FOUNDATIONS OF APPLIED ELECTRODYNAMICS

Wen Geyi
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To my parents

To Jun and Lan

Contents

Preface

Electrodynamics is an important course in both physics and electrical engineering curricula. The graduate students majoring in applied electromagnetics are often confronted with a large number of new concepts and mathematical techniques found in a number of courses, such as *Advanced Electromagnetic Theory*, *Field Theory of Guided Waves*, *Advanced Antenna Theory*, *Electromagnetic Wave Propagation*, *Network Theory and Microwave Circuits*, *Computational Electromagnetics*, *Relativistic Electronics*, and *Quantum Electrodynamics*. Frequently, students have to consult a large variety of books and journals in order to understand and digest the materials in these courses, and this turns out to be a time-consuming process. For this reason, it would be helpful for the students to have a book that gathers the essential parts of these courses together and treats them according to the similarity of mathematical techniques.

Engineers, applied mathematicians and physicists who have been doing research for many years often find it necessary to renew their knowledge and want a book that contains the fundamental results of these courses with a fresh and advanced approach. With this goal in mind, inevitably this is beyond the conventional treatment in these courses. For example, the completeness of eigenfunctions is a key result in mathematical physics but is often mentioned without rigorous proof in most books due to the involvement of generalized function theory. As a result, many engineers lack confidence in applying the theory of eigenfunction expansions to solve practical problems. In order to fully understand the theory of eigenfunction expansions, it is imperative to go beyond the classical solutions of partial differential equations and introduce the concept of generalized solutions.

The contents of this book have been selected according to the above considerations, and many topics are approached in contemporary ways. The book intends to provide a whole picture of the fundamental theory of electrodynamics in most active areas of engineering applications. It is self-contained and is adapted to the needs of graduate students, engineers, applied physicists and mathematicians, and is aimed at those readers who wish to acquire more advanced analytical techniques in studying applied electrodynamics. It is hoped that the book will be a useful tool for readers saving them time and effort consulting a wide range of books and technical journals. After reading this book, the readers should be able to pursue further studies in applied electrodynamics without too much difficulty.

The book consists of ten chapters and four appendices. Chapter 1 begins with experimental laws and reviews Maxwell equations, constitutive relations, as well as the important properties derived from them. In addition, the basic electromagnetic theorems are summarized. Since most practical electromagnetic signals can be approximated by a temporal or a spatial wavepacket, the theory of wavepackets and various propagation velocities of wavepackets are also examined.

In applications, the solution of a partial differential equation is usually understood to be a classical solution that satisfies the smooth condition required by the highest derivative in the equation. This requirement may be too stringent in some situations. A rectangular pulse is not smooth in the classical sense yet it is widely used in digital communication systems. The first derivative of the Green's function of a wave equation is not continuous, but is broadly accepted by physicists and engineers. Chapter 2 studies the solutions of Maxwell equations. Three main analytical methods for solving partial differential equations are discussed: (1) the separation of variables; (2) the Green's function; and (3) the variational method. In order to be free of the constraint of classical solutions, the theory of generalized solutions of differential equation is introduced. The Lagrangian and Hamiltonian formulations of Maxwell equations are the foundations of quantization of electromagnetic fields, and they are studied through the use of the generalized calculus of variations. The integral representations of the solutions of Maxwell equations and potential theory are also included.

Eigenvalue problems frequently appear in physics, and have their roots in the method of separation of variables. An eigenmode of a system is a possible state when the system is free of excitation, and the corresponding eigenvalue often represents an important quantity of the system, such as the total energy and the natural oscillation frequency. The theory of eigenvalue problems is of fundamental importance in physics. One of the important tasks in studying the eigenvalue problems is to prove the completeness of the eigenmodes, in terms of which an arbitrary state of the system can be expressed as a linear combination of the eigenmodes. To rigorously investigate the completeness of the eigenmodes, one has to use the concept of generalized solutions of partial differential equations. Chapter 3 discusses the eigenvalue problems from a unified perspective. The theory of symmetric operators is introduced and is then used to study the interior eigenvalue problems in electromagnetic theory, which involves metal waveguides and cavity resonators. This chapter also treats the mode theory of spherical waveguides and the method of singular function expansion for scattering problems, which are useful in solving exterior boundary value problems.

