(x)$ is continuous, X is said to be a continuous random variable. When it is the case, it can take any value on the real axis as the result of the experiment. However, when $F_X(x)$ is a step function, X is said to be a discrete random variable. X may be a combination of continuous and discrete segments.

2.1.2 Probability Density Function

To keep it simple, consider X to be a continuous random variable. By analogy with the physical world, we can define its probability density function such that the integral of such a function on the event defined on the real axis gives the probability of this event.

This density function can be obtained from the cumulative distribution function when looking at an infinitely small event. To see that, consider the event

$$A = \{x < X \leq x + \Delta x\}. \tag{2.3}$$

On the one hand, the cumulative function gives

$$\text{Prob}(A) = F_X(x + \Delta x) - F_X(x), \tag{2.4}$$

and on the other hand we have

$$\text{Prob}(A) = \int_x^{x+\Delta x} f_X(\alpha) d\alpha, \tag{2.5}$$

where f_X is the probability density function of the random variable X.

On the infinitesimal interval $[x, x + \Delta x]$, since $f_X(.)$ is continuous, it remains constant so that we have:

$$f_X(x)\Delta x \approx \text{Prob}(A) = F_X(x + \Delta x) - F_X(x), \tag{2.6}$$

where

$$f_X(x) \approx \frac{F_X(x + \Delta x) - F_X(x)}{\Delta x}. \tag{2.7}$$

When Δx becomes infinitesimal, we see that the probability density function is the derivative of the cumulative distribution function:

$$f_X(x) = \frac{dF_X(x)}{dx}. \tag{2.8}$$

Property 1.1 *Function $f_X(x)$ is non negative. Since integrating the density function on an event gives us the probability of the event, if it were negative on a particular interval, integrating on this interval would give us a negative probability. This would violate our probability axioms. This property can be proved easily since the probability density function is the derivative of the cumulative distribution function. This cumulative function being a non decreasing function, its derivative can never be negative.*

Property 1.2

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha) d\alpha \leq 1, \tag{2.9}$$

$$F_X(-\infty) = 0 \quad \textit{and} \quad F_X(+\infty) = 1, \tag{2.10}$$

which leads to

$$f_X(x) \underset{|x| \to \infty}{\longrightarrow} 0 \tag{2.11}$$

and

$$\int_{-\infty}^{+\infty} f_X(\alpha) d\alpha = 1. \tag{2.12}$$

For a discrete random variable X, since X takes values in a finite (or countably-infinite) set, we prefer to use the term probability mass function. The probability mass is the probability that X takes a precise value in this finite or countably-infinite set:

$$\text{Prob}(X = k). \tag{2.13}$$

We present below examples of widely used random variables.

Example 1.3 We toss a dice and the random variable is defined by the face which is shown on the dice. We define the random variable X taking the values 1, 2, 3, 4, 5, and 6. We have

$$\text{Prob}(X = k) = \frac{1}{6}, \quad \forall k \in \{1, 2, 3, 4, 5, 6\}. \tag{2.14}$$

Example 1.4 We discussed above the number of calls received at a telephone exchange. Let X be this random variable. Then X can be 0, 1, 2, ..., ∞. This phenomenon follows a distribution known as the Poisson distribution and its probability density function is defined by

$$\text{Prob}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \tag{2.15}$$

with $k = 0, 1, 2, \ldots, \infty$, where λ is a positive constant depicting the average number of calls observed.

This distribution is often used in finance in credit risk modeling, especially to describe credit default. In that case X can be the number of defaults in a given period and λ is the average number of defaults.

Example 1.5 The most common probability density functions are

(i) The Gaussian normal distribution having the probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \tag{2.16}$$

where μ and σ are constants, σ being positive. We show in the next section that μ and σ are respectively the mean and standard deviation of the random variable X. This distribution is often used in finance to represent asset returns. It is also a key distribution in statistical inference. Figure 2.1 plots the probability density function for a variable following a normal distribution.

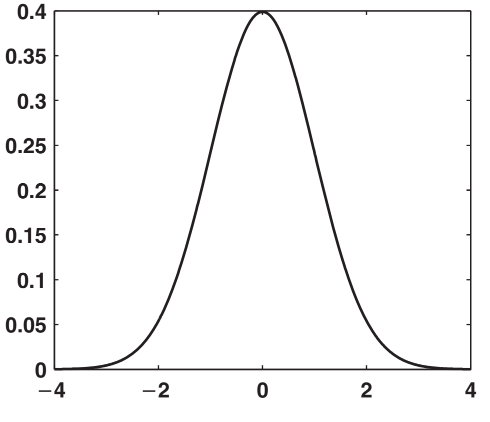

Figure 2.1 Gaussian density function

(ii) The exponential density:

$$f_X(x) = \alpha e^{-\alpha x}, \quad \forall x \in [0, +\infty], \tag{2.17}$$

where α is a positive constant. Figure 2.2 plots the probability density function for a variable following an exponential distribution.