Vibrations and Waves in Continuous Mechanical Systems

Peter Hagedorn
TU Darmstadt, Germany

Anirvan DasGupta
IIT Kharagpur, India

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Contents

Preface

This book is a successor to the book written by the first author (with the help of Dr Klaus Kelkel, now at ZF Friedrichshafen), *Technische Schwingungslehre: Lineare Schwingungen kontinuierlicher mechanischer Systeme*, published in 1989 in German. The German book, which has been out of print for many years now, was developed from a course on the vibrations of continuous systems delivered regularly by the first author at the Technische Universität Darmstadt over the last 30 years to fourth and fifth year students of Applied Mechanics, Mechanical Engineering, and other engineering curricula. This course deals exclusively with linear continuous systems and structures, including wave propagation in different media, in particular acoustic waves. The students come from a course on the vibrations of discrete systems, or at least with rudimentary knowledge of discrete vibrations. Over the years, the course content has changed more and more. The plan for a new text came up in 2004 when the second author was spending a year in Darmstadt as an Alexander von Humboldt Research Fellow. It was then that we started to work on the present book. Later, we had a chance to get together again for some time in the Mathematisches Forschungsinstitut Oberwolfach, in the Black Forest, in Germany. In this stimulating and pleasant environment we worked out many details that have found their place in the present book.

From the beginning, in the Darmstadt vibration course we aimed at presenting both the modal solutions and the traveling wave solutions, showing the relations between the two types of representations of solutions. We have found time and again in different engineering problems involving the vibrations of elastic structures, that one and the same problem can be handled in both ways, and this dual approach gives new insights. This is particularly useful whenever the spectra are rather dense, as for example in the vortex-excited vibrations of overhead transmission lines. We believe that stressing the duality between modal representation and a wave-type solution often leads to better understanding of the system's dynamics.

In a time when most of the structural vibrations problems in industry are dealt with by commercial finite-element and/or multi-body codes, often used as black boxes, it may seem that analytical solutions to vibration problems have become superfluous. True, in general it is hopeless to search for analytical solutions for vibrations problems in systems with complex geometry, for example. On the other hand, it can also be extremely dangerous to solve vibration problems using finite-element codes as black boxes without properly checking the applicability and convergence for the problem at hand. Often, for example, gyroscopic terms, non-classical damping and other effects may not be properly handled by the codes if these are used naively. There are worked problems in this book that clearly demonstrate this point. It is therefore important to have benchmark solutions for a large number of vibration problems. Such benchmark results are precisely given by the analytical solutions. Moreover, certain qualitative aspects, such as dependence on parameters, asymptotic behavior, or the basic physics of the problem can be easily recognized from the analytical solutions, and are difficult to find by purely numerical methods. In certain cases, a theoretical/analytical handle can also help in extracting the numerical solution accurately and efficiently. The authors therefore believe that analytical solutions for linear vibrations of continuous systems even today are of great relevance to engineering curricula.

This book deals mainly with the derivation of the linear equations of motion of continuous mechanical systems such as strings, rods, beams, plates and membranes as well as with their solution, both via modal decomposition, and by the wave approach. The equations are derived using the elementary Newton–Euler approach, as well as using variational techniques. Both the free vibrations and forced damped and undamped vibrations are studied. The eigenvalue problems are solved analytically wherever possible, and orthogonality conditions are derived. Problems with non-homogeneous boundary conditions and systems involving simultaneously distributed and lumped parameters are discussed in detail. Eigenvalue problems for systems in which the eigenvalue appears explicitly in the boundary conditions are examined, and the orthogonality of eigenfunctions is also derived for such systems. The forced vibrations are also studied through different solution techniques. Important discretization methods are discussed in a systematic fashion, including the Rayleigh–Ritz and the Galerkin methods. Scattering of waves, and energetics of wave propagation in continuous media are examined in detail. The wave approach is used to explain certain phenomena, such as dispersion, wave propagation during impact and radiation damping.

The dynamics of the aforementioned elastic structural elements are dealt with in the first five chapters. In each of these chapters, a number of free and forced vibration problems are solved, using both exact and also approximate techniques, modal and wave representation. Almost no attention is given to the numerical solution of matrix eigenvalue problems resulting from the discretization of continuous systems, since tools such as MATLAB or Mathematica are readily available for their solution. Among some topics less commonly found in vibration books are dynamics of systems involving continuous and lumped parameters, dynamics and wave propagation in traveling continua, wave propagation during impacts, and the phenomenon of radiation damping.

In Chapter 6, the self-adjoint boundary value problems of continuous elastic systems are dealt with in a somewhat more abstract manner, and general results, such as the expansion theorem and Rayleigh's quotient, are stated and discussed in general form. A formulation for the eigenvalue problem in terms of integral equations using Green's functions is also given. The same chapter also deals with the class of discretization methods in which the solution is written as a series of products of chosen shape-functions with unknown time functions (generalized coordinates). The different ways of minimizing the error then lead to the different methods such as the Rayleigh–Ritz method, the Galerkin method and the collocation method. This also includes finite-element methods, which can be regarded as a particular case of the Rayleigh–Ritz methods.

Chapter 7 is in two parts. The first part is devoted to waves in fluids, including acoustic media, propagation in wave guides and also in slightly viscous fluids. Radiation from membranes and plates is also examined. The second part deals with surface waves in incompressible liquids, sloshing of liquids in partially filled tanks, and surface waves in channels. Chapter 8 deals with elements of wave motion in three-dimensional elastic continua, and includes a short introduction to Rayleigh surface waves.

