THINKING ABOUT EQUATIONS

A Practical Guide for Developing Mathematical Intuition in the Physical Sciences and Engineering

MATT A. BERNSTEIN, PhD
WILLIAM A. FRIEDMAN, PhD

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PREFACE

Equations play a central role in day-to-day problem solving for the physical sciences, engineering, and related fields. Those of us who pursue these technical careers must learn to understand and to solve equations. It is often a challenge to figure out what an equation means and then to form a strategy to solve it. One aim of this book is to illustrate that by investing a little effort into thinking about equations, your task can become easier and your solutions can become more reliable.

On a deeper level, studying the topics presented here could provide you with greater insight into your own field of study, and you may even derive some intellectual pleasure in the process. Today, a staggering amount of knowledge is readily available through the Internet. With this abundance of information comes an ever-greater premium on understanding the interconnections and themes embedded within this information. This book can help you gain that intuitive understanding while you acquire some practical mathematical tools. We hope that our selection of topics and approaches will open your eyes to this useful and rewarding process.

One might ask why we wrote a book about equations, i.e., focusing on analytic methods, when today so many problems are solved using numerical methods. In our own scientific research, we rely heavily on numerical computation. Equations, however, are the basis for almost all numerical solutions and simulations. A deeper understanding of the underlying equations often can improve the strategy for a numerical approach, and importantly, it helps one to check whether the resulting numerical answers are reasonable.

With the advent and widespread availability of powerful computational packages such as MATLAB®, MAPLE™, IDL®, and others, some students have spent less time studying and thinking about equations and have not yet developed an intuitive "feel" for this topic that experience brings. This book can serve as a resource and workbook for students who wish to improve those skills in an efficient manner. We believe that the powerful graphics capabilities of those computational packages can complement the material presented here, and users of symbolic manipulation software such as Mathematica® will also find this book relevant to their work.

The core of our intended audience is undergraduate students majoring in any discipline within engineering or any of the physical sciences including physics, geology, astronomy, and chemistry, as well as related fields such as applied mathematics and biophysics. Even students in peripherally related fields, such as computational finance, will find the book useful, particularly when they apply methods that originated in the physical sciences and engineering to their own work. Graduate students with an undergraduate background in a field like biology who are transitioning into a more quantitative area such as biomedical engineering could also find the techniques described in this book quite useful.

We have tried to target the level of the book so it is accessible to undergraduate students who have a background in basic calculus and some familiarity with differential equations. An undergraduate course in introductory physics is also recommended. Some of the worked example problems draw on concepts and results from linear algebra and complex analysis, but this background is not essential to follow most of the material. We are not mathematicians, and there are no formal mathematical proofs in the book. Instead, the derivations, where provided, are presented at a level of rigor that is typical of work in the physical sciences. A few of the sections and example problems are labeled with an asterisk (*), because either they assume a higher level of mathematical background or else the material contained in them is not central to the rest of the book. These starred sections and problems can be skipped without loss of continuity.

This book is *not* a comprehensive "how-to" manual of mathematical methods. Instead, selected concepts are introduced by a short discussion in each section of each chapter. Then, the concepts are illustrated and further developed with example problems followed by detailed solutions. These worked example problems form the backbone of this book.

Our own backgrounds are in physics, with specialization in magnetic resonance imaging, medical physics, and biomedical engineering (MAB) and theoretical nuclear physics (WAF). While we have chosen example problems drawn from subjects that are familiar to us, we have tried to avoid problems that require highly specialized knowledge. We also have tried to provide explanations of terms that might be unfamiliar and to avoid the use of jargon. We hope you will read and work through example problems that are *outside* of your immediate area of interest. If you can recognize concepts that can be applied to your own work, then you will have made great progress toward our ultimate goal. At the end of each chapter, there are a number of exercises designed to test and further develop your understanding of the material. We also encourage you to devise and to solve some of your own exercises, perhaps related to your own specific field of interest.

