Mesh Generation

Application to Finite Elements

Second Edition

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Contents

Introduction

Mesh generation techniques are widely employed in various engineering fields including those related to physical models described by partial differential equations (PDE). Numerical simulations of such models are intensively used for design, dimensioning and validation purposes. One of the most frequently used methods, among many others, is the *finite element* method (FEM). In this method, a continuous problem (the initial PDE model) is replaced by a discrete problem that can actually be computed thanks to the power of currently available computers. The solution to this discrete problem is an approximate solution to the initial problem whose accuracy is based on the various choices that were made in the numerical process.

The first step (in terms of actual computation) of such a simulation involves constructing a mesh of the computational domain (i.e., the domain where the physical phenomenon under interest occurs and evolves) so as to replace the continuous region by means of a finite union of (geometrically simple and bounded) elements such as triangles, quadrilaterals, tetrahedra, pentahedra, prisms, hexahedra, etc., based on the spatial dimension of the domain. For this reason, mesh construction is an essential pre-requisite for any numerical simulation of a PDE problem. Moreover, mesh construction could be seen as a bottleneck for a numerical process in the sense that a failure in this mesh construction step jeopardizes any subsequent numerical simulation.

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Mesh construction in general and more precisely for numerical simulation purposes involves several different fields and domains. These include (classical) geometry, so-called computational geometry and numerical simulation (engineering) topics coupled with advanced knowledge about what is globally termed computer science. The above classification in terms of disciplines which can interact in mesh construction for numerical simulation clearly shows why this topic is not so straightforward. Indeed, people with a geometrical, a computational geometry or a numerical background may not have the same perception of what a mesh (and, *a fortiori*, a computational mesh) should be, and subsequently do not share a common idea of what a mesh construction method could be.

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To give a rough idea of this problem, we mention, without in any way claiming to be exhaustive, some commonly accepted ideas about meshes based on the background of those considering the issue.

From a purely geometrical point of view, meshes are mostly of interest for the properties enjoyed by such or such geometrical item, a triangle for instance. In this respect, various issues have been investigated regarding the properties of such an element including aspect ratios, angle measures, orthogonality properties, affine properties and various related constructions (centroids, circumcenters, circumcircles, incircles, particular (characteristic) points, projections, intersections, etc.).

A computational geometry point of view mainly focuses on theoretical properties about triangulation methods including a precise analysis of the corresponding complexity. In this respect, Delaunay triangulation and its dual, the Voronoï diagram, have received much attention since nice theoretical foundations exist and lead to interesting theoretical results. However, triangulation methods are not necessarily suitable for general meshing purposes and must, to some extent, be adapted or modified.

Mesh construction from a purely numerical point of view (where, indeed, meshes are usually referred to as triangulations or grids) tends to reduce the mesh to a finite union of (simply shaped) elements whose size tends towards 0:
" Let $\mathcal{T}_h$ be a triangulation where h tends to 0, then ..., "
where $\mathcal{T}_h$ is provided in some way or other (with no further details given on this point). The construction of $\mathcal{T}_h$ is no longer a relevant problem if a theoretical study is envisaged (such as a convergence issue for a given numerical scheme).

In contrast to all the previous aspects, people actually involved in mesh construction methods face a different problem. Provided with some data, the problem is to develop methods capable of constructing a mesh (using a computer) that conforms to the needs of "numerical" and more generally "engineering" people. With regard to this, the above subscript h does not vanish, the domain geometry that must be handled could be of arbitrary complexity and a series of requirements may be demanded based on the subsequent use of the mesh once it has been constructed. On the one hand, theoretical results about triangulation algorithms (mainly obtained from computational geometry) may not be so realistic when viewed in terms of actual computer implementation. On the other hand, engineering requirements may differ slightly from what the theory states or needs to assume.

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As a brief conclusion, people involved in "meshing" must make use of knowledge from various disciplines, mainly geometry and computation, then combine this knowledge with numerical requirements (and computational limitations) to decide whether or not an *a priori* attractive aspect (for a particular discipline) is relevant in a meshing process. In other words, good candidates for mesh construction activities must have a sound knowledge in various disciplines in order to be able to select from these what they really require for a given goal.