An antenna is a device for radiating or receiving radio waves. It is an overpass connecting a feeding line in a wireless system to free space. The antenna is characterized by a number of parameters such as gain, bandwidth, and radiation pattern. The free space may be viewed as a spherical waveguide, and the spherical wave modes excited by the antenna depend on the antenna size. The bigger the antenna size, the more the propagating modes are excited. For a small antenna, most spherical modes turn out to be evanescent, making the stored energy around the antenna very large and the gain of the antenna very low. For this reason, most of the antenna parameters are subject to certain limitations. From time to time, there arises a question of how to achieve better antenna performance than previously obtained. Chapter 4 attempts to answer this question and deals with the fundamentals of radiation theory. The most important antenna parameters are reviewed and summarized. A complete theory of spherical vector wave functions is introduced, and is then used to study the upper bounds of the product of gain and bandwidth for an arbitrary antenna. In this chapter, the Foster reactance theorem for an ideal antenna without Ohmic loss, and the relationship between antenna bandwidth and antenna quality factor are investigated. In addition, the methods for evaluating antenna quality factor are also developed.

Electromagnetic boundary value problems can be characterized either by a differential equation or an integral equation. The integral equation is most appropriate for radiation and scattering problems, where the radiation condition at infinity is automatically incorporated in the formulation. The integral equation formulation has certain unique features that a differential equation formulation does not have. For example, the smooth requirement for the solution of integral equation is weaker than the corresponding differential equation. Another feature is that the discretization error of the integral equation is limited on the boundary of the solution region, which leads to more accurate numerical results. Chapter 5 summarizes integral equations for various electromagnetic field problems encountered in microwave and antenna engineering, including waveguides, metal cavities, radiation, and scattering problems by conducting and dielectric objects. The spurious solutions of integral equations are examined. Numerical methods generally applicable to both differential equations and integral equations are introduced.

Field theory and circuit theory are complementary to each other in electromagnetic engineering, and the former is the theoretical foundation of the latter while the latter is much easier to master. The circuit formulation has removed unnecessary details in the field problem and has preserved most useful overall information, such as the terminal voltages and currents. Chapter 6 studies the network representation of electromagnetic field systems and shows how the network parameters of multi-port microwave systems can be calculated by the field theory through the use of reciprocity theorem, which provides a deterministic approach to wireless channel modeling. Also discussed in this chapter is the optimization of power transfer between antennas, a foundation for wireless power transfer.

The wave propagation in an inhomogeneous medium is a very complicated process, and it is characterized by a partial differential equation with variable coefficients. The inhomogeneous waveguides are widely used in microwave engineering. If the waveguides are bounded by a perfect conductor, only a number of discrete modes called guided modes can exist in the waveguides. If the waveguides are open, an additional continuum of radiating modes will appear. In order to obtain a complete picture of the modes in the inhomogeneous waveguides, one has to master a sophisticated tool called spectral analysis in operator theory. Chapter 7 investigates the wave propagation problems in inhomogeneous media and contains an introduction to spectral analysis. It covers the propagation of plane waves in inhomogeneous media, inhomogeneous metal waveguides, optical fibers and inhomogeneous metal cavity resonators.

Time-domain analysis has become a vital research area in recent years due to the rapid progress made in ultra-wideband technology. The traditional time-harmonic field theory is based on an assumption that a monotonic electromagnetic source turns on at $t = -\infty$ so that the initial conditions or causality are ignored. This assumption does not cause any problems if the system has dissipation or radiation loss. When the system is lossless, the assumption may lead to physically unacceptable solutions. In this case, one must resort to time-domain analysis. Chapter 8 discusses the time-domain theory of electromagnetic fields, including the transient fields in waveguides and cavity resonators, spherical wave expansion in time domain, and time-domain theory for radiation and scattering.