Chapter 1 deals with units and dimensions of physical quantities. Although this is a very basic topic, it has some surprisingly profound implications that are developed more fully in Chapter 6, which deals with dimensional analysis and scaling. Chapter 2 illustrates several common pitfalls to avoid when dealing with equations, along with a few handy techniques and tricks that are used in later chapters. Chapter 3 discusses special cases and their use to check equations and to guess at the solution of difficult problems. Chapter 4 focuses on pictorial and graphical methods, and provides a basic introduction to the concept of symmetry and its use for simplifying equations. Chapter 5 discusses a variety of estimation and approximation techniques. Chapter 7 discusses how and when to generalize equations. Finally, Chapter 8 provides a few more instructive examples that illustrate several problem-solving techniques while reinforcing some of the concepts introduced in earlier chapters.

There are many interrelations and cross-references among the chapters. Generalization, which is discussed in Chapter 7, is closely related to the subject of special cases introduced in Chapter 3. Similarly, dimensional analysis and scaling (Chapter 6) are closely related to the study of units and dimensions (Chapter 1). Notice that these linked chapters are spaced apart in the book. This placement is intended to give the reader time to think about the material before it is revisited. Despite the interconnections among them, the chapters are, for the most part, self-contained and need not necessarily be read in numerical sequence. The book can be used to construct modules for a quick study of a more focused topic. Two such possible modules are:

(1) dimensional analysis and scaling: Chapters 1 and 6;
(2) special cases, approximation, and generalization: Chapters 3, 5, and 7.

We have also tried to enliven the discussion by identifying the nationality and approximate era of some of the luminaries who made major contributions to the scientific and mathematical literature that we have drawn upon, and by providing a few historical anecdotes. While this material is not necessary for dealing with the equations themselves, we hope that you find it entertaining and that it will provide some insight into how science often has progressed along an unexpected or a roundabout path.

If you have suggestions for additional material that could be included in future editions of this book, find errors or descriptions that are unclear, or wish to provide feedback of any other type, we would be happy to hear from you.

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Individually, we would like to express our gratitude.

MAB: I thank my friends and coworkers at the Mayo Clinic and also in the broader magnetic resonance imaging community. It has been a privilege to work with such outstanding people during such an exciting time. I am also indebted to my entire family and especially to my wife Rhoda Lichy for her unwavering support and patience during the writing of this book.

WAF: I am grateful to the many students at all levels with whom I have had contact during my nearly 40 years of teaching, and to my late wife Cheryl who was by my side during that journey.

LIST OF WORKED-OUT EXAMPLE PROBLEMS

* Indicates a worked-out example problem that uses somewhat more advanced mathematical techniques compared to the others.

1

EQUATIONS REPRESENTING PHYSICAL QUANTITIES

Some equations model systems or processes that occur in the real, physical world. Most of the variables that appear in these equations have dimensions, and they carry certain physical units. For example, a variable d describing distance has the dimension of length and carries a specific unit such as meters, microns, or miles. The numerical value of the variable d is given as a multiple of the unit we choose, and the specific unit is usually chosen so that the numerical values are convenient to work with.

Without a unit, the physical meaning of the numerical value associated with a dimensioned variable contains no useful information. For example, to say the distance between points A and B is "$d = 8$" is not useful for scientific and engineering purposes. We also have to specify a unit of length, such as $d = 8\,\text{in}$, $d = 8\,\text{m}$, or $d = 8$ light-years, each of which describes very different quantities in the physical world.

A number that does not carry any physical units, e.g., 1, –2.23, or π, is said to be *dimensionless*. There are some dimensionless quantities that nonetheless can carry units. One well-known example is an angle θ. Angles are dimensionless because they represent the ratio of two lengths, namely the subtended arc length on a circle divided by the radius r of that circle. The natural unit for θ is the radian. The value of the angle for one complete circular revolution is 2π radians, which follows from the fact that the corresponding arc length is the circumference of the circle, or $2\pi r$. Alternatively, the degree is a commonly

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used unit to measure angles. There are 360° in one complete circular revolution, so the conversion factor between radians and degrees is

$$1\,\text{rad} = 360°/2\pi \approx 57.3°. \tag{1.1}$$

We have a choice of which of these units to use.