Three appendices complement the text. The first one is on Hamilton's principle and the variational formulation of dynamics, the second one on harmonic waves, Fourier representation of waves and dispersion, and the third one is on the variational formulation of plate dynamics.

Each chapter comes with a number of problems of different degrees of difficulty, most of which have been used as homework problems in the course. There are many others which are new. Some of the exercise problems are intended to motivate the reader to explore some of the more advanced topics that are available in scientific journals or more advanced texts.

The authors believe that this book will fill a void as a textbook for a course on the linear vibrations of continuous systems. The sections of the book are carefully planned so that they may be used selectively in an undergraduate course, or a post-graduate course. It is hoped that the presence of some of the advanced topics (all of which may not be possible to cover in one course) will inspire the students to explore beyond the limits of a formal course. This book also should be of use to engineers working in the field of structural vibrations and dynamics.

The authors thank the staff of the Dynamics and Vibrations group in Darmstadt, in particular Dr Daniel Hochlenert and Dr Gottfried Spelsberg-Korspeter. They not only participated in the Oberwolfach project and gave important inputs, but also spent some time at IIT Kharagpur with the second author, where they helped in setting up the Latex environment for producing the book. The second author thanks Professor Sandipan Ghosh Moulic for providing useful comments on Chapter 7, and Mr Miska Venu Babu for his help in preparing the figures. The authors also thank the Alexander von Humboldt Foundation, the DAAD (German Academic Exchange Service), which made possible the visit of Darmstadt staff to IIT Kharagpur, the Mathematisches Forschungsinstitut Oberwolfach, as well as Wiley staff, who were extremely helpful in producing this book.

March 2007

Peter Hagedorn
Darmstadt

Anirvan DasGupta
Kharagpur

1

Vibrations of strings and bars

A one-dimensional continuous system, whose configuration at any time requires only one space dimension for description, is the simplest model of a class of continua with boundaries. Strings in transverse vibration, and bars of certain geometries in axial and torsional vibrations may be adequately described by one-dimensional continuous models. In this chapter, we will consider such models that are not only simple to study, but also are useful in developing the basic framework for analysis of continuous systems of one or more dimensions.

1.1 DYNAMICS OF STRINGS AND BARS: THE NEWTONIAN FORMULATION

1.1.1 Transverse dynamics of strings

A string is a one-dimensional elastic continuum that does not transmit or resist bending moment. Such an idealization may be justified even for cable-like components when the ratio of the thickness of the cable to its length (or wavelength of waves in the cable) is small compared to unity. In deriving the elementary equation of motion, it is assumed that the motion of the string is planar, and transverse to its length, i.e., longitudinal motion is neglected. Further, the amplitude of motion is assumed to be small enough so that the change in tension is negligible.

Consider a string, stretched along the x-axis to a length l by a tension T, as shown in Figure 1.1. Arbitrary distributed forces are assumed to act over the length of the string. The transverse motion of any point on the string at the coordinate position x is represented by the field variable $w(x, t)$ where t is the time. Consider the free body diagram of a small element of the string between two closely spaced points x and $x + \Delta x$, as shown in Figure 1.2. Let the element have a mass $\Delta m(x)$, and a deformed length Δs. The tensions at the two ends are $T(x, t)$ and $T(x + \Delta x, t)$, respectively, and the external force densities (force per unit length) are $p(x, t)$ in the transverse direction, and $n(x, t)$ in the longitudinal direction, as shown in the figure. Neglecting the inertia force in the longitudinal direction of the string, we can write the force balance equation for the small element in the longitudinal direction as

$$0 = T(x + \Delta x, t)\cos[\alpha(x + \Delta x, t)] - T(x, t)\cos[\alpha(x, t)] + n(x, t)\Delta s, \qquad (1.1)$$

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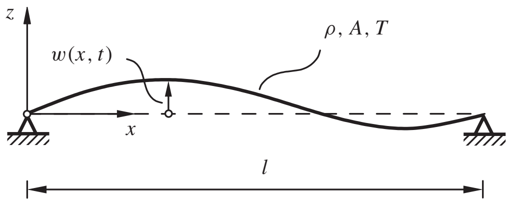

Figure 1.1 Schematic representation of a taut string

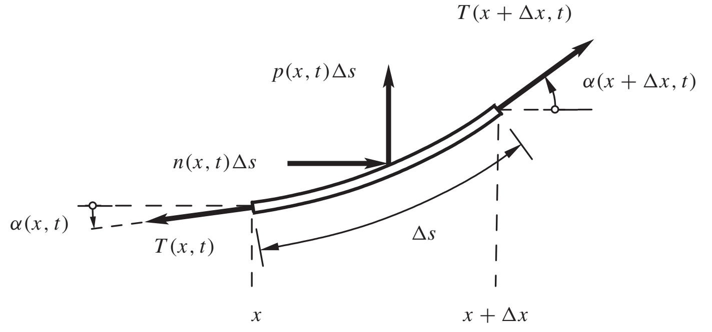

Figure 1.2 Free body diagram of a string element

where $\alpha(x, t)$ represents the angle between the tangent to the string at x and the x-axis, as shown in Figure 1.2. Dividing both sides of (1.1) by Δx and taking the limit $\Delta x \to 0$ yields

$$[T(x, t)\cos\alpha(x, t)]_{,x} = -n(x, t)\frac{\mathrm{d}s}{\mathrm{d}x}, \tag{1.2}$$

where $[\cdot]_{,x}$ represents partial derivative with respect to x. From geometry, one can write

