Scaling, Fractals and Wavelets

Edited by
Patrice Abry
Paulo Gonçalves
Jacques Lévy Véhel

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Table of Contents

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Preface

It is a common scheme in many sciences to study systems or signals by looking for *characteristic scales* in time or space. These are then used as references for expressing all measured quantities. Physicists may for instance employ the size of a structure, while signal processors are often interested in correlation lengths: (blocks of) samples whose distance is several times the correlation lengths are considered statistically independent. The concept of *scale invariance* may be considered to be the converse of this approach: it means that there is no characteristic scale in the system. In other words, all scales contribute to the observed phenomenon. This “non-property” is also loosely referred to as *scaling law* or *scaling behavior*. Note that we may reverse the perspective and consider scale invariance as the signature of a strong organization in the system. Indeed, it is well known in physics that invariance laws are associated with fundamental properties. It is remarkable that phenomena where scaling laws have been observed cover a wide range of fields, both in natural and artificial systems. In the first category, these include for instance hydrology, in relation to the variability of water levels, hydrodynamics and the study of turbulence, statistical physics with the study of long-range interactions, electronics with the so-called $1/f$ noise in semiconductors, geophysics with the distribution of faults, biology, physiology and the variability of human body rhythms such as the heart rate. In the second category, we may mention geography with the distribution of population in cities or in continents, Internet traffic and financial markets.

From a signal processing perspective, the aim is then to study *transfer mechanisms* between scales (also called “cascades”) rather than to identify relevant scales. We are thus led to forget about scale-based models (such as Markov models), and to focus on models allowing us to study correspondences between many scales. The central notion behind scaling laws is that of *self-similarity*. Loosely speaking, this means that each part is (statistically) the same as the whole object. In particular, information gathered from observing the data should be independent of the scale of observation.

There is considerable variety in observed self-similar behaviors. They may for instance appear through scaling laws in the Fourier domain, either at all frequencies or in a finite but large range of frequencies, or even in the limit of high or low frequencies. In many cases, studying second-order quantities such as spectra will prove insufficient for describing scaling laws. Higher-order moments are then necessary. More generally, the fundamental model of self-similarity has to be adapted in many settings, and to be generalized in various directions, so that it becomes useful in real-world situations. These include self-similar stochastic processes, $1/f$ processes, long memory processes, multifractal and multifractional processes, locally self-similar processes and more. Multifractal analysis, in particular, has developed as a method allowing us to study complex objects which are not necessarily “fractal”, by describing the variations of local regularity. The recent change of paradigm consisting of using *fractal methods* rather than studying *fractal objects* is one of the reasons for the success of the domain in applications.

We are delighted to invite our reader for a promenade in the realm of scaling laws, its mathematical models and its real-world manifestations. The 14 chapters have all been written by experts. The first four chapters deal with the general mathematical tools allowing us to measure fractional dimensions, local regularity and scaling in its various disguises. Wavelets play a particular role for this purpose, and their role is emphasized. Chapters 5 and 6 describe advanced stochastic models relevant in our area. Chapter 7 deals with fractional calculus, and Chapter 8 explains how to synthesize certain fractal models. Chapter 9 gives a general introduction to IFS, a powerful tool for building and describing fractals and other complex objects, while Chapter 10, of applied nature, considers the application of IFS to image compression. The four remaining chapters also deal with applications: various signal and image processing tasks are considered in Chapter 11. Chapter 12 deals with Internet traffic, and Chapter 13 with financial data analysis. Finally, Chapter 14 describes a fractal space-time in the frame of cosmology.

It is a great pleasure for us to thank all the authors of this volume for the quality of their contribution. We believe they have succeeded in exposing advanced concepts with great pedagogy.

Chapter 1

Fractal and Multifractal Analysis in Signal Processing

1.1. Introduction

The aim of this chapter is to describe some of the fundamental concepts of fractal analysis in view of their application. We will thus present a simple introduction to the concepts of fractional dimension, regularity exponents and multifractal analysis, and show how they are used in signal and image processing.

Since we are interested in applications, most theoretical results are given without proofs. These are available in the references mentioned where appropriate. In contrast, we will pay special attention to the practical aspects. In particular, almost all the notions explained below are implemented in the FracLab toolbox. This toolbox is freely available from the following site: `http://complex.futurs.inria.fr/FracLab/`, so that interested readers may perform hands-on experiments.