At first, it might seem like keeping track of the units associated with each variable in an equation is an inconvenience, akin to carrying extra baggage. As explored in this chapter, however, the use of units in fact can help us to better understand equations that contain variables representing physical quantities. Keeping track of dimensions and units can also uncover errors and can simplify work. This theme recurs elsewhere in this book, especially in Chapter 6, where the topics of dimensional analysis and scaling are discussed.

Some units have long and interesting histories, which illustrate their importance in science, engineering, and commerce. In ancient times, balance scales were commonly used to measure weight. The unknown weight of an object was measured by counting the number of unit weights required to counterbalance it. The carob tree is grown in the Mediterranean region, and its fruit is a pod that contains multiple seeds. It was found that the weight of the carob seeds varied little from one to the next. Also, it was relatively easy to get a uniform set of seeds. The heavier or lighter seeds could be eliminated from the collection because their weight correlated well with their size.

So it became convenient to use a group of carob seeds of uniform size to counterbalance the unknown quantity on the other side of the scale. The weight of the carob seeds was also of a convenient magnitude for weighing small objects like gemstones. The relative weight of the unknown object was quite accurately expressed in terms of the equivalent number of carob seeds, and this practice became a standard for commerce. Measured in modern units, a typical carob seed has a mass of approximately 0.20 g. Today, the unit *carat* is used to measure the mass (or the equivalent weight) of gemstones. A carat is defined to be exactly 0.20 g, and its name is derived from the name of the carob tree and its seeds.

1.1 SYSTEMS OF UNITS

Many different systems of units have been devised. For most scientific and engineering work today, the preferred units are in the "SI" system. This designation comes from the French "Système International d'Unités" (International System of Units). SI units are based on quantities with the seven fundamental dimensions listed in Table 1.1. Note that three of these fundamental units, the meter, kilogram, and second, were carried over from the older MKS system of units for the quantities of length, mass, and time when the SI system was developed in 1960.

TABLE 1.1. The Seven Fundamental SI Units.

Dimension	SI Unit
Length	meter or metre (m)
Mass	kilogram (kg)
Time	second (s)
Electric current	ampere (A)
Thermodynamic temperature	kelvin (K)
Amount of substance	mole (mol)
Luminous intensity	candela (cd)

SI units have gained popularity for several reasons. First, they use prefixes (e.g., nano, milli, kilo, mega) based on powers of 10. Prefixes allow the introduction of related units that are appropriate over a wide range of scales. For example, the unit of 1 nm is equal to 10^{-9} m. The powers of 10 also make conversion relatively simple, for example, converting units of area:

$$1\,\mathrm{m}^2 = \left(1\times 10^9\,\mathrm{nm}\right)^2 = 10^{18}\,\mathrm{nm}^2. \tag{1.2}$$

In contrast, the imperial (sometimes called "British") system of units contains conversion factors that are usually not integer powers of 10. For example, to express 1 yd^2 in terms of square inches, we have to calculate 36×36:

$$1\,\mathrm{yd}^2 = (36\,\mathrm{in})^2 = 1296\,\mathrm{in}^2. \tag{1.3}$$

Another advantage of the SI system is that it contains many named, derived units such as the watt to measure power. The addition of these derived units is one of the major changes between the MKS and SI systems. The derived units in the SI system are *coherent*, that is, each one can be expressed in terms of a product of the fundamental units (or other derived units) and a numerical multiplier that is equal to 1. For example, the SI unit of electrical charge is the coulomb, and

$$\begin{aligned}
&1 \text{ coulomb} = 1 \text{ ampere} \times 1 \text{ second} = 1\ \mathrm{A\cdot s} \\
&1 \text{ coulomb} = 1 \text{ farad} \times 1 \text{ volt} = 1\ \mathrm{F\cdot V} \\
&1 \text{ coulomb} = 1 \text{ volt} \times 1 \text{ second} \div 1 \text{ ohm} = 1\ \mathrm{V\cdot s\cdot \Omega^{-1}}.
\end{aligned} \tag{1.4}$$