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+w_{,x}^2}}, \qquad \text{and} \qquad \frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1+w_{,x}^2}. \tag{1.3}$$

Substituting (1.3) in (1.2), and assuming $w_{,x} \ll 1$, yields on simplification

$$[T(x, t)]_{,x} = -n(x, t). \tag{1.4}$$

Therefore, when $n(x, t) \equiv 0$, (1.4) implies that the tension $T(x, t)$ is a constant. On the other hand, for a hanging string, shown in Figure 1.3, one has $n(x, t) = \rho A(x)g$, where ρ is the density, A is the area of cross-section, and g is the acceleration due to gravity. Then, using the boundary condition of zero tension at the free end, i.e., $T(l, t) \equiv 0$ (for constant ρA), (1.4) yields $T(x, t) = \rho Ag(l - x)$. In general, the tension in a string may also depend on time. However, in the following discussions, it will be assumed to depend at most on x.

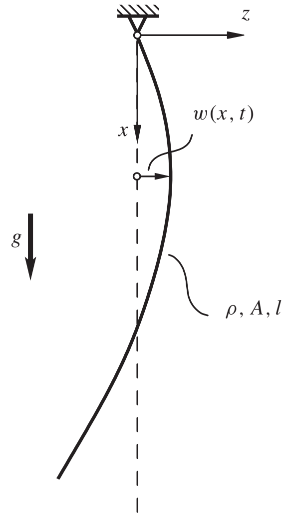

Figure 1.3 Schematic representation of a hanging string

Now, consider the transverse dynamics of the string element shown in Figure 1.1. The equation of motion of the small element in the transverse direction can be written from Newton's second law of motion as

$$\Delta m w_{,tt}(x+\theta\Delta x, t) = T(x+\Delta x)\sin[\alpha(x+\Delta x, t)] \\ -T(x)\sin[\alpha(x,t)] + p(x,t)\Delta s, \quad (1.5)$$

where Δm is the mass of the element, $\theta \in [0, 1]$, and $(\cdot)_{,tt}$ indicates double partial differentiation with respect to time. Again assuming $w_{,x} \ll 1$, one can write $\sin\alpha \approx \tan\alpha = w_{,x}$. Further, $\Delta m = \rho A(x)\Delta s$. Using these expressions in (1.5) and dividing by Δx on both sides, one can write after taking the limit $\Delta x \to 0$

$$\rho A(x) w_{,tt} - [T(x) w_{,x}]_{,x} = p(x,t), \quad (1.6)$$

where, based on the previous considerations, we have assumed $\mathrm{d}s/\mathrm{d}x \approx 1$. The linear partial differential equation (1.6), along with (1.4), represents the dynamics of a taut string. When the external force is not distributed but a concentrated force acting at, say $x = a$, the forcing function on the right hand side of (1.6) can be written using the *Dirac delta function* as

$$p(x,t) = f(t)\delta(x-a), \quad (1.7)$$

where $f(t)$ is the time-varying force, and $\delta(\cdot)$ represents the Dirac delta function.

Let us consider the hanging string shown in Figure 1.3 once again. The expression of tension derived earlier was $T(x) = \rho A g(l-x)$. Substituting this expression in (1.6) and assuming $p(x,t) \equiv 0$, one obtains on simplification

$$w_{,tt} - g[(l-x)w_{,x}]_{,x} = 0. \quad (1.8)$$

This case will be considered again later.

An important particular form of (1.6) is obtained for $p(x, t) \equiv 0$, and T and ρA not depending on x. We can rewrite (1.6) as

$$w_{,tt} - c^2 w_{,xx} = 0, \tag{1.9}$$

where $c = \sqrt{T/\rho A}$ is a constant having the dimension of speed. This represents the unforced transverse dynamics of a uniformly tensioned string. The hyperbolic partial differential equation (1.9) is known as the linear one-dimensional *wave equation*, and c is known as the wave speed. In the case of a taut string, c is the speed of transverse waves on the string, as we shall see later. This implies that a disturbance created at any point on the string propagates with a speed c. It should be clear that the wave speed c is distinct from the transverse material velocity (i.e., the velocity of the particles of the string) which is given by $w_{,t}(x, t)$. The solution and properties of the wave equation will be discussed in detail in Chapter 2.

The complete solution of the second-order partial differential equation (1.6) (or (1.9)) requires specification of two boundary conditions, and two initial conditions. For example, for a taut string shown in Figure 1.1, the appropriate boundary conditions are $w(0, t) \equiv 0$ and $w(l, t) \equiv 0$. For the case of a hanging string, the boundary conditions are $w(0, t) \equiv 0$ and $w(l, t)$ is finite. The initial conditions are usually specified in terms of the initial shape of the string, and initial velocity of the string, i.e., in the forms $w(x, 0) = w_0(x)$, and $w_{,t}(x, 0) = v_0(x)$, respectively. These will be discussed further later in this chapter.