Modern physics has its origins deeply rooted in electrodynamics. A cornerstone of modern physics is relativity, which is composed of both special relativity and general relativity. The special theory of relativity studies the physical phenomena perceived by different observers traveling at a constant speed relative to each other, and it is a theory about the structure of space–time. The general theory studies the phenomena perceived by different observers traveling at an arbitrary relative speed and is a theory of gravitation. The relativity, especially the special relativity, is usually considered as an integral part of electrodynamics. Relativity has many practical applications. For example, in the design of the global positioning system

(GPS), the relativistic effects predicted by the special and general theories of relativity must be taken into account to enhance the positioning precision. Chapter 9 deals with both special relativity and general relativity. The tensor algebra and tensor analysis on manifolds are used throughout the chapter.

Another cornerstone of modern physics is quantum mechanics. Quantum electrodynamics is a quantum field theory of electromagnetics, which describes the interaction between light and matter or between two charged particles through the exchange of photons. It is remarkable for its extremely accurate predictions of some physical quantities. Quantum electrodynamics is especially needed in today's research and education activities in order to understand the interactions of new electromagnetic materials with the fields. Chapter 10 provides a short introduction to quantum electrodynamics and a review of the fundamental concepts of quantum mechanics. The interactions of fields with charged particles are investigated by use of the perturbation method, in terms of which the dielectric constant for atom media is derived. Furthermore, the Klein–Gordon equation and the Dirac equation in relativistic mechanics are briefly discussed.

The book features a wide coverage of the fundamental topics in applied electrodynamics, including microwave theory, antenna theory, wave propagation, relativistic and quantum electrodynamics, as well as the advanced mathematical techniques that often appear in the study of theoretical electrodynamics. For the convenience of readers, four appendices are also included to present the fundamentals of set theory, vector analysis, special functions, and the SI unit system. The prerequisite for reading the book is advanced calculus. The SI units are used throughout the book. A $e^{j\omega t}$ time variation is assumed for time-harmonic fields. A special symbol $\square$ is used to indicate the end of an example or a remark.

During the writing and preparation of this book, the author had the pleasure of discussing the book with many colleagues and cannot list them all here. In particular, the author would like to thank Prof. Robert E. Collin of Case Western Reserve University for his comments and input on many topics discussed in the book, and Prof. Thomas T. Y. Wong of Illinois Institute of Technology for his useful suggestions on the selection of the contents of the book.

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1

Maxwell Equations

> Ten thousand years from now, there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics.
>
> —Richard Feynman (American physicist, 1918–1988)

To master the theory of electromagnetics, we must first understand its history, and find out how the notions of electric charge and field arose and how electromagnetics is related to other branches of physical science. Electricity and magnetism were considered to be two separate branches in the physical sciences until Oersted, Ampère and Faraday established a connection between the two subjects. In 1820, Hans Christian Oersted (1777–1851), a Danish professor of physics at the University of Copenhagen, found that a wire carrying an electric current would change the direction of a nearby compass needle and thus disclosed that electricity can generate a magnetic field. Later the French physicist André Marie Ampère (1775–1836) extended Oersted's work to two parallel current-carrying wires and found that the interaction between the two wires obeys an inverse square law. These experimental results were then formulated by Ampère into a mathematical expression, which is now called Ampère's law. In 1831, the English scientist Michael Faraday (1791–1867) began a series of experiments and discovered that magnetism can also produce electricity, that is, electromagnetic induction. He developed the concept of a magnetic field and was the first to use lines of force to represent a magnetic field. Faraday's experimental results were then extended and reformulated by James Clerk Maxwell (1831–1879), a Scottish mathematician and physicist. Between 1856 and 1873, Maxwell published a series of important papers, such as 'On Faraday's line of force' (1856), 'On physical lines of force' (1861), and 'On a dynamical theory of the electromagnetic field' (1865). In 1873, Maxwell published 'A Treatise on Electricity and Magnetism' on a unified theory of electricity and magnetism and a new formulation of electromagnetic equations since known as Maxwell equations. This is one of the great achievements of nineteenth-century physics. Maxwell predicted the existence of electromagnetic waves traveling at the speed of light and he also proposed that light is an electromagnetic phenomenon. In 1888, the German physicist Heinrich Rudolph Hertz (1857–1894) proved that an electric signal can travel through the air and confirmed the existence of electromagnetic waves, as Maxwell had predicted.