The derived SI unit for power is the watt, which is equal to 1 joule per second. On the other hand, a unit of power in the imperial system, 1 horsepower, equals to 550 ft·lb/s. The simple conversion factors in the SI system also make it easy to decompose all of the derived units back into integer powers of the fundamental units. For example,

$$\begin{aligned} 1\text{ newton} &= 1\,\text{m}\cdot\text{kg}\cdot\text{s}^{-2}\,(\text{force}), \\ 1\text{ watt} &= 1\,\text{m}^2\cdot\text{kg}\cdot\text{s}^{-3}\,(\text{power}), \\ 1\text{ ohm} &= 1\,\text{m}^2\cdot\text{kg}\cdot\text{s}^{-3}\cdot\text{A}^{-2}\,(\text{electrical resistance}). \end{aligned} \tag{1.5}$$

As discussed in Chapter 6, the decomposition illustrated in Equation 1.5 is particularly useful for dimensional analysis.

1.2 CONVERSION OF UNITS

Scientists and engineers are trained to work with SI units. Inevitably, however, we encounter units from other dimensional systems that require conversion back and forth to the SI system. For example, the speed of a car is commonly expressed in miles per hour or kilometers per hour, but rarely in the SI unit of meter per second. Similarly, household electrical energy usage is billed in kilowatt-hours rather than the SI unit of joules. Sometimes, the scale of the SI unit is not very convenient. For example, a kilogram is a very large unit in which to express the mass of an individual molecule, and a meter is a very short unit for interstellar distances. Rather than relying solely on the power-of-10 prefixes mentioned previously, more convenient, non-SI units like the atomic mass unit (amu or u) or light-year are sometimes used. Fortunately, the conversion of units is straightforward, as illustrated in the following example.

EXAMPLE 1.1

Convert 1 mi/h to the SI unit for speed, meter per second. There are 5280 ft per mile, 12 in per foot, and 2.54 cm per inch.

ANSWER

Unit conversion is readily accomplished with multiplication by a string of conversions factors, each of which is dimensionless and equals to 1, such as 1 = (2.54 cm)/(1.00 in) = 2.54 cm/in. We multiply together powers (or inverse powers) of the conversion factors so that all of the units "cancel," except for the desired result:

$$1\frac{\text{mi}}{\text{h}} = 1\frac{\text{mi}}{\text{h}} \times \frac{5280\,\text{ft}}{\text{mi}} \times \frac{12\,\text{in}}{\text{ft}} \times \frac{2.54\,\text{cm}}{\text{in}} \times \frac{1\,\text{m}}{100\,\text{cm}} \times \frac{1\,\text{h}}{60\,\text{min}} \times \frac{1\,\text{min}}{60\,\text{s}}. \tag{1.6}$$

Gathering the numerical terms,

$$1\,\text{mi/h} = \frac{12 \times 5280 \times 2.54}{100 \times 60 \times 60}\,\text{m/s} = 0.44704\,\text{m/s}. \tag{1.7}$$

As elementary as Example 1.1 seems, errors in unit conversion are not uncommon and can have disastrous consequences. A well-known example occurred on September 23, 1999, when an unmanned orbiting satellite approached Mars at too low an altitude and crashed into the red planet. A subsequent investigation by NASA revealed that engineers failed to properly convert the imperial system unit of force used to measure rocket thrust (the pound-force) into the SI unit force, the newton.

Another example of confusion caused by the improper conversion of units occurred over 300 years earlier in connection with Sir Isaac Newton's work on the theory of gravitation. In 1679, Newton consulted a sailor's manual to obtain numerical values that he used to check the predictions of his theory for the speed of the Moon as it orbits the Earth. That speed was known in Newton's time from the Moon's observed orbital period and its estimated distance from the Earth, which was deduced from the observation of eclipses. Newton, however, did not know that the term "mile" in the sailor's manual referred to a nautical mile, which is approximately 15% longer than the statute mile (5280 ft) with which he was familiar. This confusion led to a 15% discrepancy between his prediction for the speed of the Moon and the accepted value. Discouraged by this, Newton abandoned his correct approach and searched for an alternative theory. This detour delayed Newton's work on gravity by approximately 5 years. Eventually, of course, Newton discovered the error concerning units, and his theory of gravitation has become the basis for much of modern space flight.