Boundary conditions are classified into two types, namely *geometric* (or *essential*) boundary conditions, and *dynamic* (or *natural*) boundary conditions. A geometric boundary condition is one that imposes a kinematic constraint on the system at the boundary. The forces at such a boundary adjust themselves to maintain the constraint. On the other hand, a dynamic boundary condition imposes a condition on the forces, and the geometry adjusts itself to maintain the force condition. For example, in Figure 1.4, the right-end boundary condition is obtained from the consideration that the component of the tension in the transverse direction is zero, the roller being assumed massless. This implies $Tw_{,x}(l, t) \equiv 0$, which is a natural boundary condition. As a consequence of this force condition, the slope of the string remains zero. At the left-end boundary, the condition $w(0, t) \equiv 0$ is a geometric boundary condition, and the transverse force from the support point (which can be computed as $Tw_{,x}(0, t)$) will adjust itself appropriately to prevent any transverse motion of the right end of the string. Classification of boundary conditions based on their mathematical structure is discussed in Section 6.1.1.

When a string, in addition to the distributed mass, carries lumped masses (i.e., particles of finite mass) and is subjected to concentrated elastic restoring forces, these can be

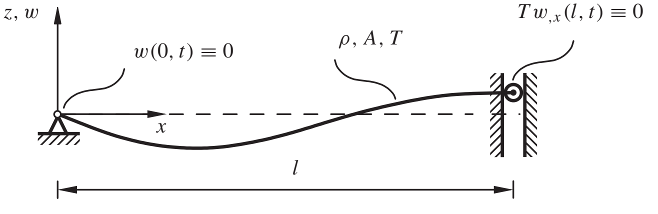

Figure 1.4 A taut string with geometric and natural boundary conditions

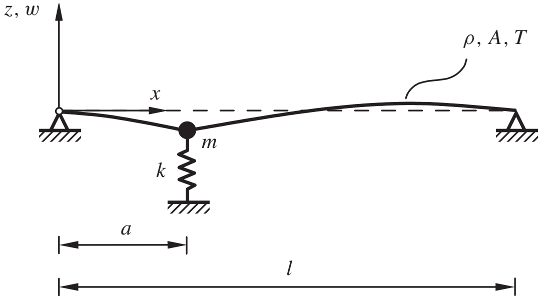

Figure 1.5 A taut string with lumped elements

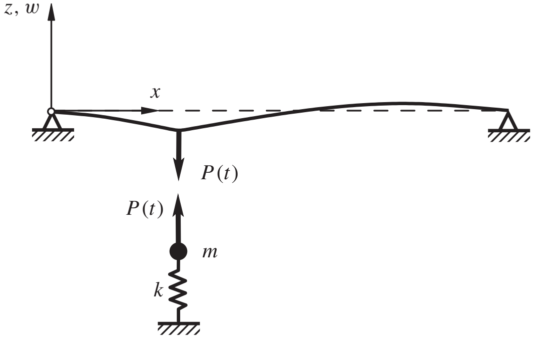

Figure 1.6 The interaction force diagram

easily incorporated into the equation of motion as follows. Consider the system shown in Figure 1.5, and the interaction force diagram shown in Figure 1.6. The force $P(t)$ at the interface between the string and the particle of mass m can be written from Newton's second law for the mass–spring system as $P(t) = mw_{,tt}(a, t) + kw(a, t)$, where $x = a$ is the location of the lumped system. Using the Dirac delta function, one can represent $P(t)$ as a distributed force

$$p(x, t) = mw_{,tt}(x, t)\delta(x - a) + kw(x, t)\delta(x - a). \tag{1.10}$$

Therefore, the equation of motion of the combined system can be written as

$$\rho A(x)w_{,tt} - [T(x)w_{,x}]_{,x} = -p(x, t),$$

or

$$[\rho A(x) + m\delta(x - a)]w_{,tt} - [T(x)w_{,x}]_{,x} + k\delta(x - a)w = 0. \tag{1.11}$$

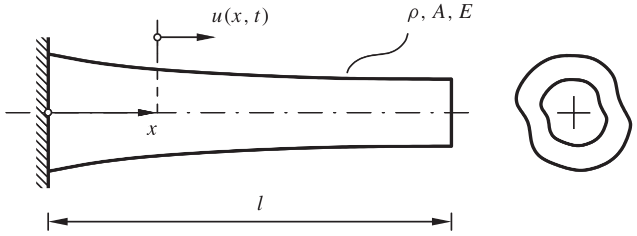

Figure 1.7 Schematic representation of a bar

1.1.2 Longitudinal dynamics of bars

Let us consider the longitudinal dynamics of a bar of arbitrary cross-section, as shown in Figure 1.7. We assume that the centroid of each cross-section lies on a straight line which is perpendicular to the cross-section. Under such assumptions, we can study the pure longitudinal motion of the bar. Such cases include bars which are solids of revolution (for example, cylinders and cones), and other standard structural elements.

Consider the free body diagram of an element of length Δx of the bar, as shown in Figure 1.8. We assume the displacement of any point of the bar to be along the x-axis, so that it can be represented by a single field variable $u(x, t)$. Using Newton's second law, one can write the equation of longitudinal motion of the element as

$$\rho A(x)\Delta x u_{,tt}(x+\theta\Delta x, t) = \sigma_x(x+\Delta x, t)A(x+\Delta x) - \sigma_x(x, t)A(x), \tag{1.12}$$

where ρ is the density, $A(x)$ is the cross-sectional area at x, $\theta \in [0, 1]$, and $\sigma_x(x, t)$ is the normal stress over the cross-section. Dividing (1.12) by Δx, and taking the limit $\Delta x \to 0$, yields

