



# Fundamentals of Acoustics

Michel Bruneau

Thomas Scelo  
Translator and Contributor

*Series Editor*  
*Société Française d'Acoustique*

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## Preface

The need for an English edition of these lectures has provided the original author, Michel Bruneau, with the opportunity to complete the text with the contribution of the translator, Thomas Scelo.

This book is intended for researchers, engineers, and, more generally, postgraduate readers in any subject pertaining to “physics” in the wider sense of the term. It aims to provide the basic knowledge necessary to study scientific and technical literature in the field of acoustics, while at the same time presenting the wider applications of interest in acoustic engineering. The design of the book is such that it should be reasonably easy to understand without the need to refer to other works. On the whole, the contents are restricted to acoustics in fluid media, and the methods presented are mainly of an analytical nature. Nevertheless, some other topics are developed succinctly, one example being that whereas numerical methods for resolution of integral equations and propagation in condensed matter are not covered, integral equations (and some associated complex but limiting expressions), notions of stress and strain, and propagation in thick solid walls are discussed briefly, which should prove to be a considerable help for the study of those fields not covered extensively in this book.

The main theme of the 11 chapters of the book is acoustic propagation in fluid media, dissipative or non-dissipative, homogeneous or non-homogeneous, infinite or limited, etc., the emphasis being on the “theoretical” formulation of problems treated, rather than on their practical aspects. From the very first chapter, the basic equations are presented in a general manner as they take into account the nonlinearities related to amplitudes and media, the mean-flow effects of the fluid and its inhomogeneities. However, the presentation is such that the factors that translate these effects are not developed in detail at the beginning of the book, thus allowing the reader to continue without being hindered by the need for in-depth understanding of all these factors from the outset. Thus, with the exception of

Chapter 10 which is given over to this problem and a few specific sections (diffusion on inhomogeneities, slowly varying media) to be found elsewhere in the book, developments are mainly concerned with linear problems, in homogeneous media which are initially at rest and most often dissipative.

These dissipative effects of the fluid, and more generally the effects related to viscosity, thermal conduction and molecular relaxation, are introduced in the fundamental equations of movement, the equations of propagation and the boundary conditions, starting in the second chapter, which is addressed entirely to this question. The richness and complexity of the phenomena resulting from the taking into account of these factors are illustrated in Chapter 3, in the form of 13 related “exercises”, all of which are concerned with the fundamental problems of acoustics. The text goes into greater depth than merely discussing the dissipative effects on acoustic pressure; it continues on to shear and entropic waves coupled with acoustic movement by viscosity and thermal conduction, and, more particularly, on the use that can be made of phenomena that develop in the associated boundary layers in the fields of thermo-acoustics, acoustic gyrometry, guided waves and acoustic cavities, etc.

Following these three chapters there is coverage (Chapters 4 and 5) of fundamental solutions for differential equation systems for linear acoustics in homogenous dissipative fluid at rest: classic problems are both presented and solved in the three basic coordinate systems (Cartesian, cylindrical and spherical). At the end of Chapter 4, there is a digression on boundary-value problems, which are widely used in solving problems of acoustics in closed or unlimited domain.

The presentation continues (Chapter 6) with the integral formulation of problems of linear acoustics, a major part of which is devoted to the Green’s function (previously introduced in Chapters 3 and 5). Thus, Chapter 6 constitutes a turning point in the book insofar as the end of this chapter and through Chapters 7 to 9, this formulation is extensively used to present several important classic acoustics problems, namely: radiation, resonators, diffusion, diffraction, geometrical approximation (rays theory), transmission loss and structural/acoustic coupling, and closed domains (cavities and rooms).

Chapter 10 aims to provide the reader with a greater understanding of notions that are included in the basic equations presented in Chapters 1 and 2, those which concern non-linear acoustics, fluid with mean flow and aero-acoustics, and can therefore be studied directly after the first two chapters.

Finally, the last chapter is given over to modeling of the strong coupling in acoustics, emphasizing the coupling between electro-acoustic transducers and the acoustic field in their vicinity, as an application of part of the results presented earlier in the book.