1.3 DIMENSIONAL CHECKS AND THE USE OF SYMBOLIC PARAMETERS

Anytime we equate one term to another, they both must have the same dimensions for the expression to make physical sense. We cannot equate a term with the dimension of length to a term with the dimension of mass. Using the basic rules of algebra, we can extend this principle to say that whenever we add or subtract terms, they must also have the same dimensions. We say that such an expression is dimensionally consistent or dimensionally *homogeneous*. We can always add zero to or subtract zero from any equation. Whenever we do so, we will assume that the zero carries the appropriate dimensions.

If an equation is dimensionally inconsistent, we can recognize immediately that it must be flawed. The inconsistency might have arisen because the equation's construction was based on faulty principles, or because its derivation contained an algebraic error. The converse is not necessarily true. If an equation is dimensionally consistent, it does not mean that it is necessarily correct, only that it *could be* correct.

Consider the following alternative expressions both intended to describe the height y of a ball thrown in the air, as a function of time t,

$$y(t) = 3 + 2t - 4.9t^2 \tag{1.8}$$

and

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2, \quad y_0 = 3\,\mathrm{m},\, v_{0y} = 2\ \mathrm{m/s}, \text{and } g = 9.8\,\mathrm{m/s}^2, \tag{1.9}$$

where t is measured in seconds. These two expressions might appear equivalent at first glance, and Equation 1.8 is more compact. Retaining symbols as in Equation 1.9, however, has several important advantages. First, a quick, visual check confirms that we are adding and subtracting terms that all have the same units, in this case meters. Equation 1.8 is not dimensionally homogeneous unless we assume that the units are implied, i.e., in the first term, "3" really means "3 m," and similarly for the other numerical coefficients. It is easy to apply this assumption inconsistently, leading to errors. On the other hand, retaining symbols and checking units can bring our attention to a typographic or careless error. For example, the incorrect exponent can be spotted easily

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^3 \quad \text{[incorrect]}, \tag{1.10}$$

because the last term has the incorrect units of m·sec instead of meters. Common errors like these can be quite difficult to uncover when the numerical format of Equation 1.8 is used, where the dimensions of the numerical factors are implied, rather than given explicitly. The use of the symbolic format as in Equation 1.9 avoids many of these problems.

The symbolic format of Equation 1.9 also allows the acceleration and initial height and initial velocity for the trajectory of the ball to be easily extracted. The initial height is $y(t)|_{t=0} = y_0$, and differentiating $y(t)$ with respect to time yields the initial velocity and the acceleration:

$$\begin{array}{c} \left.\dfrac{dy}{dt}\right|_{t=0} = v_{0y} \\ \dfrac{d^2y}{dt^2} = -g \end{array}. \tag{1.11}$$

Equation 1.8 can also be differentiated with respect to time. With a proper choice of notation, however, symbols such as v_{0y} generally do a better job evoking the physical meaning of the parameters than the numerical values appearing in Equation 1.8. Usually, the more complicated the analytic expression, the greater the advantage of retaining the symbolic notation becomes.

If we assign numerical values to quantities appearing in equations, we also have to be careful about their units. For example, the expression (1 m + 1 cm) combines two quantities that have the dimension of length but are measured in different units. To reduce the expression correctly to 1.01 m or 101 cm (instead of "2") naturally requires proper unit conversion.

Finally, the basic expression for $y(t)$ in Equation 1.9 remains valid regardless of which system of units is chosen, which is not true for Equation 1.8. To convert the expression for $y(t)$ in Equation 1.9 into units of feet, we only need to convert the given parameters to a new set with the desired units. This is easy to do using the method illustrated in Example 1.1. Using Equation 1.9, the values of the coefficients become $y_0 = 9.84\,\text{ft}$, $v_{0y} = 6.56\,\text{ft/s}$, and $g = 32.2\,\text{ft/s}^2$.

