



# Advanced Ceramic Coatings and Interfaces III

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*A Collection of Papers Presented at the  
32nd International Conference on Advanced  
Ceramics and Composites  
January 27–February 1, 2008  
Daytona Beach, Florida*

Editors

Hua-Tay Lin

Dongming Zhu

Volume Editors

Tatsuki Ohji

Andrew Wereszczak



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# Preface

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The symposium on Advanced Ceramic Coatings for Structural, Environmental and Functional Applications was held during The American Ceramic Society's 32nd International Conference on Advanced Ceramics and Composites in Daytona Beach, Florida, January 28 to Feb 1, 2008. A total of 69 papers, including 8 invited talks, were presented at the symposium, covering broad ceramic coating and interface topic areas and emphasizing the latest advancement in coating processing, characterization, development, and applications.

The present volume contains fourteen contributed papers from the symposium, with topics including damping and erosion coatings, wear and tribological coatings, nanostructured coatings, and thermal barrier coating processing, development, modeling and life prediction, which highlights the state-of-the-art ceramic coatings technologies for various critical engineering applications.

We are greatly indebted to the members of the symposium organizing committee, including Drs. Yutaka Kagawa, Anette Karlsson, Irene Spitsberg, Dileep Singh, Yong-Ho Sohn, Xingbo Liu, Uwe Schulz, Robert Vaßen, and Jennifer Sample, for their assistance in developing and organizing this vibrant and cutting-edge symposium. We also would like to express our sincere thanks to manuscript authors and reviewers, all the symposium participants and session chairs for their contributions to a successful meeting. Finally, we are also grateful to the staff of The American Ceramic Society for their time and effort in ensuring an enjoyable conference and the high-quality publication of the proceeding volume.

H. T. Lin  
*Oak Ridge National Laboratory*

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# Introduction

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Organized by the Engineering Ceramics Division (ECD) in conjunction with the Basic Science Division (BSD) of The American Ceramic Society (ACerS), the 32nd International Conference on Advanced Ceramics and Composites (ICACC) was held on January 27 to February 1, 2008, in Daytona Beach, Florida. 2008 was the second year that the meeting venue changed from Cocoa Beach, where ICACC was originated in January 1977 and was fostered to establish a meeting that is today the most preeminent international conference on advanced ceramics and composites

The 32nd ICACC hosted 1,247 attendees from 40 countries and 724 presentations on topics ranging from ceramic nanomaterials to structural reliability of ceramic components, demonstrating the linkage between materials science developments at the atomic level and macro level structural applications. The conference was organized into the following symposia and focused sessions:

Symposium 1	Mechanical Behavior and Structural Design of Monolithic and Composite Ceramics
Symposium 2	Advanced Ceramic Coatings for Structural, Environmental, and Functional Applications
Symposium 3	5th International Symposium on Solid Oxide Fuel Cells (SOFC): Materials, Science, and Technology
Symposium 4	Ceramic Armor
Symposium 5	Next Generation Bioceramics
Symposium 6	2nd International Symposium on Thermoelectric Materials for Power Conversion Applications
Symposium 7	2nd International Symposium on Nanostructured Materials and Nanotechnology: Development and Applications
Symposium 8	Advanced Processing & Manufacturing Technologies for Structural & Multifunctional Materials and Systems (APMT): An International Symposium in Honor of Prof. Yoshinari Miyamoto
Symposium 9	Porous Ceramics: Novel Developments and Applications

Symposium 10	Basic Science of Multifunctional Ceramics
Symposium 11	Science of Ceramic Interfaces: An International Symposium Memorializing Dr. Rowland M. Cannon
Focused Session 1	Geopolymers
Focused Session 2	Materials for Solid State Lighting

Peer reviewed papers were divided into nine issues of the 2008 Ceramic Engineering & Science Proceedings (CESP); Volume 29, Issues 2-10, as outlined below:

- Mechanical Properties and Processing of Ceramic Binary, Ternary and Composite Systems, Vol. 29, Is 2 (includes papers from symposium 1)
- Corrosion, Wear, Fatigue, and Reliability of Ceramics, Vol. 29, Is 3 (includes papers from symposium 1)
- Advanced Ceramic Coatings and Interfaces III, Vol. 29, Is 4 (includes papers from symposium 2)
- Advances in Solid Oxide Fuel Cells IV, Vol. 29, Is 5 (includes papers from symposium 3)
- Advances in Ceramic Armor IV, Vol. 29, Is 6 (includes papers from symposium 4)
- Advances in Bioceramics and Porous Ceramics, Vol. 29, Is 7 (includes papers from symposia 5 and 9)
- Nanostructured Materials and Nanotechnology II, Vol. 29, Is 8 (includes papers from symposium 7)
- Advanced Processing and Manufacturing Technologies for Structural and Multifunctional Materials II, Vol. 29, Is 9 (includes papers from symposium 8)
- Developments in Strategic Materials, Vol. 29, Is 10 (includes papers from symposia 6, 10, and 11, and focused sessions 1 and 2)

The organization of the Daytona Beach meeting and the publication of these proceedings were possible thanks to the professional staff of ACerS and the tireless dedication of many ECD and BSD members. We would especially like to express our sincere thanks to the symposia organizers, session chairs, presenters and conference attendees, for their efforts and enthusiastic participation in the vibrant and cutting-edge conference.

ACerS and the ECD invite you to attend the 33rd International Conference on Advanced Ceramics and Composites (<http://www.ceramics.org/daytona2009>) January 18–23, 2009 in Daytona Beach, Florida.

TATSUKI OHJI and ANDREW A. WERESZCZAK, Volume Editors  
*July 2008*

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# Damping and Erosion Coatings

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## COATINGS FOR ENHANCED PASSIVE DAMPING

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### ABSTRACT

The amplitude of vibration in a structure undergoing resonant vibration is governed by the total damping of the system. As the inherent damping of materials suitable for use in the fabrication of structures and machine components is often quite low, increasing the total system damping by including a dissipative material or mechanism can often provide significant reductions in the peak values of response (stress, strain, and displacement), enabling more efficient designs and enhanced performance. Available methodologies include active dampers and such passive techniques as friction and impact dampers and constrained layer treatments. It has also been found that metals and ceramics applied as free-layer hard coatings by plasma spray or electron beam physical vapor deposition add significant damping to vibrating members. In order to incorporate the influence of a damping coating in a prediction of system response during a preliminary design, it is essential that properties of the coating be known. The relevant damping characteristic is a measure of the energy dissipated by a homogeneous unit volume of material undergoing a completely reversed cycle of oscillation. A useful metric for this is the loss modulus. As all of these materials are inherently non-linear, as evidenced by amplitude-dependent measures of damping, determinations of properties must be made at the levels of strain appropriate to the application. Methods for determining the damping properties of materials are discussed, and comparisons are made of the damping of various classes of materials with those of ceramic coatings deposited by plasma spray or electron beam physical vapor deposition.

### I. THE NEED FOR DAMPING

Although the assumption of a perfectly elastic material is very convenient for use in the analysis of structures, and adequate in most cases, no structural materials are truly elastic. A system given an initial perturbation will eventually come to rest unless the dissipation is offset by the addition of energy. Cyclic motion of a structure can be sustained at constant amplitude only if the energy lost through dissipation is offset by work done on the system. Energy dissipation, or damping, can be advantageous to the performance of a system as it governs the maximum amplitude achieved under resonance and the rate at which a perturbed system progresses to a satisfactorily quiescent state. In addition to lowering the probability of failure due to fatigue, a reduction of amplitude can have other benefits such as reducing a visible vibration, reducing the sound transmitted from a valve cover, or reducing the signature of a vibrating submarine propeller. Damping, however, can also be disadvantageous, contributing as it does to such unwanted phenomena as shaft whirl, instrument hysteresis, and temperature increases due to self-heating.

The term damping (not dampening) refers to the dissipation of energy in a material or structure under cyclic stress or strain through a process of converting mechanical energy (strain and kinetic) to heat. When such dissipation occurs locally within the material, the process is referred to as material damping, taken as inclusive of the dissipative mechanisms variously referred to as mechanical hysteresis, anelasticity, or internal friction.