## Chapter 1

# Equations of Motion in Non-dissipative Fluid

The objective of the two first chapters of this book is to present the fundamental equations of acoustics in fluids resulting from the thermodynamics of continuous media, stressing the fact that thermal and mechanical effects in compressible fluids are absolutely indissociable.

This chapter presents the fundamental phenomena and the partial differential equations of motion in non-dissipative fluids (viscosity and thermal conduction are introduced in Chapter 2). These equations are widely applicable as they can deal with non-linear motions and media, non-homogeneities, flows and various types of acoustic sources. Phenomena such as cavitation and chemical reactions induced by acoustic waves are not considered.

Chapter 2 completes the presentation by introducing the basic phenomenon of dissipation associated to viscosity, thermal conduction and even molecular relaxation.

### **1.1. Introduction**

The first paragraph presents, in no particular order, some fundamental notions of thermodynamics.

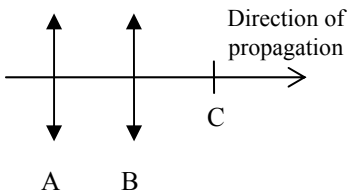
#### **1.1.1. *Basic elements***

The domain of physics acoustics is simply part of the fast science of thermomechanics of continuous media. To ensure acoustic transmission, three fundamental elements are required: one or several emitters or sources, one receiver

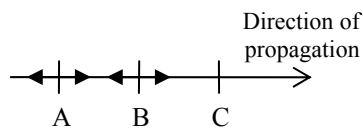
and a propagation medium. The principle of transmission is based on the existence of “particles” whose position at equilibrium can be modified. All displacements related to any types of excitation other than those related to the transmitted quantity are generally not considered (i.e. the motion associated to Brownian noise in gases).

### 1.1.2. Mechanisms of transmission

The waves can either be transverse or longitudinal (the displacement of the particle is respectively perpendicular or parallel to the direction of propagation). The fundamental mechanisms of wave transmission can be qualitatively simplified as follows. A particle B, adjacent to a particle A set in a time-dependent motion, is driven, with little delay, via the bonding forces; the particle A is then acting as a source for the particle B, which acts as a source for the adjacent particle C and so on (Figure 1.1).



**Figure 1.1.** *Transverse wave*



**Figure 1.2.** *Longitudinal wave*

The double bolt arrows represent the displacement of the particles.

In solids, acoustic waves are always composed of a longitudinal and a transverse component, for any given type of excitation. These phenomena depend on the type of bonds existing between the particles.

In liquids, the two types of wave always coexist even though the longitudinal vibrations are dominant.

In gases, the transverse vibrations are practically negligible even though their effects can still be observed when viscosity is considered, and particularly near walls limiting the considered space.

### 1.1.3. Acoustic motion and driving motion

The motion of a particle is not necessarily induced by an acoustic motion (audible sound or not). Generally, two motions are superposed: one is qualified as acoustic (A) and the other one is “anacoustic” and qualified as “driving” (E); therefore, if  $g$  defines an entity associated to the propagation phenomenon (pressure, displacement, velocity, temperature, entropy, density, etc.), it can be written as

$$g(x, t) = g_{(A)}(x, t) + g_{(E)}(x, t).$$

This field characteristic is also applicable to all sources. A fluid is said to be at rest if its driving velocity is null for all particles.

### 1.1.4. Notion of frequency

The notion of frequency is essential in acoustics; it is related to the repetition of a motion which is not necessarily sinusoidal (even if sinusoidal dependence is very important given its numerous characteristics). The sound-wave characteristics related to the frequency (in air) are given in Figure 1.3. According to the sound level, given on the dB scale (see definition in the forthcoming paragraph), the “areas” covered by music and voice are contained within the audible area.