Foundations of Applied Electrodynamics Geyi Wen
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Maxwell's theory is the foundation for many future developments in physics, such as special relativity and general relativity. Today the words 'electromagnetism', 'electromagnetics' and 'electrodynamics' are synonyms and all represent the merging of electricity and magnetism. Electromagnetic theory has greatly developed to reach its present state through the work of many scientists, engineers and mathematicians. This is due to the close interplay of physical concepts, mathematical analysis, experimental investigations and engineering applications. Electromagnetic field theory is now an important branch of physics, and has expanded into many other fields of science and technology.

1.1 Experimental Laws

It is known that nature has four fundamental forces: (1) the strong force, which holds a nucleus together against the enormous forces of repulsion of the protons, and does not obey the inverse square law and has a very short range; (2) the weak force, which changes one flavor of quark into another and regulates radioactivity; (3) gravity, the weakest of the four fundamental forces, which exists between any two masses and obeys the inverse square law and is always attractive; and (4) electromagnetic force, which is the force between two charges. Most of the forces in our daily lives, such as tension forces, friction and pressure forces are of electromagnetic origin.

1.1.1 Coulomb's Law

Charge is a basic property of matter. Experiments indicate that certain objects exert repulsive or attractive forces on each other that are not proportional to the mass, therefore are not gravitational. The source of these forces is defined as the charge of the objects. There are two kinds of charges, called positive and negative charge respectively. Charges are quantitized and come in integer multiples of an **elementary charge**, which is defined as the magnitude of the charge on the electron or proton. An arrangement of one or more charges in space forms a charge distribution. The **volume charge density**, the **surface charge density** and the **line charge density** describe the amount of charge per unit volume, per unit area and per unit length respectively. A net motion of electric charge constitutes an electric current. An electric current may consist of only one sign of charge in motion or it may contain both positive and negative charge. In the latter case, the current is defined as the net charge motion, the algebraic sum of the currents associated with both kinds of charges.

In the late 1700s, the French physicist Charles-Augustin de Coulomb (1736–1806) discovered that the force between two charges acts along the line joining them, with a magnitude proportional to the product of the charges and inversely proportional to the square of the distance between them. Mathematically the force $\mathbf{F}$ that the charge q_1 exerts on q_2 in vacuum is given by **Coulomb's law**

$$\mathbf{F} = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2}\mathbf{u}_R \tag{1.1}$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance between the two charges with $\mathbf{r}'$ and $\mathbf{r}$ being the position vectors of q_1 and q_2 respectively; $\mathbf{u}_R = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ is the unit vector pointing from q_1 to q_2, and $\varepsilon_0 = 8.85 \times 10^{-12}$ is the permittivity of the medium in vacuum. In order that the distance between the two charges can be clearly defined, strictly speaking, Coulomb's law applies only to the point charges, the charged objects of zero size. Dividing (1.1) by q_2 gives a force exerting on a unit charge, which is defined as the **electric field intensity E** produced by the charge q_1. Thus the electric field produced by an arbitrary charge q is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 R^2}\mathbf{u}_R = -\nabla\phi(\mathbf{r}) \tag{1.2}$$

where $\phi(\mathbf{r}) = q/4\pi\varepsilon_0 R$ is called the **Coulomb potential**. Here $R = \left|\mathbf{r} - \mathbf{r}'\right|$, $\mathbf{r}'$ is the position vector of the point charge q and $\mathbf{r}$ is the observation point. For a continuous charge distribution in a finite volume V with charge density $\rho(\mathbf{r})$, the electric field produced by the charge distribution is obtained by superposition

$$\mathbf{E}(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{4\pi\varepsilon_0 R^2}\mathbf{u}_R dV(\mathbf{r}') = -\nabla\phi(\mathbf{r}) \tag{1.3}$$

where

$$\phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{4\pi\varepsilon_0 R} dV(\mathbf{r}')$$

is the potential. Taking the divergence of (1.3) and making use of $\nabla^2(1/R) = -4\pi\delta(R)$ leads to