The distinction between material damping and system damping should be observed. Material damping, the inherent ability of a substance to dissipate energy under cyclic stress or strain, is a material property and can be expressed in absolute units of energy dissipated per unit volume per cycle or as a dimensionless measure formed from a ratio of the energy dissipated to the energy stored. System damping, on the other hand, is a measure of the influence of the total dissipation of all the components of the system on the overall response of a structure or structural component.

At the present time, there is a particularly high level of interest in reducing the amplitude of vibration of the blades in gas turbine engines. As each blade in a rotating component passes a blade (stator) in the static component, a pressure pulse is generated. As the blades rotate at a high cyclic rate (~10,000 RPM) and there may well be several dozen stators, excitation rates of several thousand impulses per second are common. As a response at around 3K Hz gives rise to one million cycles of vibration in only 10 engine hours, and as the maximum velocity of structural motion is proportional to maximum strain, it is evident that excitation at a resonant frequency can lead to failure of the blade by fatigue. This phenomenon is known as high cycle fatigue (HCF) and is to be distinguished from the blade damage known as low cycle fatigue (LCF) that results from the much less frequently occurring perturbations due to major changes in engine RPM and temperature.

As it is the amount of damping present in the system that governs the magnitude of the response at resonance, it would be highly advantageous to be able to apply a passive damping treatment to rotating turbine blades. However, a successful treatment must survive not only the high vibratory stresses for the service life of the engine but also the environmental challenges of high temperature, erosion, corrosion, foreign object damage, and centrifugal forces due to the high rotation rates. Further, a successful treatment must not degrade the performance of the engine by adding excessive weight, by detrimental changes in the shapes of aerodynamic surfaces, or by inducing cracks in the blade material.

A number of concepts for the reduction of vibratory response are being explored. These include the use of friction dampers, impact dampers, energy absorbers, and the inclusion of dissipative materials in constrained layer damping treatments. Of a special interest is the use of free layer coatings in the form of thin layers of high damping metals and ceramics applied through plasma spray and electron beam physical vapor deposition. The use of such ceramics as alumina, magnesium aluminate spinel, and yttria stabilized zirconia appear to have received the greatest attention.

## II. DAMPING AS A MATERIAL PROPERTY

### A. Measures of Material Damping

The most fundamental measure of the dissipative capability of a material is the specific damping energy or unit damping, defined as the energy dissipated in a unit volume of material at homogeneous strain and temperature while undergoing a fully reversed cycle of cyclic stress or strain. The specific damping energy,  $D$ , has dimension of energy per unit volume, per cycle, and is, in general, a function of the amplitude and history of stress or strain, temperature, and frequency. For some materials, the unit damping is also dependent on the mean (static) stress or is influenced by magnetic fields. The unit damping is customarily given in terms of the amplitude of a uniaxial tensile or shear stress or strain. Multiaxial states of stress are characterized by an equivalent uniaxial stress<sup>1</sup>.

Material damping may be categorized as being linear or non-linear. In the first class, the energy dissipated per cycle is dependent on the square of the amplitude of cyclic stress or strain. As the strain energy density is also normally proportional to the square of amplitude, the ratio of dissipated and stored energies, as well as other dimensionless measures of damping, are then independent of amplitude. Materials displaying these attributes are said to display linear damping. In the second class of materials, the energy dissipated per cycle varies as amplitude of cyclic stress to some power other than two. If the strain energy density varies as, or nearly as, the square of amplitude, the ratio of dissipated to stored energy is then a function of the amplitude of stress or strain. Such materials are said to display nonlinear damping.

While some important mechanisms of damping, such as viscoelastic and thermoelastic, are essentially linear, many others are not. In the case of structural materials for which the predominant damping mechanism is plastic deformation on a scale which leaves the material macroscopically linear, the specific damping energy has been found<sup>1</sup> to depend on the amplitude of cyclic stress as



$$D = J\sigma_d^n \quad (1)$$

with  $n \approx 2.4$  for cyclic stresses below about 70% of the endurance limit. For metals, the parameters  $J$  and  $n$  are generally independent of frequency, but dependent on temperature. At higher stress, the same functional form may be applied, but the damping typically increases more rapidly with stress. The parameter  $n$  is then much greater and may increase or decrease with the number of cycles.