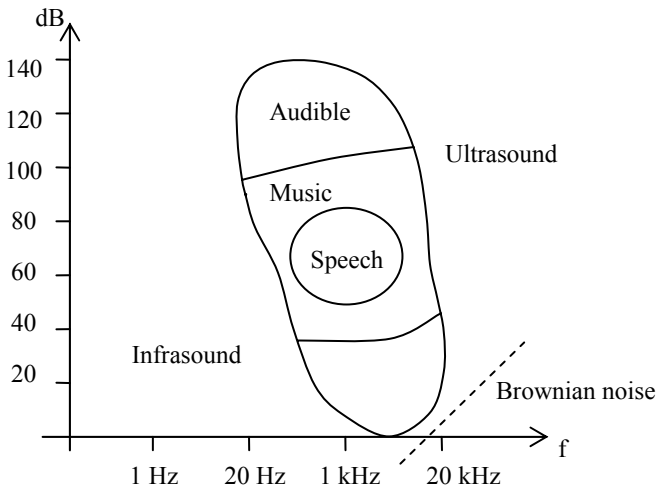


Figure 1.3. The sounds

### 1.1.5. Acoustic amplitude and intensity

The magnitude of an acoustic wave is usually expressed in decibels, which are unit based on the assumption that the ear approximately satisfies Weber-Fechner law, according to which the sense of audition is proportional to the logarithm of the intensity ( $I$ ) (the notion of intensity is described in detail at the end of this chapter). The level in decibel (dB) is then defined as follows:

$$L_{\text{dB}} = 10 \log_{10} I / I_r,$$

where  $I_r = 10^{-12} \text{ W/m}^2$  represents the intensity corresponding to the threshold of perception in the frequency domain where the ear sensitivity is maximum (approximately 1 kHz).

Assuming the intensity  $I$  is proportional to the square of the acoustic pressure (this point is discussed several times here), the level in dB can also be written as

$$L_{\text{dB}} = 20 \log_{10} p / p_r,$$

where  $p$  defines the magnitude of the pressure variation (called acoustic pressure) with respect to the static pressure (without acoustic perturbation) and where  $p_r = 2 \cdot 10^{-5} \text{ Pa}$  defines the value of this magnitude at the threshold of audibility around 1,000 Hz.

The origin 0 dB corresponds to the threshold of audibility; the threshold of pain, reached at about 120–140 dB, corresponds to an acoustic pressure equal to 20–200 Pa. The atmospheric pressure (static) in normal conditions is equal to  $1.013 \cdot 10^5 \text{ Pa}$  and is often written 1013 mbar or  $1.013 \cdot 10^6 \text{ } \mu\text{bar}$  (or baryes or  $\text{dyne/cm}^2$ ) or even 760 mm Hg.

The magnitude of an acoustic wave can also be given using other quantities, such as the particle displacement  $\xi$  or the particle velocity  $\vec{v}$ . A harmonic plane wave propagating in the air along an axis  $x$  under normal conditions of temperature (22°C) and of pressure can indifferently be represented by one of the following three variations of particle quantities

$$\begin{aligned} \xi &= \xi_0 \sin(\omega t - kx), \\ v &= \omega \xi_0 \sin(\omega t - kx), \\ p &= p_0 \sin(\omega t - kx), \end{aligned}$$

where  $p_0 = \rho_0 c_0 \omega \xi_0$ ,  $\rho_0$  defining the density of the fluid and  $c_0$  the speed of sound (these relations are demonstrated later on). For the air, in normal conditions of pressure and temperature,

$$\begin{aligned}c_0 &\cong 344.8 \text{ m s}^{-1}, \\ \rho_0 &\cong 1.2 \text{ kg m}^{-3}, \\ \rho_0 \cdot c_0 &\cong 400 \text{ kg m}^{-3} \text{ s}^{-1}\end{aligned}$$

At the threshold of audibility (0 dB), for a given frequency ( $N$ ) close to 1 kHz, the magnitudes are

$$\begin{aligned}p_0 &= 2 \cdot 10^{-5} \text{ Pa}, \\ v_0 &= \frac{p}{\rho_0 c_0} \cong 5 \cdot 10^{-8} \text{ ms}^{-1}, \\ \xi_0 &= \frac{v_0}{2\pi N} \cong 10^{-11} \text{ m}.\end{aligned}$$

It is worth noting that the magnitude  $\xi_0$  is 10 times smaller than the atomic radius of Bohr and only 10 times greater than the magnitude of the Brownian motion (which associated sound level is therefore equal to -20 dB, inaudible).