$$\rho A(x)u_{,tt}(x, t) = [\sigma_x(x, t)A(x)]_{,x}. \tag{1.13}$$

From elementary theory of elasticity (see [1]), we can relate the longitudinal strain $\epsilon_x(x, t)$ and the displacement field as $\epsilon_x(x, t) = u_{,x}(x, t)$. Using this strain–displacement relation and Hooke's law, one can write

$$\sigma_x(x, t) = E\epsilon_x(x, t) = Eu_{,x}(x, t), \tag{1.14}$$

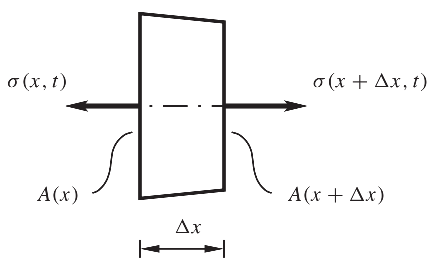

Figure 1.8 Free body diagram of a bar element

where E is the material's Young's modulus. Using (1.14) in (1.13) yields on rearrangement

$$\rho A(x)u_{,tt} - [EA(x)u_{,x}]_{,x} = 0. \tag{1.15}$$

If the bar is homogeneous and has a uniform cross-section, then (1.15) simplifies to

$$u_{,tt} - c^2 u_{,xx} = 0, \tag{1.16}$$

where $c = \sqrt{E/\rho}$ is the speed of the longitudinal waves in a uniform bar.

The boundary conditions for the bar can be written by inspection. For example, in Figure 1.7, the left-end boundary condition is $u(0, t) \equiv 0$, which is a geometric boundary condition. The right end of the bar is force-free, i.e., $EAu_{,x}(l, t) \equiv 0$. Hence, the right end of the bar has a dynamic boundary condition.

1.1.3 Torsional dynamics of bars

In this section, we make the same assumptions regarding the centroidal axis as made for the longitudinal dynamics of bars. The torsional dynamics of a bar depends on the shape of its cross-section. Complications arise due to warping of the cross-section during torsion in bars with non-circular cross-section (see [1]). In general, the torsional vibration of a bar is also coupled with its flexural vibration. Therefore, to keep the discussion simple, we will consider only torsional dynamics of bars with circular cross-section. As is known from the theory of elasticity, for bars with circular cross-section, planar sections remain planar for small torsional deformation. Further, an imaginary radial line on the undeformed cross-section can be assumed to remain straight even after deformation.

Consider a circular bar, as shown in Figure 1.9. A small sectional element of the bar between the centroidal coordinates x and $x + \Delta x$ is shown in Figure 1.10. Let $\phi(x, t)$ be the angle of twist at coordinate x, and $\phi + \Delta\phi(x, t)$ be the twist at $x + \Delta x$. From Figure 1.10, one can write, at any radius r, the kinematic relation

$$r\Delta\phi(x, t) = \Delta x\psi(r, t), \tag{1.17}$$

where $\psi(r, t)$ is the angular deformation of a longitudinal line at r, as shown in the figure. This angular deformation is the shear angle, as shown in Figure 1.11. Then, the shear stress $\tau_{x\phi}(r, t)$ is obtained from Hooke's law as

$$\tau_{x\phi}(r, t) = G\psi(r, t), \tag{1.18}$$

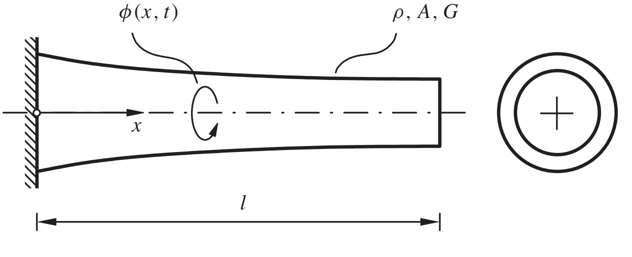

Figure 1.9 Schematic representation of a circular bar

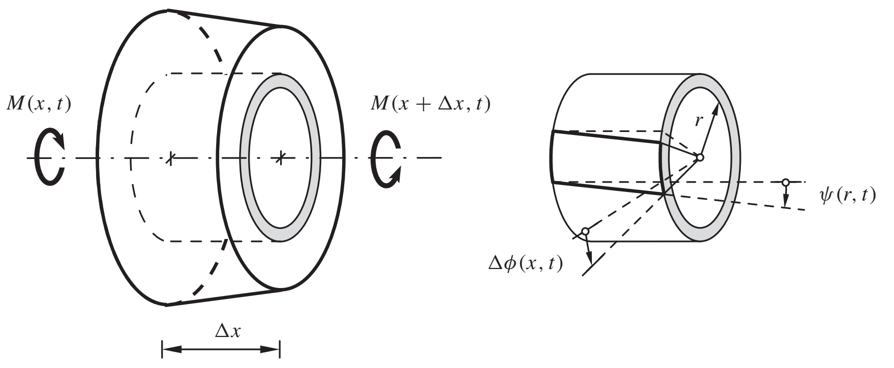

Figure 1.10 Deformation of a bar element under torsion

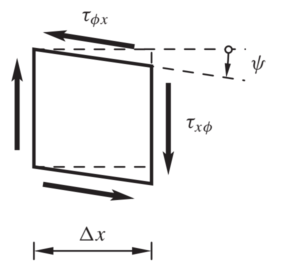

Figure 1.11 State of stress on a bar element under torsion

where G is the shear modulus. Substituting the expression of $\psi(r, t)$ from (1.17) in (1.18), one can write in the limit $\Delta x \to 0$