Before we start, we wish to emphasize the following point: recent successes of fractal analysis in signal and image processing do not generally stem from the fact that they are applied to fractal *objects* (in a more or less strict sense). Indeed, most real-world signals are neither self-similar nor display the characteristics usually associated with fractals (except for the irregularity at each scale). The relevance of fractal analysis instead results from the progress made in the development of fractal *methods*. Such methods have lately become more general and reliable, and they now allow to describe precisely the singular structure of complex signals,

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Chapter written by Jacques LÉVY VÉHEL and Claude TRICOT.

without any assumption of “fractality”: as a rule, performing a fractal analysis will be useful as soon as the considered signal is irregular and this irregularity contains meaningful information. There are numerous examples of such situations, ranging from image segmentation (where, for instance, contours are made of singular points; see section 1.4.7 and Chapter 11) to vocal synthesis [DAO 02] or financial analysis.

This chapter roughly follows the chronological order in which the various tools have been introduced. We first describe several notions of fractional dimensions. These provide a global characterization of a signal. We then introduce Hölder exponents, which supply local measures of irregularity. The last part of the chapter is devoted to multifractal analysis, a most refined tool that describes the local as well as the overall singular structure of signals. All the concepts presented here are more fully developed in [TRI 99, LEV 02].

1.2. Dimensions of sets

The concept of dimension applies to objects more general than signals. To simplify, we shall consider sets in a metric space, although the notion of dimension makes sense for more complex entities such as measures or classes of functions [KOL 61]. Several interesting notions of dimension exist. This might look like a drawback for the mathematical analysis of fractal sets. However, it is actually an advantage, since each dimension emphasizes a different aspect of an object. It is thus worthwhile to determine the specificity of each dimension. As a general rule, none of these tools outperform the other.

Let us first give a general definition of the notion of dimension.

DEFINITION 1.1.– We call *dimension* an application d defined on the family of bounded sets of $\mathbb{R}^n$ and ranging in $\mathbb{R}^+ \cup \{-\infty\}$, such that:

1) $d(\emptyset) = -\infty$, $d(\{x\}) = 0$ for any point x;

2) $E_1 \subset E_2 \Rightarrow d(E_1) \leqslant d(E_2)$ (monotonicity);

3) if E has non-zero n-dimensional volume, then $d(E) = n$;

4) if E is a diffeomorphism T of $\mathbb{R}^n$ (such as, in particular, a similarity with non-zero ratio, or a non-singular affine application), then $d(T(E)) = d(E)$ (invariance).

Moreover, we will say that d is *stable* if $d(E_1 \cup E_2) = \max\{d(E_1), d(E_2)\}$. It is said to be σ-stable if, for any countable collection of sets:

$$d\big(\cup_n E_n\big) = \sup d\big(E_n\big)$$

σ-stable dimensions may be extended in a natural way to characterize unbounded sets of $\mathbb{R}^n$.

1.2.1. *Minkowski-Bouligand dimension*

The Minkowski-Bouligand dimension was invented by Bouligand [BOU 28], who named it the *Cantor-Minkowski order*. It is now commonly referred to as the *box dimension*. Let us cover a bounded set E of $\mathbb{R}^n$ with cubes of side ε and disjoint interiors. Let $N_\varepsilon(E)$ be the number of these cubes. When E contains an infinite number of points (i.e. if it is a curve, a surface, etc.), $N_\varepsilon(E)$ tends to $+\infty$ when ε tends to 0. The box dimension Δ characterizes the rate of this growth. Roughly speaking, Δ is the real number such that:

$$N_\varepsilon(E) \simeq \left(\frac{1}{\varepsilon}\right)^\Delta,$$

assuming this number exists. More generally, we define, for all bounded E, the number:

$$\Delta(E) = \limsup_{\varepsilon \to \infty} \frac{\log N_\varepsilon(E)}{|\log \varepsilon|} \tag{1.1}$$

A lower limit may also be used:

$$\delta(E) = \liminf_{\varepsilon \to \infty} \frac{\log N_\varepsilon(E)}{|\log \varepsilon|} \tag{1.2}$$

Note that some authors refer to the *box dimension* only when both indices coincide, that is, when the limit exists.