The magnitudes at the threshold of pain (at about 120 dB at 1 kHz) are

$$\begin{aligned}p_0 &= 20 \text{ Pa}, \\ v_0 &\cong 5 \cdot 10^{-2} \text{ ms}^{-1}, \\ \xi_0 &\cong 10^{-5} \text{ m}.\end{aligned}$$

These values are relevant as they justify the equations' linearization processes and therefore allow a first order expansion of the magnitude associated to acoustic motions.

### 1.1.6. *Viscous and thermal phenomena*

The mechanism of damping of a sound wave in "simple" media, homogeneous fluids that are not under any particular conditions (such as cavitation), results generally from two, sometimes three, processes related to viscosity, thermal conduction and molecular relaxation. These processes are introduced very briefly in this paragraph; they are not considered in this chapter, but are detailed in the next one.

When two adjacent layers of fluid are animated with different speeds, the viscosity generates reaction forces between these two layers that tend to oppose the displacements and are responsible for the damping of the waves. If case dissipation is negligible, these viscous phenomena are not considered.

When the pressure of a gas is modified, by forced variation of volume, the temperature of the gas varies in the same direction and sign as the pressure (Lechatelier's law). For an acoustic wave, regions of compression and depression are spatially adjacent; heat transfer from the "hot" region to the "cold" region is induced by the temperature difference between the two regions. The difference of temperature over half a wavelength and the phenomenon of diffusion of the heat wave are very slow and will therefore be neglected (even though they do occur); the phenomena will then be considered adiabatic as long as the dissipation of acoustic energy is not considered.

Finally, another damping phenomenon occurs in fluids: the delay of return to equilibrium due to the fact that the effect of the input excitation is not instantaneous. This phenomenon, called relaxation, occurs for physical, thermal and chemical equilibriums. The relaxation effect can be important, particularly in the air. As for viscosity and thermal conduction, this effect can also be neglected when dissipation is not important.

## 1.2. Fundamental laws of propagation in non-dissipative fluids

### 1.2.1. *Basis of thermodynamics*

"Sound" occurs when the medium presents dynamic perturbations that modify, at a given point and time, the pressure  $P$ , the density  $\rho_0$ , the temperature  $T$ , the entropy  $S$ , and the speed  $\bar{v}$  of the particles (only to mention the essentials). Relationships between those variables are obtained using the laws of thermomechanics in continuous media. These laws are presented in the following paragraphs for non-dissipative fluids and in the next chapter for dissipative fluids. Preliminarily, a reminder of the fundamental laws of thermodynamics is given; useful relationships in acoustics are numbered from (1.19) to (1.23). Complementary information on thermodynamics, believed to be useful, is given in the Appendix to this chapter.

A state of equilibrium of  $n$  moles of a pure fluid element is characterized by the relationship between its pressure  $P$ , its volume  $V$  (volume per unit of mass in acoustics), and its temperature  $T$ , in the form  $f(P, T, V) = 0$  (the law of perfect gases,  $PV - nRT = 0$ , for example, where  $n$  defines the number of moles and

$R = 8.32$  the constant of perfect gases). This thermodynamic state depends only on two, independent, thermodynamic variables.

The quantity of heat per unit of mass received by a fluid element  $dQ = T dS$  (where  $S$  represents the entropy) can then be expressed in various forms as a function of the pressure  $P$  and the volume per unit of mass  $V$  – reciprocal of the density  $\rho_0$  ( $V = 1/\rho_0$ )

$$T dS = C_p dT + h dP, \quad (1.1)$$

$$T dS = C_V dT + \ell dV, \quad (1.2)$$

where  $C_p$  and  $C_V$  are the heat capacities per unit of mass at respectively constant pressure and constant volume and where  $h$  and  $\ell$  represent the calorimetric coefficients defined by those two relations.