$$\tau_{x\phi}(r, t) = Gr\phi_{,x}. \tag{1.19}$$

Now, the torque at any cross-section x can be computed as

$$M(x, t) = \int_{A(x)} r\tau_{x\phi}(r, t)\,\mathrm{d}A = G\phi_{,x} \int_{A(x)} r^2 \mathrm{d}A = GI_\mathrm{p}(x)\phi_{,x}, \tag{1.20}$$

where $A(x)$ represents the cross-sectional area, and $I_\mathrm{p}(x)$ is the polar moment of the area. Writing the moment of momentum equation for the element yields

$$\left[\int_{A(x+\theta\Delta x)} \rho r^2 \Delta x\,\mathrm{d}A\right] \phi_{,tt}(x, t) = GI_\mathrm{p}(x + \Delta x)\phi_{,x}(x + \Delta x, t) - GI_\mathrm{p}(x)\phi_{,x}(x, t) + n_\mathrm{E}(x, t)\Delta x, \tag{1.21}$$

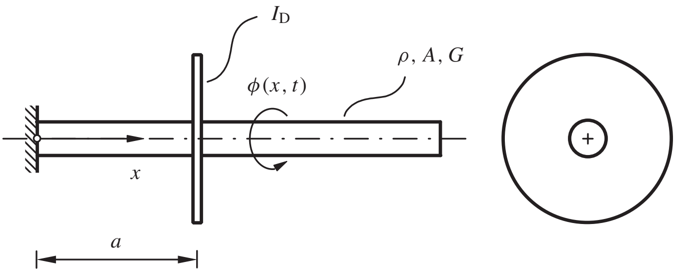

Figure 1.12 A circular bar with a disc

where $n_E(x, t)$ is an externally applied torque distribution. Dividing both sides in (1.21) by Δx and taking the limit $\Delta x \to 0$, we obtain

$$\rho I_p \phi_{,tt} - (G I_p \phi_{,x})_{,x} = n_E(x, t). \tag{1.22}$$

The partial differential equation (1.22) represents the torsional dynamics of a circular bar. For a bar with uniform cross-section (i.e., I_p independent of x), and $n_E(x, t) \equiv 0$, we obtain the wave equation

$$\phi_{,tt} - c^2 \phi_{,xx} = 0, \tag{1.23}$$

where $c = \sqrt{G/\rho}$ is the speed of torsional waves in the bar.

The boundary conditions for the fixed–free bar shown in Figure 1.9 can be written as $\phi(0, t) \equiv 0$, and $M(l, t) = G I_p \phi_{,x}(l, t) \equiv 0$. We can easily identify the first boundary condition as geometric, while the second is a natural boundary condition.

As an example, consider the torsional dynamics of a uniform circular bar with a massive disc at $x = a$, as shown in Figure 1.12. The disc can be considered as having a lumped rotational inertia. Therefore, the bar experiences an external torque due to the rotational inertia of the disc given by $n_E(x, t) = -I_D \phi_{,tt}(x, t) \delta(x - a)$, where I_D is the rotational inertia of the disc. Substituting this expression of external moment in (1.22), the complete equation of torsional dynamics of the bar can then be written as

$$[\rho I_p + I_D \delta(x - a)] \phi_{,tt} - G I_p \phi_{,xx} = 0. \tag{1.24}$$

1.2 DYNAMICS OF STRINGS AND BARS: THE VARIATIONAL FORMULATION

The variational formulation presents an elegant and powerful method of deriving the equations of motion of a dynamical system. Through this formulation, all the boundary conditions of a system are revealed. This is clearly an advantage especially for continuous mechanical systems. As will be discussed later, this approach also yields very useful methods of obtaining approximate solutions of vibration problems. The fundamentals of the variational approach for continuous systems is presented in Appendix A. In the following, we directly use the procedure discussed in Appendix A in deriving the equation of motion for strings and bars.

1.2.1 Transverse dynamics of strings

Consider a string of length l, as shown in Figure 1.1. The kinetic energy $\mathcal{T}$ of the string is

$$\mathcal{T} = \frac{1}{2}\int_0^l \rho A w_{,t}^2 \,\mathrm{d}x. \tag{1.25}$$

The potential energy can be written from the consideration that the unstretched length Δx is stretched to $\Delta s = \sqrt{1 + w_{,x}^2}\,\Delta x$ under a constant tension T. Therefore, the potential energy $\mathcal{V}$ stored in the string is given by

$$\begin{aligned}\mathcal{V} &= \int_0^l T(\mathrm{d}s - \mathrm{d}x) \approx \int_0^l T\left[\left(1 + \frac{1}{2}w_{,x}^2\right) - 1\right]\mathrm{d}x \\ &= \frac{1}{2}\int_0^l T w_{,x}^2 \,\mathrm{d}x.\end{aligned} \tag{1.26}$$

Defining the Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$, Hamilton's principle can be written as

$$\delta \int_{t_1}^{t_2} \mathcal{L}\,\mathrm{d}t = 0 \tag{1.27}$$

or

$$\delta \int_{t_1}^{t_2} \frac{1}{2}\int_0^l \left[\rho A w_{,t}^2 - T w_{,x}^2\right]\mathrm{d}x. \tag{1.28}$$