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}. \tag{1.4}$$

This is called **Gauss's law**, named after the German scientist Johann Carl Friedrich Gauss (1777–1855). Taking the rotation of (1.3) gives

$$\nabla\times\mathbf{E}(\mathbf{r}) = 0. \tag{1.5}$$

The above results are valid in a vacuum. Consider a dielectric placed in an external electric field. If the dielectric is ideal, there are no free charges inside the dielectric but it does contain bound charges which are caused by slight displacements of the positive and negative charges of the dielectric's atoms or molecules induced by the external electric field. These slight displacements are very small compared to atomic dimensions and form small electric dipoles. The **electric dipole moment** of an induced dipole is defined by $\mathbf{p} = ql\mathbf{u}_l$, where l is the separation of the two charges and $\mathbf{u}_l$ is the unit vector directed from the negative charge to the positive charge (Figure 1.1).

Example 1.1: Consider the dipole shown in Figure 1.1. The distances from the charges to a field point P are denoted by R_+ and R_- respectively, and the distance from the center of the

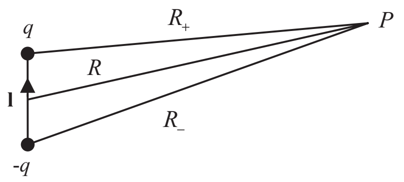

Figure 1.1 Induced dipole

dipole to the field point P is denoted by R. The potential at P is

$$\phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right).$$

If $l \ll R$, we have

$$\frac{1}{R_+} = \frac{1}{\sqrt{(l/2)^2 + R^2 - lR\mathbf{u}_l \cdot \mathbf{u}_R}} \approx \frac{1}{R}\left(1 + \frac{1}{2}\frac{l}{R}\mathbf{u}_l \cdot \mathbf{u}_R\right),$$
$$\frac{1}{R_-} = \frac{1}{\sqrt{(l/2)^2 + R^2 + lR\mathbf{u}_l \cdot \mathbf{u}_R}} \approx \frac{1}{R}\left(1 - \frac{1}{2}\frac{l}{R}\mathbf{u}_l \cdot \mathbf{u}_R\right),$$

where $\mathbf{u}_R$ is the unit vector directed from the center of the dipole to the field point P. Thus the potential can be written as

$$\phi \approx \frac{1}{4\pi\varepsilon_0 R^2}\mathbf{p} \cdot \mathbf{u}_R. \tag{1.6}$$

□

The dielectric is said to be polarized when the induced dipoles occur inside the dielectric. To describe the macroscopic effect of the induced dipoles, we define the **polarization vector P** as

$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{1}{\Delta V}\sum_i \mathbf{p}_i \tag{1.7}$$

where ΔV is a small volume and $\sum_i \mathbf{p}_i$ denotes the vector sum of all dipole moments induced inside ΔV. The polarization vector is the volume density of the induced dipole moments. The dipole moment of an infinitesimal volume dV is given by $\mathbf{P}dV$, which produces the potential (see (1.6))

$$d\phi \approx \frac{dV}{4\pi\varepsilon_0 R^2}\mathbf{P} \cdot \mathbf{u}_R.$$

The total potential due to a polarized dielectric in a region V bounded by S may be expressed as

$$\begin{aligned}\phi(\mathbf{r}) &\approx \int\limits_V \frac{\mathbf{P}\cdot\mathbf{u}_R}{4\pi\varepsilon_0 R^2} dV(\mathbf{r}') = \frac{1}{4\pi\varepsilon_0}\int\limits_V \mathbf{P}\cdot\nabla'\frac{1}{R}dV(\mathbf{r}') \\ &= \frac{1}{4\pi\varepsilon_0}\int\limits_V \nabla'\cdot\left(\frac{\mathbf{P}}{R}\right)dV(\mathbf{r}') + \frac{1}{4\pi\varepsilon_0}\int\limits_V \frac{-\nabla'\cdot\mathbf{P}}{R}dV(\mathbf{r}') \\ &= \frac{1}{4\pi\varepsilon_0}\int\limits_S \frac{\mathbf{P}\cdot\mathbf{u}_n(\mathbf{r}')}{R}dV(\mathbf{r}') + \frac{1}{4\pi\varepsilon_0}\int\limits_V \frac{-\nabla'\cdot\mathbf{P}}{R}dV(\mathbf{r}')\end{aligned} \tag{1.8}$$