The entropy is a function of state; consequently,  $dS$  is an exact total differential, thus

$$\frac{C_p}{T} = \left( \frac{\partial S}{\partial T} \right)_P, \quad \frac{h}{T} = \left( \frac{\partial S}{\partial P} \right)_T \quad (1.3)$$

$$\frac{C_V}{T} = \left( \frac{\partial S}{\partial T} \right)_V, \quad \frac{\ell}{T} = \left( \frac{\partial S}{\partial V} \right)_T. \quad (1.4)$$

Applying Cauchy's conditions to the differential of the free energy  $F$  ( $dF = -SdT - PdV$ ) gives

$$\left( \frac{\partial P}{\partial T} \right)_V = \left( \frac{\partial S}{\partial V} \right)_T, \quad (1.5)$$

which, defining the increase of pressure per unit of temperature at constant density as  $\beta P = (\partial P / \partial T)_V$  and considering equation (1.4), gives

$$P\beta = \ell / T. \quad (1.6)$$

Similarly, Cauchy's conditions applied to the exact total differential of the enthalpy  $G$  ( $dG = -SdT + V dP$ ) gives

$$\left( \frac{\partial V}{\partial T} \right)_P = - \left( \frac{\partial S}{\partial P} \right)_T, \quad (1.7)$$

which, defining the increase of volume per unit of temperature at constant pressure as  $\alpha V = (\partial V / \partial T)_P$  and considering equation (1.3), gives

$$V\alpha = -h / T. \quad (1.8)$$

Reporting the relation

$$dV = (\partial V / \partial T)_P dT + (\partial V / \partial P)_T dP$$

Into

$$dS = (\partial S / \partial T)_V dT + (\partial S / \partial V)_T dV$$

leads to

$$\begin{aligned} dS &= [(\partial S / \partial T)_V + (\partial S / \partial V)_T (\partial V / \partial T)_P] dT + (\partial S / \partial V)_T (\partial V / \partial P)_T dP \\ \Rightarrow (\partial S / \partial T)_P &= (\partial S / \partial T)_V + (\partial S / \partial V)_T (\partial V / \partial T)_P. \end{aligned} \quad (1.9)$$

Finally, combining equations (1.3) to (1.8) yields

$$C_P - C_V = PVT\alpha\beta. \quad (1.10)$$

In the particular case where  $n$  moles of a perfect gas are contained in a volume  $V$  per unit of mass,

$$V\alpha = \frac{nR}{P} = \frac{V}{T} \quad \text{and} \quad P\beta = \frac{nR}{V} \quad \text{so} \quad C_P - C_V = nR. \quad (1.11)$$

Adopting the same approach as above and considering that

$$dT = (\partial T / \partial V)_P dV + (\partial T / \partial P)_V dP,$$

the quantity of heat per unit of mass  $dQ = TdS$  can be expressed in the forms

$$dQ = C_V dT + \ell dV = C_V (\partial T / \partial P)_V dP + [\ell + C_V (\partial T / \partial V)_P] dV \quad (1.12)$$

$$\text{or } dQ = C_P dT + h dV,$$

$$= C_P (\partial T / \partial V)_P dV + [h + C_P (\partial T / \partial P)_V] dP, \quad (1.13)$$

$$\text{or } dQ = \lambda dP + \mu dV. \quad (1.14)$$



Comparing equation (1.14) with equation (1.12) (considering, for example, an isochoric transformation followed by an isobaric transformation) directly gives

$$\lambda = C_V \left( \frac{\partial T}{\partial P} \right)_V = \frac{C_V}{P\beta} \quad \text{and} \quad \mu = C_P \left( \frac{\partial T}{\partial V} \right)_P = \frac{C_P}{V\alpha} = \frac{\rho C_P}{\alpha}. \quad (1.15)$$

Considering the fact that  $(\partial V / \partial P)_T (\partial T / \partial V)_P (\partial P / \partial T)_V = -1$  (directly obtained by eliminating the exact total differential of  $T(P, V)$  and also written as  $\alpha = \beta \chi_T P$ ) the ratio  $\lambda / \mu$  is defined by

$$\frac{\lambda}{\mu} = -\frac{1}{\gamma} \left( \frac{V}{\partial P} \right)_T = \frac{V \chi_T}{\gamma} = \frac{\chi_T}{\rho \gamma}, \quad (1.16)$$

where the coefficient of isothermal compressibility  $\chi_T$  is

$$\chi_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial P} \right)_T, \quad (1.17)$$

and the ratio of specific heats is

$$\gamma = C_P / C_V.$$

For an adiabatic transformation  $dQ = \lambda dP + \mu dV = 0$ , the coefficient of adiabatic compressibility  $\chi_S$  defined by  $\chi_S V = -(\partial V / \partial P)_S$  can also be written as

$$-\chi_S V = (\partial V / \partial P)_S = -\frac{\lambda}{\mu} = -\frac{V \chi_T}{\gamma}.$$