While rooted in the concept of a linear viscoelastic material, the concept of the complex modulus may be adapted to characterize the dissipation of other materials undergoing cyclic loading. In the case of a nonlinear material, we may define amplitude-dependent effective values of a storage and a loss modulus,  $E_1$  and  $E_2$ , by

$$E_1(\omega, T, \varepsilon_d) \equiv \frac{2U(\omega, T, \varepsilon_d)}{\varepsilon_d^2} \quad \text{and} \quad E_2(\omega, T, \varepsilon_d) \equiv \frac{D(\omega, T, \varepsilon_d)}{\pi \varepsilon_d^2} \quad (2a, b)$$

where  $U$  is the stored (strain) energy in the unit volume,  $D$  is the specific damping energy, and  $\varepsilon_d$  is the amplitude of cyclic strain. For structural materials, the values of storage and loss modulus are typically independent of frequency, but vary with amplitude and temperature. In the case of viscoelastic materials, the moduli are typically independent of amplitude, but vary strongly with both frequency and temperature. A material loss factor may also be defined as the ratio of energy dissipated in the unit volume per radian of oscillation to the peak energy stored.

$$\eta \equiv \frac{D}{2\pi U} = \frac{1}{2\pi} \frac{\pi E_2(\omega, T, \varepsilon_d)}{E_1(\omega, T, \varepsilon_d)/2} = \frac{E_2(\omega, T, \varepsilon_d)}{E_1(\omega, T, \varepsilon_d)} \quad (3)$$

If either or both of the components of the modulus are dependent on amplitude, then the material loss factor is also dependent on amplitude. Note that the use of the loss factor, defined in terms of energy dissipated per radian, is to be preferred over the sometimes-used specific damping capacity or damping index, computed from the energy dissipated per cycle by  $\Psi = D/U$ . This is because the unit of the radian is more truly a dimensionless quantity than is the cycle.

The complex modulus is defined as the ratio of the Fourier transform of stress to that of strain,

$$E^* = \sigma^*(\omega) / \varepsilon^*(\omega) = E_1 + jE_2 = E_1(1 + j\eta) \quad (4)$$

and is particularly convenient for analyzing the response of time-dependent materials to sinusoidal forcing functions. The storage modulus,  $E_1$ , is the customary Young's or elastic modulus; the loss modulus,  $E_2$ , is the product of the storage modulus and the loss factor. The use of a complex quantity to describe a real, physical material is troubling to some. The proper interpretation is not that the material behaves in a mathematically complex manner, but rather that the presence of dissipation is associated with an out-of-phase relationship between stress and strain. The loss factor quantifies the degree to which they are out of phase. The angle by which the strain lags the stress is given by  $\tan \phi = E_2(\omega, T, \varepsilon_d) / E_1(\omega, T, \varepsilon_d)$  and is referred to as the loss tangent.

## B. High Damping Materials

### 1. HIDAMETS

The specific damping energy of most structural materials falls in a fairly narrow range, particularly at stresses of design interest, i.e. well below the endurance limit. The mean curve for a wide range<sup>1</sup> of structural metals may be represented by  $D = 14(\sigma_d / \sigma_F)^{2.4}$  Joule/m<sup>3</sup> for applied stresses,  $\sigma_d$ , up to about 70% of the endurance limit,  $\sigma_F$ . There are, however, a few exceptions, as such magnetoelastic materials as 403 alloy and the alloys of nickel and cobalt; alloys of manganese and copper; and the shape memory (SM) alloys. The high damping of magnetoelastic materials arises because the application of stress induces rotations of the magnetic domains similar to that produced by the application of a magnetic field, a process that dissipates energy. The high damping of Mn-Cu and the shape memory alloys is associated with transitions from face centered cubic to face centered tetragonal structures. Attempts to exploit these mechanisms so as to obtain high inherent material damping while meeting the other requirements of a material for primary structures have not, to date, been successful. These, and other high damping metals, may yet find application, perhaps as coatings.