To summarize, the symbolic format of Equation 1.9 is preferred over the numerical format of Equation 1.8 because it facilitates dimensional checks, makes no assumptions about the dimensions of the parameters, and is easier to translate between systems of units. For computational work, the symbolic format of Equation 1.9 is also preferable to the "hard-coded" format of Equation 1.8, because it provides an easier and less error-prone way to pass the values of the coefficients between program modules such as subroutines.

1.4 ARGUMENTS OF TRANSCENDENTAL FUNCTIONS

Because added or subtracted terms must have the same dimensions, the following section will show that we can infer that the arguments of many transcendental functions are dimensionless. The trigonometric functions like sine, cosine, tangent, and secant are transcendental, as are the exponential functions, which also include hyperbolic sine and hyperbolic cosine. They are distinguished from algebraic functions, which include polynomials, square roots, and other simpler functions.

These functions are often expressed in terms of an infinite series expansion. Consider the well-known series expansion for the sine function:

$$\sin(u) = u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{u^7}{5040} + \ldots. \tag{1.12}$$

Because the coefficients of 1/6, 1/120, etc., are dimensionless numbers, u cannot carry any physical units either. Suppose, for example, that u had the dimensions of power, measured in units of "watts." Because we cannot subtract a term with units of watts^3 from a term with units of watts, Equation 1.12 would not make physical sense. Therefore, the argument of the transcendental function $\sin(u)$ must always be dimensionless.

Typically, the arguments of trigonometric functions are angles. As mentioned earlier, angles are dimensionless, and their SI unit is the radian. It is important to remember that many common mathematical formulas involving trigonometric functions such as the series expansion

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} + \ldots \tag{1.13}$$

or the derivative

$$\frac{d}{d\theta}\sin\theta = \cos\theta \tag{1.14}$$

are not valid unless the angle θ is measured in radians. For example, Equation 1.12 implies that the sine of a small angle ($\ll 1$ rad) is approximately equal to the angle itself, $\sin\theta \approx \theta$. With the use of a scientific calculator, the reader can easily verify that, to five significant figures, sin(0.01 rad) = 0.01000. On the other hand, sin(0.01 °) = 0.00017453, which reflects the conversion factor ($2\pi/360 \approx 0.01745$) stated in Equation 1.1.

Next, consider an exponential function and its series expansion:

$$f(t) = e^{3t} = 1 + 3t + 4.5t^2 \ldots \tag{1.15}$$

Equation 1.15 is dimensionally consistent provided that t is a dimensionless variable. If, however, the variable t represents time measured in seconds, then Equation 1.15 does not make sense unless we assume that the units are implied, i.e., "3" really means "3 s^{-1}" and "4.5" really means "4.5 s^{-2}." The discussion following Equation 1.8 showed how this type of assumption can lead to problems. Instead, it is better to write

$$f(t) = e^{\lambda t} = 1 + \lambda t + \frac{(\lambda t)^2}{2} + \ldots \tag{1.16}$$

with $\lambda = 3s^{-1}$. The argument of the transcendental function in Equation 1.16 is now the product λt, which is dimensionless, as is the third term in the expansion, ½$(\lambda t)^2$.

Among the transcendental functions, the logarithm provides an interesting special case. Consider $\log(x/a)$, where the ratio (x/a) is dimensionless. For example, both x and a might have the dimension of length, measured in units of meters. Suppose that x = 3 m and a = 2 m. Logarithms reduce the operation of division to subtraction, i.e.,

$$\log\left(\frac{x}{a}\right) = \log\left(\frac{3\,\mathrm{m}}{2\,\mathrm{m}}\right) = \log\left(\frac{3}{2}\right) = \log(3) - \log(2). \tag{1.17}$$

All of the operations in Equation 1.17 are valid, and note that Equation 1.17 holds regardless of the logarithm's base, e.g., 10, 2, or *e*.

We might encounter a symbolic expression containing a term log(x), where x is not dimensionless. We cannot immediately conclude that the *entire* expression is incorrect. The expression might also include another term of the form –log(a), or equivalently +log(1/a), where a has the same units as x. Then, even though the variable x appearing in log(x) is dimensioned, the entire expression can be correct.