Finally,

$$\chi_S = \chi_T / \gamma \quad (\text{Reech's formula}). \quad (1.18)$$

The variation of entropy per unit of mass is obtained from equations (1.14) and (1.15) as:

$$dS = \frac{C_V}{TP\beta} dP - \frac{C_P}{T\rho\alpha} d\rho. \quad (1.19)$$

Considering that  $\alpha = \beta \chi_T P$  and  $\chi_S = \chi_T / \gamma$ ,

$$dS = \frac{C_V}{TP\beta} \left[ dP - \frac{\gamma}{\rho \chi_T} d\rho \right] = \frac{C_V}{TP\beta} \left[ dP - \frac{1}{\rho \chi_S} d\rho \right]. \quad (1.20)$$

Moreover, equations (1.12) and (1.13) give

$$h + C_P(\partial T / \partial P)_V = C_V(\partial T / \partial P)_V \text{ and thus } h = -(C_P - C_V) / (P\beta).$$

Consequently, substituting the latter result into equation (1.13) yields

$$dS = \frac{C_P}{T} dT - \frac{C_P - C_V}{TP\beta} dP. \quad (1.21)$$

Substituting equation (1.10) and  $\gamma = \beta\chi_T P$  into equation (1.21) leads to

$$dS = \frac{C_P}{T} dT - \frac{P\beta}{\rho} \chi_T dP. \quad (1.22)$$

Lechatelier's law, according to which a gas temperature evolves linearly with its pressure, is there demonstrated, in particular for adiabatic transformations: writing  $dS = 0$  in equation (1.22) brings proportionality between  $dT$  and  $dP$ , the proportionality coefficient  $TP\beta\chi_T / (\rho C_P)$  being positive.

The differential of the density  $d\rho = (\partial\rho / \partial P)_T dP + (\partial\rho / \partial T)_P dT$  can be expressed as a function of the coefficients of isothermal compressibility  $\chi_T$  and of thermal pressure variation  $\beta$  by writing that

$$\chi_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial P} \right)_T \text{ and } P\beta\chi_T = \alpha = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_P.$$

Thus,

$$d\rho = \rho\chi_T [dP - P\beta dT]. \quad (1.23)$$

Note: according to equation (1.20), for an isotropic transformation ( $dS = 0$ ):

$$dP = \frac{\gamma}{\rho\chi_T} d\rho = \frac{\gamma}{\rho\chi_S} d\rho;$$

which, for a perfect gas, is

$$dP = \gamma \frac{RT}{M} d\rho = \gamma \frac{P}{\gamma} d\rho, \text{ where } \frac{dP}{P} + \gamma \frac{dV}{V} = 0,$$

leading, by integrating, to  $PV^\gamma = \text{cte} = P_0V_0^\gamma$  the law for a reversible adiabatic transformation.

Similarly, according to equation (1.23), for an isothermal transformation ( $dT = 0$ )

$$dP = \frac{1}{\rho\chi_T} d\rho. \quad (1.24)$$

### 1.2.2. Lagrangian and Eulerian descriptions of fluid motion

The parameters normally used to describe the nature and state of a fluid are those in the previous paragraph:  $\alpha, \beta, C_P, C_V, \gamma$ , etc. for the nature of the fluid and  $P, V$  or  $\rho, T, S$ , etc. for its state. However, the variables used to describe the dynamic perturbation of the gas are the variations of state functions, the differentials  $dP, dV$  or  $d\rho, dT, dS$ , etc. and the displacement (or velocity) of any point in the medium. The study of this motion, depending on time and location, requires the introduction of the notion of “particle” (or “elementary particle”): the set of all molecules contained in a volume chosen which is small enough to be associated to a given physical quantity (i.e. the velocity of a particle at the vicinity of a given point), but which is large enough for the hypothesis of continuous media to be valid (great number of molecules in the particle).