Both indices Δ and δ are dimensions in the sense previously defined. However, Δ is stable, contrarily to δ, so that Δ is more commonly used. Let us mention an important property: if $\bar{E}$ denotes the closure of E (the set of all limit points of sequences in E), then:

$$\Delta(\bar{E}) = \Delta(E)$$

This property shows that Δ is not sensitive to the topological type of E. It only characterizes the *density* of a set. For example, the (countable) set of the rational numbers of the interval $[0, 1]$ has one dimension, which is the dimension of the interval itself. Even discrete sequences may have non-zero dimension: let, for instance, E be the set of numbers $n^{-\alpha}$ with $\alpha > 0$. Then $\Delta(E) = 1/(\alpha + 1)$.

Equivalent definitions

It is not mandatory to use cubes to calculate Δ. The original definition of Bouligand is as follows:

– in $\mathbb{R}^n$, let us consider the *Minkowski sausage*:

$$E(\varepsilon) = \cup_{x\in E} B_\varepsilon(x)$$

which is the union of all the balls of radius ε centered at E. Denote its volume by $\mathrm{Vol}_n(E(\varepsilon))$. This volume is approximately of the order of $N_\varepsilon(E)\,\varepsilon^n$. This allows us to give the equivalent definition:

$$\Delta(E) = \limsup_{\varepsilon\to 0}\left(n - \frac{\mathrm{Vol}_n\big(E(\varepsilon)\big)}{\log\varepsilon}\right); \tag{1.3}$$

– we may also define $N'_\varepsilon(E)$, which is the smallest number of balls of radius ε covering E; or $N''_\varepsilon(E)$, the largest number of disjoint balls of radius ε centered on E. Replacing $N_\varepsilon(E)$ by any of these values in equation (1.1) still gives $\Delta(E)$.

Discrete values of ε

In these definitions, the variable ε is continuous. The results remain the same if we use a discrete sequence such as $\varepsilon_n = 2^{-n}$. More generally we may replace ε with any sequence which does not converge too quickly towards 0. More precisely, we require that:

$$\lim_{n\to\infty}\frac{\log\varepsilon_n}{\log\varepsilon_{n+1}} = 1.$$

This remark is important, as it allows us to perform numerical estimations of Δ.

Let us now give some well-known examples of calculating dimensions.

EXAMPLE 1.1.– Let (a_n) be a sequence of real numbers such that $0 < 2a_{n+1} < a_n < a_0 = 1$. Let $E_0 = [0, 1]$. We construct by induction a sequence of sets (E_n) such that E_n is made of 2^n closed disjoint intervals of length a_n, each containing exactly two intervals of E_{n+1}. The sets E_n are nested, and the sequence (E_n) converges to a compact set E such that:

$$E = \cap_n E_n.$$

Let us consider a particular case. When all the interval extremities E_n are also interval extremities of E_{n+1}, E is called a *perfect symmetric set* [KAH 63] or sometimes, more loosely, a *Cantor set*. Assume that the ratio $\log a_n/\log a_{n+1}$ tends to 1. According to the previous comment on discrete sequences, we obtain the following values:

$$\delta(E) = \liminf_{n\to\infty}\frac{n\log 2}{|\log a_n|},\qquad \Delta(E) = \limsup_{n\to\infty}\frac{n\log 2}{|\log a_n|}.$$

However, these results are true for any sequence (a_n). Even more specifically, consider the case where $a_n = a^n$, with $0 < a < \frac{1}{2}$. The ratios a_n/a_{n+1} are then constant and dimensions take the common value $\log 2/|\log a|$. This is the case of the *self-similar* set which satisfies the following relation:

$$E = f_1(E) \cup f_2(E)$$

with $f_1(x) = a\,x$ and $f_2(x) = a\,x + 1 - a$. This set is the attractor of the iterated function system $\{f_1, f_2\}$ (see Chapters 9 and 10). It is also called a *perfect symmetric set with constant ratio*.

EXAMPLE 1.2.– We construct a planar self-similar curve with extremities A and B, $A \neq B$ as follows: take $N+1$ distinct points $A_1 = A$, $A_2, \dots, A_{N+1} = B$, such that $\operatorname{dist}(A_i, A_{i+1}) < \operatorname{dist}(A, B)$. For each $i = 1, \dots, N$, define a similarity f_i (that is, a composition of a homothety, an orthogonal transformation and a translation), such that

$$f_i(AB) = A_i\,A_{i+1}.$$

The ratio of f_i is $a_i = \operatorname{dist}(A_i, A_{i+1})/\operatorname{dist}(A, B)$. Starting from the segment $\Gamma_0 = AB$, define by induction the polygonal curves $\Gamma_n = \cup_i f_i(\Gamma_{n-1})$. This sequence (Γ_n) converges to a curve Γ which satisfies the following relation:

$$\Gamma = \cup_i f_i(\Gamma).$$

In other words, Γ is the attractor of the IFS $\{f_1, \dots, f_n\}$. When Γ is simple, the dimensions δ and Δ assume a common value, which is also the *similarity dimension*, i.e. the unique solution of the equation

$$\sum_{i=1}^{N} a_i^x = 1.$$

In the particular case where all distances $\operatorname{dist}(A_i, A_{i+1})$ are the same, the ratios a_i are equal to a value a such that $Na > 1$ (necessary condition for the continuity of Γ) and $Na^2 < 1$ (necessary condition for the simplicity of Γ). Clearly, $\delta(\Gamma) = \Delta(\Gamma) = \log N/|\log a|$.

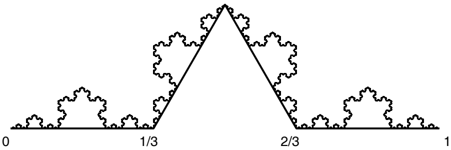

Figure 1.1. *Von Koch curve, the attractor of a system of four similarities with common ratio* $\frac{1}{3}$

Function scales

The previous definitions all involve ratios of logarithms. This is an immediate consequence of the fact that a dimension is defined as an order of growth related to the *scale of functions* $\{t^\alpha, \alpha > 0\}$. In general, a scale of functions $\mathcal{F}$ in the neighborhood of 0 is a family of functions which are all comparable in the Hardy sense, that is, for any f and g in $\mathcal{F}$, the ratio $f(x)/g(x)$ tends to a limit (possibly $+\infty$ or $-\infty$) when x tends to 0. Function scales are defined in a similar way in the neighborhood of $+\infty$. Scales other than $\{t^\alpha\}$ will yield other types of dimensions. A dimension must be considered as a Dedekind cut in a given scale of functions. The following expressions will make this clearer:

$$\Delta(E) = \inf\{\alpha \text{ such that } \varepsilon^\alpha N_\varepsilon(E) \to 0\} \tag{1.4}$$

$$\delta(E) = \sup\{\alpha \text{ such that } \varepsilon^\alpha N_\varepsilon(E) \to +\infty\} \tag{1.5}$$

these are equivalent to equations (1.1) and (1.2) (see [TRI 99]).

Complementary intervals on the line

In the particular case where the compact E lies in an interval J of the line, the complementary set of E in J is a union of disjoint open intervals, whose lengths will be denoted by c_n. Let $|E|$ be the Lebesgue measure of E (which means, for an interval, its length). The dimension of E may be written as:

$$\Delta(E) = \limsup_{\varepsilon \to 0} \left(1 - \frac{\log \big|E(\varepsilon)\big|}{\log \varepsilon}\right)$$

If $|E| = 0$, the sum of the c_n is equal to the length of J. The dimension is then equal to the *convergence exponent* of the series $\sum c_n$:

$$\Delta(E) = \inf\left\{\alpha \text{ such that } \sum_n c_n^\alpha < +\infty\right\} \tag{1.6}$$

Proof. This result may be obtained by calculating an approximation of the length of Minkowski sausage $E(\varepsilon)$. Let us assume that the complementary intervals are ranked in decreasing lengths:

$$c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n \geqslant \cdots$$

If $|E| = 0$ and if $c_n \geqslant \varepsilon > c_{n+1}$, then:

$$|E(\varepsilon)| \simeq n\varepsilon + \sum_{i \geqslant n} c_i$$

thus $\varepsilon^{\alpha-1}L(E(\varepsilon)) \simeq n\varepsilon^{\alpha} + \varepsilon^{\alpha-1}\sum_{i\geqslant n} c_i$. It may be shown that both values

$$\inf\{\alpha \text{ such that } n\varepsilon^{\alpha} < +\infty\} \quad \text{and} \quad \inf\left\{\alpha \text{ such that } \varepsilon^{\alpha-1}\sum_{i\geqslant n} c_i < +\infty\right\}$$

are equal to the convergence exponent. It is therefore equal to $\Delta(E)$. □

EXERCISE 1.1.– Verify formula (1.6) for the perfect symmetric sets of Example 1.1.