where the divergence theorem has been used. In the above, $\mathbf{u}_n$ is the outward unit normal to the surface. The first term of (1.8) can be considered as the potential produced by a surface charge density $\rho_{ps} = \mathbf{P}\cdot\mathbf{u}_n$, and the second term as the potential produced by a volume charge density $\rho_p = -\nabla\cdot\mathbf{P}$.Both ρ_{ps} and ρ_p are the bound charge densities. The total electric field inside the dielectric is the sum of the fields produced by the free charges and bound charges. Gauss's law (1.4) must be modified to incorporate the effect of dielectric as follows

$$\nabla\cdot\varepsilon_0\mathbf{E} = \rho + \rho_p.$$

This can be written as

$$\nabla\cdot\mathbf{D} = \rho \tag{1.9}$$

where $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ is defined as the **electric induction intensity**. When the dielectric is linear and isotropic, the polarization vector is proportional to the electric field intensity so that $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$, where χ_e is a dimensionless number, called **electric susceptibility**. In this case we have

$$\mathbf{D} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon_r\varepsilon_0\mathbf{E} = \varepsilon\mathbf{E}$$

where $\varepsilon_r = 1 + \chi_e = \varepsilon/\varepsilon_0$ is a dimensionless number, called relative permittivity. Note that (1.5) holds in the dielectric.

1.1.2 Ampère's Law

There is no evidence that magnetic charges or magnetic monopoles exist. The source of the magnetic field is the moving charge or current. **Ampère's law** asserts that the force that a current element $\mathbf{J}_2 dV_2$ exerts on a current element $\mathbf{J}_1 dV_1$ in vacuum is

$$d\mathbf{F}_1 = \frac{\mu_0}{4\pi}\frac{\mathbf{J}_1 dV_1 \times (\mathbf{J}_2 dV_2 \times \mathbf{u}_R)}{R^2} \tag{1.10}$$

where R is the distance between the two current elements, $\mathbf{u}_R$ is the unit vector pointing from current element $\mathbf{J}_2 dV_2$ to current element $\mathbf{J}_1 dV_1$, and $\mu_0 = 4\pi \times 10^{-7}$ is the permeability in vacuum. Equation (1.10) can be written as

$$d\mathbf{F}_1 = \mathbf{J}_1 dV_1 \times d\mathbf{B}$$

where $d\mathbf{B}$ is defined as the **magnetic induction intensity** produced by the current element $\mathbf{J}_2 dV_2$

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{J}_2 dV_2 \times \mathbf{u}_R}{R^2}.$$

By superposition, the magnetic induction intensity generated by an arbitrary current distribution $\mathbf{J}$ is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}') \times \mathbf{u}_R}{R^2} dV(\mathbf{r}'). \tag{1.11}$$

This is called the **Biot-Savart law**, named after the French physicists Jean-Baptiste Biot (1774–1862) and Félix Savart (1791–1841). Equation (1.11) may be written as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

where $\mathbf{A}$ is known as the **vector potential** defined by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{R} dV(\mathbf{r}').$$

Thus

$$\nabla \cdot \mathbf{B} = 0. \tag{1.12}$$

This is called Gauss's law for magnetism, which says that the magnetic flux through any closed surface S is zero

$$\int_S \mathbf{B} \cdot \mathbf{u}_n dS = 0.$$

Taking the rotation of magnetic induction intensity and using $\nabla^2(1/R) = -4\pi\delta(R)$ and $\nabla \cdot \mathbf{J} = 0$ yields

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}(\mathbf{r}). \tag{1.13}$$

This is the differential form of Ampère's law.