Finding the equations of motion requires the attention to be focused on a given particle. Therefore, two different, but equivalent, descriptions are possible: the Lagrangian description, in which the observer follows the evolution of a fluid element, differentiated from the others by its location  $X$  at a given time  $t_0$  (for example, its location can be defined as  $\chi(X, t)$  with  $\chi(X, t_0) = X$  and its velocity  $\dot{\chi} = \partial\chi(X, t)/\partial t$ ), and the Eulerian description, in which the observer is not interested in following the evolution of an individual fluid element over a period of time, but at a given location, defined by  $\bar{r}$  and considered fixed or at least with infinitesimal displacements (for the differential calculus). The Lagrangian description has the advantage of identifying the particles and giving their trajectories directly; however, it is not straightforward when studying the dynamic of a continuous fluid in motion. Therefore, Euler’s description, which uses variables that have an immediate meaning in the actual configuration, is most often used in acoustics. It is this description that will be used herein. It implies that the differential of an ordinary quantity  $q$  is written either as

$$\begin{aligned} dq &= q(\bar{r} + d\bar{r}, t + dt) - q(\bar{r}, t), \\ \text{or } dq &= q(\bar{r} + d\bar{r}, t + dt) - q(\bar{r}, t + dt) + q(\bar{r}, t + dt) - q(\bar{r}, t), \\ \text{or } dq &= \text{grad } q(\bar{r}, t + dt) d\bar{r} + \frac{\partial}{\partial t} q(\bar{r}, t) dt. \end{aligned}$$

The differential  $dq$  represents the material derivative (noted  $Dq$  in some works) if the observer follows the particle in infinitesimal motion with instantaneous velocity  $\vec{v}$ , that is  $d\vec{r} = \vec{v} dt$ . Then, considering the fact that  $q(t+dt)dt \approx q(t)dt$  by neglecting the 2<sup>nd</sup> order term  $(\partial q / \partial t)_0 dt dt$ ,

$$dq = \text{grad } q(\vec{r}, t) \vec{v} dt + \frac{\partial}{\partial t} q(\vec{r}, t) dt,$$

or, using the operator formalism,

$$\frac{d}{dt} = \vec{v} \text{grad} + \frac{\partial}{\partial t}. \quad (1.25)$$

The following brief comparison between those two descriptions highlights their respective practical implications. The superscripts (E) and (L) distinguish Euler's from the Lagrangian approaches.

The instantaneous location  $\vec{r}$  of a particle is a function of  $\vec{r}_0$  and  $t$ , where  $\vec{r}_0$  is the location of the considered particle at  $t = t_0$  ( $\vec{r}_0$  is often representing the initial position).

Using Lagrangian variables, any quantity is expressed as a function of two variables  $\vec{r}_0$  and  $t$ . For example, the acceleration is represented by the function  $\vec{\Gamma}^{(L)}(\vec{r}_0, t)$ .

Using Eulerian variables, any quantity (the acceleration is used here as an example) is expressed as a function of the actual location  $\vec{r}$  and  $t$ , noted  $\vec{\Gamma}^{(E)}(\vec{r}, t)$ . This function can be expressed in such form that the expression of  $\vec{r}$  as a function of  $\vec{r}_0$  and  $t$  appears; it is then written as  $\vec{\Gamma}^{(E)}(\vec{r}(\vec{r}_0, t), t)$ , but still represents the same function  $\vec{\Gamma}^{(E)}(\vec{r}, t)$ .