As detailed in Appendix A, one obtains from (1.28)

$$\begin{aligned}&\int_0^l \rho A w_{,t}\delta w\Big|_{t_1}^{t_2}\mathrm{d}x - \int_{t_1}^{t_2} T w_{,x}\delta w\Big|_0^l \mathrm{d}t \\ &\quad - \int_{t_1}^{t_2}\int_0^l [\rho A w_{,tt} - (T w_{,x})_{,x}]\,\delta w\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned} \tag{1.29}$$

The first term in (1.29) is always zero since the variations of the field variable at the initial and final times are zero, i.e., $\delta w(x, t_0) \equiv 0$, and $\delta w(x, t_1) \equiv 0$. Following the arguments in Appendix A, the integrand of the third term in (1.29) has to be zero, i.e.,

$$\rho A w_{,tt} - (T w_{,x})_{,x} = 0, \tag{1.30}$$

which yields the equation of transverse dynamics of the string. The second term in (1.29) is zero if, for example,

$$Tw_{,x}(0,t) \equiv 0 \qquad \text{or} \qquad w(0,t) \equiv 0 \tag{1.31}$$

and

$$Tw_{,x}(l,t) \equiv 0 \qquad \text{or} \qquad w(l,t) \equiv 0, \tag{1.32}$$

which represent possible boundary conditions. For a fixed–fixed string, the conditions $w(0,t) \equiv 0$ and $w(l,t) \equiv 0$ hold, while for a fixed–sliding string (see Figure 1.4), $w(0,t) \equiv 0$ and $Tw_{,x}(l,t) \equiv 0$.

In the case of a string with discrete elements shown in Figure 1.5, the kinetic and potential energies can be written as, respectively,

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}\int_0^l \rho A w_{,t}^2(x,t)\,\mathrm{d}x + \frac{1}{2} m w_{,tt}^2(a,t) \\ &= \frac{1}{2}\int_0^l [\rho A + m\delta(x-a)] w_{,tt}^2(x,t)\,\mathrm{d}x, \end{aligned} \tag{1.33}$$

$$\begin{aligned} \mathcal{V} &= \frac{1}{2}\int_0^l T w_{,x}^2(x,t)\,\mathrm{d}x + \frac{1}{2} k w^2(a,t) \\ &= \frac{1}{2}\int_0^l [T w_{,x}^2(x,t) + k\delta(x-a) w^2(x,t)]\,\mathrm{d}x. \end{aligned} \tag{1.34}$$

Substituting $\mathcal{L} = \mathcal{T} - \mathcal{V}$ in the variational form (1.27) and taking the variation yields on simplification

$$\begin{aligned} &\int_0^l [\rho A + m\delta(x-a)] w_{,t}\delta w \Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} T w_{,x}\delta w \Big|_0^l \mathrm{d}t \\ &- \int_{t_1}^{t_2}\int_0^l [(\rho A + m\delta(x-a)) w_{,tt} - (T w_{,x})_{,x} + k\delta(x-a) w]\,\delta w\,\mathrm{d}x\,\mathrm{d}t = 0. \end{aligned}$$

The equation of motion is obtained from the third term above which is the same as (1.11). The boundary conditions remain the same as in (1.31)–(1.32). When external forces are present, one can use the extended Hamilton's principle discussed in Appendix A to obtain the equations of motion.

1.2.2 Longitudinal dynamics of bars

In the case of longitudinal vibration of a bar, the kinetic energy is given by

$$\mathcal{T} = \frac{1}{2}\int_0^l \rho A u_{,t}^2\,\mathrm{d}x. \tag{1.35}$$

Defining σ_x and ϵ_x as the longitudinal stress and strain, respectively, the potential energy can be computed from the theory of elasticity as

$$\begin{aligned}\mathcal{V} &= \frac{1}{2}\int_0^l \sigma_x \epsilon_x A \,\mathrm{d}x = \frac{1}{2}\int_0^l EA\epsilon_x^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_0^l EAu_{,x}^2 \,\mathrm{d}x. \end{aligned} \tag{1.36}$$

Writing the Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$, Hamilton's principle assumes the form

$$\delta \int_{t_1}^{t_2} \mathcal{L} \,\mathrm{d}t = 0,$$

or

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \frac{1}{2} \int_0^l \left(\rho A u_{,t}^2 - EAu_{,x}^2\right) \mathrm{d}x \,\mathrm{d}t = 0, \\ \Rightarrow &\int_0^l \rho A \delta u\Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} EAu_{,x} \delta u\Big|_0^l \,\mathrm{d}t \\ &- \int_{t_1}^{t_2} \int_0^l [\rho A u_{,tt} - (EAu_{,x})_{,x}] \,\delta u \,\mathrm{d}x \,\mathrm{d}t = 0. \end{aligned} \tag{1.37}$$

Since by definition $\delta u(x, t_0) = \delta u(x, t_1) \equiv 0$, the first term in (1.37) vanishes identically. The third term in (1.37) yields the equation of motion

$$\rho A u_{,tt} - (EAu_{,x})_{,x} = 0, \tag{1.38}$$

and the boundary conditions are obtained from the second term. For example, the boundary conditions can be written as

$$EAu_{,x}(0, t) \equiv 0 \qquad \text{or} \qquad u(0, t) \equiv 0, \tag{1.39}$$

and

$$EAu_{,x}(l, t) \equiv 0 \qquad \text{or} \qquad u(l, t) \equiv 0. \tag{1.40}$$

It can be seen that the first condition in both (1.39) and (1.40) is the longitudinal force condition (natural boundary condition) at the two ends of the bar, while the second condition is the displacement condition (geometric boundary condition). Thus, for a fixed–fixed bar, $u(0, t) \equiv 0$, and $u(l, t) \equiv 0$, while for a fixed–free bar, $u(0, t) \equiv 0$ and $EAu_{,x}(l, t) \equiv 0$. In the case of a free–free bar, the boundary conditions are $EAu_{,x}(0, t) \equiv 0$, and $EAu_{,x}(l, t) \equiv 0$.