1.5 DIMENSIONAL CHECKS TO GENERALIZE EQUATIONS

The use of dimensional checks allows us to generalize equations and even generate new ones. Suppose we use integration by parts or a table of integrals and find

$$F(x) = \int xe^{x}dx = e^{x}(x-1) + C, \tag{1.18}$$

where C is a constant. We know that the variable x in Equation 1.18 must be dimensionless, because it is the argument of the exponential function. Using dimensional checks, we can generalize Equation 1.18 to evaluate

$$G(x,a) = \int xe^{ax}dx \tag{1.19}$$

without the need for additional integration. The product (ax) must be dimensionless because it appears as the argument of the exponential Equation 1.19. We are free to assume that the variable x has dimensions of length, measured in the unit of meters, and we will do so. In that case, a must have units of m^{-1}. So the task is to use dimensional checks to place the appropriate power of a into each term of the right-hand side of Equation 1.18. Recognizing that each of the terms of $G(x,a)$ must have units of meters squared (because the differential dx carries the same units as x, see exercise 1.12), we can infer that

$$G(x,a) = \int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right) + C'. \tag{1.20}$$

Along with the dimensional check, we also used the fact that $G(x, a)$ reduces to $F(x)$ when $a = 1\,\mathrm{m}^{-1}$. Naturally, for a further check, we can differentiate the right-hand side of Equation 1.20. Although we do not know the values of the integration constants C and C', we do know their units. C is dimensionless, and C' carries the same units as x/a, namely meters squared.

Equation 1.20, which was generated with the aid of a dimensional check, can be extended to evaluate related integrals, up to the additive constant (see exercises 1.2 and 1.13). The partial derivative of $G(x, a)$ with respect to a yields

$$\begin{aligned}\frac{\partial G(x,a)}{\partial a} &= \int x\left(\frac{\partial e^{ax}}{\partial a}\right)dx = \int x^2 e^{ax}\,dx \\ &= \frac{\partial}{\partial a}\left[\frac{e^{ax}}{a}\left(x-\frac{1}{a}\right)+C'\right] = \frac{e^{ax}}{a}\left(x^2-\frac{2x}{a}+\frac{2}{a^2}\right)\end{aligned} \tag{1.21}$$

Equation 1.21 implies that

$$\int x^2 e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^2-\frac{2x}{a}+\frac{2}{a^2}\right)+C''. \tag{1.22}$$

Starting with Equation 1.18, the use of dimensional checks followed by partial differentiation yielded a new expression given in Equation 1.22. This sequence of operations can be quite useful for evaluating a variety of indefinite and definite integrals, but we always have to be very careful to follow the rules of calculus. In the example of Equation 1.21, it would *not* be valid to differentiate with respect to x, because x is the integration variable.

1.6 OTHER TYPES OF UNITS

Up to this point, Chapter 1 has focused on variables that carry physical units that describe length, mass, electrical charge, etc. This allowed us to draw useful inferences about equations composed of these variables. There are many other types of units, however, that are not associated with the physical sciences. One example is a monetary unit, like a dollar or a euro. Another example is a unit like the number of soldiers per battalion, which might be used in a logistical calculation to find the required amount of rations for a month.

Although units like \$ or battalion^{-1} are not part of the SI system, they are often very convenient (for example, see exercise 1.14). Equations containing variables that carry these types of units still must be dimensionally homogeneous, provided that the system of units is applied in a consistent manner. Thus, the dimensional checks and unit conversion methods introduced previously in this chapter apply.

Some equations seem to model the physical world quite well but appear to be dimensionally inconsistent. For example, we might find that the time t it takes to finish a task in the office is well described by the equation "t = 20 min + three times the number of phone call interruptions received." This equation seems to be dimensionally inconsistent, because an apparently dimensionless quantity (three times the number of phone calls) is added to a dimensioned quantity (20 min). Whenever this type of expression accurately models the physical world, however, there is an implied conversion factor. In this case, the implied conversion factor is 3 min/phone call.