These definitions result in the following relationships

$$\begin{aligned} \vec{\Gamma}^{(L)} &= \frac{\partial}{\partial t} \vec{v}^{(L)}(\vec{r}_0, t) = \frac{\partial^2}{\partial t^2} \vec{\chi}(\vec{r}_0, t), \\ \vec{\Gamma}^{(E)} &= \frac{d}{dt} \vec{v}^{(E)}(\vec{r}(\vec{r}_0, t), t) = \frac{\partial}{\partial t} \vec{v}^{(E)} + \vec{v}^{(E)} \text{grad } \vec{v}^{(E)}, \\ &= \frac{\partial}{\partial t} \vec{v}^{(E)} + \sum_j \frac{\partial}{\partial t} x_j(\vec{r}_0, t) \frac{\partial}{\partial x_j} \vec{v}^{(E)}, \end{aligned}$$

where  $\bar{v}^{(E)}(\bar{r}, t) = \frac{\partial}{\partial t} \bar{r}(\bar{r}_0, t)$ .

The physical quantity “acceleration” can either be expressed by  $\bar{\Gamma}^{(L)}(\bar{r}_0, t)$  or by  $\bar{\Gamma}^{(E)}(\bar{r}, t)$ .

### 1.2.3. Expression of the fluid compressibility: mass conservation law

A certain compressibility of the fluid is necessary to the propagation of an acoustic perturbation. It implies that the density  $\rho$ , being a function of the location  $\bar{r}$  and the time  $t$ , depends on spatial variations of the velocity field (which can intuitively be conceived), and eventually on the volume velocity of a local source acting on the fluid. This must be expressed by writing that a relation, easily obtained by using the mass conservation law, exists between the density  $\rho(\bar{r}, t)$  and the variations of the velocity field

$$\frac{d}{dt} \iiint_{D(t)} \rho dD = \iiint_{D(t)} \rho q(\bar{r}, t) dD, \quad (1.26)$$

The integral is calculated over a domain  $D(t)$  in motion, consequently containing the same particles, and the fluid input from a source  $q(\bar{r}, t)$  is expressed per unit of volume per unit of time ( $[q] = s^{-1}$ ). In the right hand side of equation (1.26), the factor  $\rho q$  denotes the mass of fluid introduced in  $D(t)$  per unit of volume and of time ( $[\rho \cdot q] = \text{kg} \cdot \text{m}^{-3} \cdot \text{s}^{-1}$ ). Without any source or outside its influence, the second term is null ( $q = 0$ ).

This mass conservation law can be equivalently expressed by considering a domain  $D_0$  fixed in space (the domain  $D_0$  can, for example, represent the previously defined domain  $D(t)$  at the initial time  $t = t_0$ ). The sum of the mass of fluid entering the domain  $D_0$  through the fixed surface  $S_0$ , per unit of time,

$$- \iint_{S_0} \rho \bar{v} d\bar{S}_0 \equiv - \iiint_{D_0} \text{div}(\rho \bar{v}) dD_0,$$

(where  $\bar{v}$  defines the particle velocity,  $d\bar{S}_0$  being parallel to the outward normal to the domain), and the mass of fluid introduced by an eventual source represented by

the factor  $\rho q$ , is equal to the increase of mass of fluid within the domain  $D_0$  per unit of time,

$$\frac{\partial}{\partial t} \iiint_{D_0} \rho dD_0 \equiv \iiint_{D_0} \frac{\partial \rho}{\partial t} dD_0,$$

Thus,

$$\iiint_{D_0} \left[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) \right] dD_0 = \iiint_{D_0} \rho q dD_0. \quad (1.27)$$

This equation must be valid for any domain  $D_0$ , implying that

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) = \rho q. \quad (1.28)$$

Substituting equation (1.25) and the general relation:

$$\text{div}(\rho \vec{v}) = \rho \text{div} \vec{v} + \vec{v} \text{grad} \rho,$$

leads to the following form of equation (1.27)

$$\frac{d\rho}{dt} + \rho \text{div} \vec{v} = \rho q. \quad (1.29)$$

One can show that equation (1.26) can also be written as

$$\iiint_{D(t)} \left[ \frac{d\rho}{dt} + \rho \text{div} \vec{v} \right] dD = \iiint_{D(t)} \rho q dD, \quad \forall D(t). \quad (1.30)$$

Equation (1.30) is equivalent to equation (1.29) since it is verified for any considered domain  $D(t)$ . Equations (1.26) to (1.30) are all equivalent and express the mass conservation law for a compressible fluid (incompressibility being defined by  $d\rho/dt = 0$ ).