1.2.3 Torsional dynamics of bars

The kinetic energy of a circular bar undergoing torsional oscillations can be written in the notations used previously in Section 1.1.3 as

$$\begin{aligned}\mathcal{T} &= \frac{1}{2}\int_0^l \int_0^R \int_0^{2\pi} \rho\phi_{,t}^2 r^3 \,\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^l \rho I_{\mathrm{p}}\phi_{,t}^2 \,\mathrm{d}x. \end{aligned} \tag{1.41}$$

The potential energy can be written from elasticity theory as

$$\mathcal{V} = \frac{1}{2}\int_0^l \int_0^R \int_0^{2\pi} \tau_{x\phi}\psi r \,\mathrm{d}\phi\,\mathrm{d}r\mathrm{d}x. \tag{1.42}$$

Using the definitions of $\tau_{r\phi}$ and $\psi(x,t)$ from (1.17) and (1.18), respectively, in (1.42), we have

$$\begin{aligned}\mathcal{V} &= \frac{1}{2}\int_0^l \int_0^R \int_0^{2\pi} G\phi_{,x}^2 r^3 \,\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^l GI_{\mathrm{p}}\phi_{,x}^2 \,\mathrm{d}x. \end{aligned} \tag{1.43}$$

Hamilton's principle can then be written as

$$\begin{aligned} &\delta\int_{t_1}^{t_2} \frac{1}{2}\int_0^l \left[\rho I_{\mathrm{p}}\phi_{,t}^2 - GI_{\mathrm{p}}\phi_{,x}^2\right] \mathrm{d}x = 0 \\ \Rightarrow &\int_0^l \rho I_{\mathrm{p}}\phi_{,t}\delta\phi\Big|_{t_1}^{t_2} \mathrm{d}x - \int_{t_1}^{t_2} GI_{\mathrm{p}}\phi_{,x}\delta\phi\Big|_0^l \mathrm{d}t \\ &\quad - \int_{t_1}^{t_2}\int_0^l \left[\rho I_{\mathrm{p}}\phi_{,tt} - (GI_{\mathrm{p}}\phi_{,x})_{,x}\right] \delta\phi\,\mathrm{d}x = 0. \end{aligned} \tag{1.44}$$

The first term in (1.44) is zero by definition of the variational formulation. The third term in (1.44) yields the equation of motion

$$\rho I_{\mathrm{p}}\phi_{,tt} - (GI_{\mathrm{p}}\phi_{,x})_{,x} = 0, \tag{1.45}$$

while the second term provides information on the boundary conditions. For example, the possible boundary conditions could be

$$GI_{\mathrm{p}}\phi_{,x}(0,t) \equiv 0 \qquad \text{or} \qquad \phi(0,t) \equiv 0, \tag{1.46}$$

and

$$GI_p\phi_{,x}(l,t) \equiv 0 \qquad \text{or} \qquad \phi(l,t) \equiv 0. \tag{1.47}$$

The first condition in (1.46) and (1.47) can be easily identified to be the torque condition (natural boundary condition) at the ends of the bar, while the second condition is on the angular displacement (geometric boundary condition).

1.3 FREE VIBRATION PROBLEM: BERNOULLI'S SOLUTION

Vibration analysis of a system almost always starts with the free or natural vibration analysis. This leads us to the important concepts of natural frequency and mode of vibration of the system. These two concepts form the starting point of any quantitative and qualitative analysis and understanding of a vibratory system.

It was observed in the above discussions that, under certain assumptions of uniformity, the one-dimensional wave equation represents the transverse dynamics of a string, and longitudinal and torsional dynamics of a bar. The wave equation is one of the most important equations that appear in the study of vibrations of continuous systems. The solution and properties of the wave equation are fundamental in understanding vibration and propagation of vibration in continuous media, and will be taken up in detail in later chapters. In this section, we will discuss a simple solution procedure for the one-dimensional wave equation and study some of the solution properties.

Consider the wave equation

$$w_{,tt} - c^2 w_{,xx} = 0, \qquad x \in [0, l], \tag{1.48}$$

with the boundary conditions

$$w(0,t) \equiv 0, \qquad \text{and} \qquad w(l,t) \equiv 0. \tag{1.49}$$

Such a problem corresponds to, for example, a fixed-fixed string or bar.

Let us first look for separable solutions of (1.48) in the form

$$w(x,t) = p(t)W(x). \tag{1.50}$$

Substituting (1.50) in (1.48) yields on rearrangement

$$\frac{\ddot{p}}{p} - c^2\frac{W''}{W} = 0. \tag{1.51}$$

It is easily observed that the first term in (1.51) is solely a function of t, while the second term is solely a function of x. Therefore, (1.51) will hold identically if and only if both the terms are constant, i.e.,

$$\frac{\ddot{p}}{p} = -\omega^2 \qquad \text{and} \qquad c^2\frac{W''}{W} = -\omega^2 \tag{1.52}$$

$$\Rightarrow \ddot{p} + \omega^2 p = 0 \tag{1.53}$$