If $|E| \neq 0$, then the convergence exponent of $\sum c_n$ still makes sense. It characterizes a degree of *proximity* of the exterior with the set E. More precisely, we obtain

$$\inf\left\{\alpha \text{ such that } \sum_n c_n^{\alpha} < +\infty\right\} = \limsup_{\varepsilon\to 0}\left(1 - \frac{\log|E(\varepsilon) - E|}{\log \varepsilon}\right) \qquad (1.7)$$

where the set $E(\varepsilon) - E$ refers to the Minkowski sausage of E deprived of the points of E.

How can we generalize the study of the complementary set in $\mathbb{R}^n$ with $n \geqslant 2$? The open intervals must be replaced with an appropriate paving. The results connecting the elements of this paving to the dimension depend both on the geometry of the tiles and on their respective positions. The topology of the complementary set must be investigated more deeply [TRI 87]. The index that generalizes (1.7) (replacing the 1 of the space dimension by n) is the *fact fractal exponent*, studied in [GRE 85, TRI 86b]. In the case of a zero area curve in $\mathbb{R}^2$, this also leads to the notion of *lateral dimension*. Note that the dimensions corresponding to each side of the curve are not necessarily equal [TRI 99].

1.2.2. *Packing dimension*

The packing dimension is, to some extent, a regularization of the box dimension [TRI 82]. Indeed, Δ is not σ-stable, but we may derive a σ-stable dimension from any index thanks to the operation described below.

PROPOSITION 1.1.– Let $\mathcal{B}$ be the family of all bounded sets of $\mathbb{R}^n$ and $\alpha : \mathcal{B} \longrightarrow \mathbb{R}^+$. Then, the function $\hat{\alpha}$ defined for any subsets of $\mathbb{R}^n$ as:

$$\hat{\alpha}(E) = \inf\{\sup \alpha(E_i) / E = \cup E_i, E_i \in \mathcal{B}\}$$

is monotonous and σ-stable.

Proof. Any subset E of $\mathbb{R}^n$ is a union of bounded sets. If $E_1 \subset E_2$, then any covering of E_1 may be completed with a covering of E_2. This entails monotonicity. Now, let $\varepsilon > 0$ and a sequence $(E_k)_{k\geqslant 1}$ of sets whose union is E. For any k, there exists a decomposition $(E_{i,k})$ of E_k such that $\sup \alpha(E_{i,k}) \leqslant \hat{\alpha}(E_k) + \varepsilon 2^{-k}$. Since $E = \cup_{i,k} E_{i,k}$, we deduce that:

$$\hat{\alpha}(E) \leqslant \sup_k \hat{\alpha}(E_k) + \varepsilon \sum 2^{-k} = \sup_k \hat{\alpha}(E_k) + \varepsilon$$

Thus, the inequality $\hat{\alpha}(E) \leqslant \sup_k \hat{\alpha}(E_k)$ holds. The converse inequality stems from monotonicity. □

The packing dimension is the result of this operation on Δ. We set

$$\mathrm{Dim} = \hat{\Delta}$$

The term *packing* will be explained later. The new index Dim is indeed a dimension, and it is σ-stable. Therefore, contrarily to Δ, it vanishes for countable sets. The inequality:

$$\mathrm{Dim}(E) \leqslant \Delta(E)$$

is true for any bounded set. This becomes an equality when E presents a *homogenous* structure in the following sense:

THEOREM 1.1.– Let E be a compact set such that, for all open sets U intersecting E, $\Delta(E \cap U) = \Delta(E)$. Then $\Delta(E) = \mathrm{Dim}(E)$.

Proof. Let E_i be a decomposition of E. Since E is compact, a Baire theorem entails that the E_i are not all nowhere dense in E. Therefore, there exist an index i_0 and an open set U intersecting E such that $\overline{E_{i_0}} \cap U = \bar{E} \cap U$, which yields:

$$\Delta(E_{i_0}) = \Delta(\overline{E_{i_0}}) \geqslant \Delta(\overline{E_{i_0}} \cap U) = \Delta(\bar{E} \cap U) \geqslant \Delta(E \cap U) = \Delta(E)$$

As a result, $\Delta(E) \leqslant \sup_i \Delta(E_i)$, and thus $\Delta(E) \leqslant \mathrm{Dim}(E)$. The converse inequality is always true. □

EXAMPLE 1.3.– All self-similar sets are of this type, including those presented above: Cantor sets and curves. For these sets, the packing dimension has the same value as $\Delta(E)$.