Example 1.2: Consider a small circular loop of radius a that carries current I. The center of the loop is chosen as the origin of the spherical coordinate system as shown in Figure 1.2. The

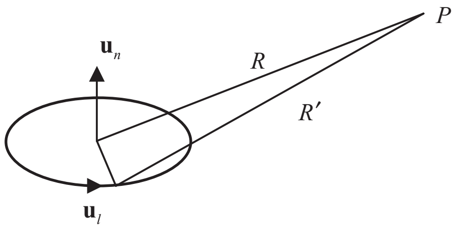

Figure 1.2 Small circular loop

vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_l \frac{1}{R'}\mathbf{u}_l dl(\mathbf{r}')$$

where $\mathbf{u}_l$ is the unit vector along current flow and l stands for the loop. Due to the symmetry, the vector potential is independent of the angle φ of the field point P. Making use of the following identity

$$\int_l \phi\mathbf{u}_l dl = \int_S \mathbf{u}_n \times \nabla\phi dS$$

where S is the area bounded by the loop l, the vector potential can be written as

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi}\int_S \mathbf{u}_n \times \nabla'\frac{1}{R'}dS(\mathbf{r}')\\ &= -\frac{\mu_0 I}{4\pi}\int_S \mathbf{u}_n \times \nabla\frac{1}{R'}dS(\mathbf{r}') = \frac{\mu_0 I}{4\pi}\nabla\times\int_S \mathbf{u}_n\frac{1}{R'}dS(\mathbf{r}').\end{aligned}$$

If the loop is very small, we can let $R' \approx R$. Thus

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi}\nabla\times\int_S \mathbf{u}_n\frac{1}{R'}dS(\mathbf{r}')\\ &\approx \frac{\mu_0}{4\pi}\nabla\times\frac{\mathbf{m}}{R} = \frac{\mu_0}{4\pi R^2}\mathbf{m}\times\mathbf{u}_R\end{aligned} \tag{1.14}$$

where $\mathbf{u}_R$ is the unit vector from the center of the loop to the field point P and

$$\mathbf{m} = I\int_S \mathbf{u}_n(\mathbf{r}')dS(\mathbf{r}') = I\mathbf{u}_n\pi a^2$$

is defined as the **magnetic dipole moment** of the loop. □

The above results are valid in a vacuum. All materials consist of atoms. An orbiting electron around the nucleus of an atom is equivalent to a tiny current loop or a magnetic dipole. In the absence of external magnetic field, these tiny magnetic dipoles have random orientations for most materials so that the atoms show no net magnetic moment. The application of an external magnetic field causes all these tiny current loops to be aligned with the applied magnetic field, and the material is said to be magnetized and the **magnetization current** occurs. To describe the macroscopic effect of magnetization, we define a **magnetization vector M** as

$$\mathbf{M} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i \mathbf{m}_i \tag{1.15}$$

where ΔV is a small volume and $\sum_i \mathbf{m}_i$ denotes the vector sum of all magnetic dipole moments induced inside ΔV. The magnetization vector is the volume density of the induced magnetic dipole moments. The magnetic dipole moments of an infinitesimal volume dV is given by $\mathbf{M}dV$, which produces a vector potential (see (1.14))

$$d\mathbf{A} = \frac{\mu_0}{4\pi R^2}\mathbf{M} \times \mathbf{u}_R dV(\mathbf{r}') = \frac{\mu_0}{4\pi}\mathbf{M} \times \nabla' \frac{1}{R} dV(\mathbf{r}').$$

The total vector potential due to a magnetized material in a region V bounded by S is then given by