2. Viscoelastic Materials

Another class of materials finding extensive application in structural additions for enhanced damping includes the elastomers and polymers. These are characterized by stress-strain relationships incorporating rate dependence. In the classical form, the constitutive relationship is taken to be a linear operator on stress proportional to a linear operator on strain. While this provides an adequate description of observed material properties, it has been found that many terms (necessitating many experimentally determined parameters) are required. More recently, it has been shown<sup>2</sup> that modifying the form by replacing the integer derivatives in the linear operators with fractional derivatives enables an excellent characterization over as many as 11 orders of magnitude in frequency. A characteristic of viscoelastic materials is that, in addition to having a strong dependence on frequency, they display a strong dependence on temperature. Fortunately, these are related. The concept of the reduced frequency has been accepted as a means of jointly describing both effects. A complete discussion is available elsewhere.<sup>3</sup>

As the ability to dissipate energy is dependent on the loss modulus, Eq. 2b, the loss modulus is an appropriate metric for identifying high damping materials. The solid line of Figure 1 represents a constant value of loss modulus (product of loss factor and storage modulus), with 1 GPa used as a reference value. Structural metals are seen to generally fall below this line; materials considered for treatments to enhance damping, such as the high damping alloys and magnetoelastic materials, fall on or above this line.

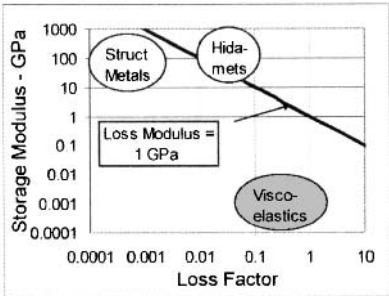


Figure 1. Categories of Damping Materials

Dissipation depends not only on the loss modulus, but also on the square of strain. As many of the viscoelastic materials can safely withstand repetitive strain on the order of unity, they most truly earn their reputation as high damping materials when used in a damping treatment with configuration designed so as induce strains in the viscoelastic material that are 2-3 orders of magnitude more than that of the base structure.

### III. DAMPING AS A SYSTEM PROPERTY

The damping of systems containing one or more dissipating elements is of interest for two reasons. In order to incorporate damping into predictions of the system performance during the preliminary design process, a mathematical model of the system, including damping, is necessary. Secondly, as it is typically very difficult to perform materials testing on samples of a homogeneous material in a homogenous state of strain, as is necessary for the determination of a true material property, material testing is typically done on a system containing the sample with material properties then extracted from the system response.

Although continuous systems have an infinity of vibratory modes, unless the system is highly nonlinear or the modes very closely spaced, the response near a resonant or natural frequency is quite well represented by the single mode dominant at that frequency. And, although the systems of interest may not be truly linear nor is the damping likely to be viscous, we base the analysis of system damping on the well-known responses of the classical one-degree-of-freedom system consisting of a mass, a linear spring, and a viscous damping element.

Our objective is to relate observable measures of system response to the damping present in the system and then to use, with caution, these same measures in the characterization of systems for which the origin of damping is of a more general nature.

#### A. The Prototype Damped System

The familiar linear oscillator with linear spring, linear Newtonian dashpot, and mass has a displacement  $x(t)$  in response to a time dependent force  $F(t)$  that is given by:

$$M \frac{d^2 x(t)}{dt^2} + C \frac{dx(t)}{dt} + Kx(t) = F(t) \quad (5)$$

$M$ ,  $C$ , and  $K$  are presumed to be real constants, independent of frequency or amplitude. The solution may be written in terms of two parameters: a natural frequency,  $\omega_n = \sqrt{K/M}$ , and a fraction of critical damping (damping ratio),  $\xi = C/(2\sqrt{KM})$ . The homogeneous solution (free vibration)

$$x(t) = C \exp(-\xi\omega_n t) \cos(\omega_d t - \phi_0) \quad (6)$$

is a decaying sinusoid, shown in Figure 2a for several values of the fraction of critical damping.

The frequency response function, i.e., the magnitude of the response to a harmonic excitation at frequency  $\Omega$ , is shown in Figure 2b. The displacement is normalized by the response in the limit as forcing frequency goes to zero (static displacement). Values for three levels of damping ratio,  $\xi = 1\%$ ,  $2\%$ , and  $5\%$ , are given as functions of a dimensionless frequency ratio,  $f = \Omega/\omega_n$ .

$$R = \frac{x(t)}{x_s} = \frac{1}{\sqrt{(1-f^2)^2 + 4\xi^2 f^2}} \quad (7)$$