EXAMPLE 1.4.– Dense sets in $[0, 1]$, when they are not compact, do not necessarily have a packing dimension equal to 1. Let us consider, for any real p, $0 < p < 1$, the set E_p of p-normal numbers, that is, those numbers whose frequency of zeros in their dyadic expansion is equal to p. Any dyadic interval of $[0, 1]$, however small it may be, contains points of E_p, so E_p is dense in $[0, 1]$. As a consequence, $\Delta(E_p) = 1$. In contrast, the value of $\mathrm{Dim}(E_p)$ is:

$$\mathrm{Dim}(E_p) = \frac{1}{\log 2}\big|p \log p + (1-p)\log(1-p)\big|.$$

This result will be derived in section 1.3.2.

1.2.3. *Covering dimension*

The covering dimension was introduced by Hausdorff [HAU 19]. Here we adopt the traditional approach through Hausdorff measures; a direct approach, using Vitali's covering convergence exponent, may be used to calculate the dimension without using measures [TRI 99].

Covering measures

Originally, the covering measures were defined to generalize and, most of all, to precisely define the concepts of length, surface, volume, etc. They constitute an important tool in geometric measure theory.

Firstly, let us consider a *determining function* ϕ: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$, which is increasing and continuous in the neighborhood of 0, and such that $\phi(0) = 0$. Let E be a set in a metric space (that is, a space where a distance has been defined). For every $\varepsilon > 0$, we consider all the coverings of E by bounded sets U_i of diameter $\mathrm{diam}(U_i) \leqslant \varepsilon$. Let

$$H_\varepsilon^\phi(E) = \inf\Big\{\sum \phi\big(\mathrm{diam}(U_i)\big)/E \subset \cup_i U_i, \mathrm{diam}(E_i) \leqslant \varepsilon\Big\}.$$

When ε tends to 0, this quantity (possibly infinite) cannot decrease. The limit corresponds to the *ϕ-Hausdorff measure*:

$$H^\phi(E) = \lim_{\varepsilon \to 0} H_\varepsilon^\phi(E)$$

In this definition, the covering sets U_i can be taken in a more restricted family. If we suppose that U_i are open, or convex, the result remains unchanged. The main properties are that of any Borel measure:

– $E_1 \subset E_2 \Longrightarrow H^\phi(E_1) \leqslant H^\phi(E_2)$;

– if (E_i) is a collection of countable sets, then

$$H^\phi(\cup E_i) \leqslant \sum_i H^\phi(E_i)$$

– if E_1 and E_2 are at non-zero distance from each other, any ε-covering of E_1 is disjoint from any ε-covering of E_2 when ε is sufficiently small. Then $H^\phi(E_1 \cup E_2) = H^\phi(E_1) + H^\phi(E_1)$. This implies that H^ϕ is a *metric measure*. The Borel sets are H^ϕ-measurable and for any collection (E_i) of disjoint Borel sets, $H^\phi(\cup_i E_i) = \sum_i H^\phi(E_i)$.

The scale of functions t^α

In the case where $\phi(t) = t^\alpha$ with $\alpha > 0$, we use the simple notation $H^\phi = H^\alpha$.

Consider the case $\alpha = 1$. For any curve Γ the value $H^1(\Gamma)$ is equal to the length of Γ. Therefore H^1 is a generalization of the concept of length: it may be applied to any subset of the metric space.

Now let $\alpha = 2$. For any plane surface S, the value of $H^2(S)$ is proportional to the area of S. For non-plane surfaces, H^2 provides an appropriate mathematical definition of area – using a triangulation of S is not acceptable from a theoretical point of view.

More generally, when α is an integer, H^α is proportional to the α-dimensional volume.

However, α can also take non-integer values, which makes it possible to define the *dimension* of any set. The use of the term *dimension* is justified by the following property: if aE is the image of E by a homothety of ratio a, then

$$H^\alpha(aE) = a^\alpha H^\alpha(E)$$

Measures estimated using boxes

If we want to restrict the class of sets from which coverings are taken even more, one option would be to cover E with centered balls or dyadic boxes. In each case, the result is a measure $H^{*\alpha}$ which is generally not equal to $H^\alpha(E)$; nevertheless, it is an equivalent measure in the sense that we can find two non-zero constants, c_1 and c_2, such that for any E:

$$c_1\, H^\alpha(E) \leqslant H^{*\alpha}(E) \leqslant c_2\, H^\alpha(E)$$

Clearly the $H^{*\alpha}$ measures give rise to the same dimension.