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0}{4\pi} \int_V \mathbf{M} \times \nabla' \frac{1}{R} dV(\mathbf{r}') \\ &= \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \times \mathbf{M}}{R} dV(\mathbf{r}') - \frac{\mu_0}{4\pi} \int_V \nabla' \times \frac{\mathbf{M}}{R} dV(\mathbf{r}') \\ &= \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \times \mathbf{M}}{R} dV(\mathbf{r}') + \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{M} \times \mathbf{u}_n(\mathbf{r}')}{R} dS(\mathbf{r}') \end{aligned} \tag{1.16}$$

where $\mathbf{u}_n$ is the unit outward normal of S. The first term of (1.16) can be considered as the vector potential produced by a volume current density $\mathbf{J}_M = \nabla \times \mathbf{M}$, and the second term as the vector potential produced by a surface current density $\mathbf{J}_{Ms} = \mathbf{M} \times \mathbf{u}_n$. Both $\mathbf{J}_M$ and $\mathbf{J}_{Ms}$ are magnetization current densities. The total magnetic field inside the magnetized material is the sum of the fields produced by the conduction current and the magnetized current and Ampère's law (1.13) must be modified as

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J}_M).$$

This can be rewritten as

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.17}$$

where $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ is called **magnetic field intensity**. When the material is linear and isotropic, the magnetization vector is proportional to the magnetic field intensity so that $\mathbf{M} = \chi_m \mathbf{H}$, where χ_m is a dimensionless number, called **magnetic susceptibility**. In this case we have

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu_r \mu_0 \mathbf{H} = \mu \mathbf{E}$$

where $\mu_r = 1 + \chi_m = \mu/\mu_0$ is a dimensionless number, called relative permeability. Notice that (1.12) holds in a magnetized material.

1.1.3 Faraday's Law

Faraday's law asserts that the induced electromotive force in a closed circuit is proportional to the rate of change of magnetic flux through any surface bounded by that circuit. The direction of the induced current is such as to oppose the change giving rise to it. Mathematically, this can be expressed as

$$\int_\Gamma \mathbf{E} \cdot \mathbf{u}_t d\Gamma = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{u}_n dS$$

where Γ is a closed contour and S is the surface spanning the contour as shown in Figure 1.3; $\mathbf{u}_n$ and $\mathbf{u}_t$ are the unit normal to S and unit tangent vector along Γ respectively, and they satisfy the right-hand rule.

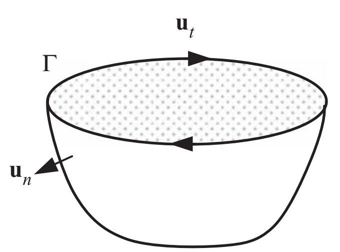

Figure 1.3 A two-sided surface

Loosely speaking, Faraday's law says that a changing magnetic field produces an electric field. The differential form of Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.18}$$

1.1.4 Law of Conservation of Charge

The law of conservation of charge states that the net charge of an isolated system remains constant. Mathematically, the amount of the charge flowing out of the surface S per second is equal to the decrease of the charge per second in the region V bounded by S

$$\int_S \mathbf{J} \cdot \mathbf{u}_n dS = -\frac{\partial}{\partial t} \int_V \rho dV.$$

The law of charge conservation is also known as the **continuity equation**. The differential form of the continuity equation is

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{1.19}$$

1.2 Maxwell Equations, Constitutive Relation, and Dispersion

From (1.18) and (1.17), one can find that a changing magnetic field produces an electric field by magnetic induction, but a changing electric field would not produce a magnetic field. In addition, equation (1.17) implies $\nabla \cdot \mathbf{J} = 0$, which contradicts the continuity equation for a time-dependent field. To solve these problems, Maxwell added an extra term $\mathbf{J}_d$ to Equation (1.17)

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d.$$

It then follows that

$$\nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d = 0.$$

Introducing the continuity equation yields

$$\nabla \cdot \mathbf{J}_d = \frac{\partial \rho}{\partial t}.$$

Substituting Gauss's law (1.4) into the above equation, one may obtain $\mathbf{J}_d = \partial \mathbf{D}/\partial t$. Thus (1.17) must be modified to

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}. \tag{1.20}$$

The term $\partial \mathbf{D}/\partial t$ is called the **displacement current**. Equation (1.20) implies that a changing electric field generates a magnetic field by electric induction. It is this new electric induction postulate that makes it possible for Maxwell to predict the existence of electromagnetic waves. The mutual electric and magnetic induction produces a self-sustaining electromagnetic vibration moving through space.