Fortunately, we should point out that meshing things are becoming increasingly recognized as a subject of interest in its own right, not only in engineering but also at universities as well. In practice the subject is being addressed in several places all over the world, and a numerous people are spending a great deal of time on it. A few specialized conferences and workshops do exist and papers on meshing technologies can be found in various journals. Currently a few books[1] entirely (or substantially) devoted to meshing technologies are available.

Purpose and scope

The scope of this book is multiple and so are the potential categories of intended readers. As a first remark, we like to think that the theoretical background that is strictly necessary to understand the book is anything but specialized. We are confident that a reasonable knowledge of basic geometry, a touch of computational geometry and a good guess of what a numerical simulation is (for instance, some basic notions about the finite element method) provide a sufficient background for the reader to profit from this material. With regard to this, one of our objectives has been to make most of the presentations self-contained.

One issue underlying some of the discussions developed in the book was what material the reader might expect to find in such a book. A tentative answer to this point has led us to incorporate some material that could be judged trivial by readers who are already familiar with some meshing methods, yet we believe that its inclusion may well prove useful to less experienced readers.

We have introduced some recent developments in meshing activities, even if they have not necessarily been well validated (at least to the industrial standard), so as to allow advanced readers to initiate new progress based on this material.

It might be said that constructing a mesh for a given purpose (academic or

[1] Probably the very first significant reference about mesh generation is the book by Thompson, Warsi and Mastin, [Thompson *et al.* 1985], authored in 1985, which mainly discussed structured meshes. A few years after, in 1991, a book by George, [George-1991], was written which aimed to cover both structured and unstructured mesh construction methods. More recently, a book authored in 1993 by Knupp and Steinberg, [Knupp, Steinberg-1993] together with a book by Liseikin, [Liseikin-2000], provided an updated view of structured meshes. In 1998, a book fully devoted to Delaunay meshing techniques, [George, Borouchaki-1997], appeared. Among books that contain significant parts about meshing issues, one can find the book authored by Carey in 1997, [Carey-1997].

Thus, it is now possible to find some references about mesh technology topics. In this respect, one needs to see the publication of the Handbook of Grid Generation, edited by Thompson, Soni and Weatherill, [Thompson *et al.* 1999], which, in about 37 chapters by at least the same number of contributors, provides an impressive source of information. To conclude, notice the publication of another collective work, "Maillage et Adaptation", [George-2001], in the MIM (Mécanique et Ingénierie des Matériaux) series published by Hermès, Paris, together with a concise vulgarization book, "le maillage facile",[Frey, George-2003]. More recently, the Encyclopedia of Computational Mechanics, edited by Stein, de Borst and Hughes, [Stein *et al.* 2004], offered a chapter on mesh generation.

industrial) does not strictly require knowing what the meshing technologies are. Numerous engineers confronted daily with meshing problems, as well as graduate students facing the same problem, have been able to complete what they need without necessarily having a precise knowledge of what the software package they are familiar with actually does. Obviously, this point of view can be refuted and clearly a minimum knowledge of the available meshing technologies is a key to making this mesh construction task more efficient. Finally, following the above observations, the book is intended for both academic (educational) and industrial purposes.

Synopsis

Although we could have begun by a general purpose introduction and led on to a presentation of classical methods, followed by a discussion of advanced methods, specialized topics, etc., we chose to structure the book in such a way that it may be read sequentially. Relevant ideas are introduced when they are strictly necessary to the discussion, which means that the discussion about simple notions is made easy while when more advanced discussions are made, the more advanced ideas are given at the same time. Also, some almost identical discussions can be found in several sections, in an attempt to make each section as self-contained as possible.

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The book contains 24 chapters. The first three chapters introduce some general purpose definitions (Chapter 1) and basic data structures and algorithms (Chapter 2), then classical mesh generation methods are briefly listed prior to more advanced techniques (Chapter 3). The following chapters provide a description of the various mesh generation methods that are in common use. Each chapter corresponds to one type of method. We include discussions about algebraic, PDE-based or multi-block methods (Chapter 4), quadtree-octree based methods (Chapter 5), advancing-front technique (Chapter 6), Delaunay-type methods (Chapter 7), mesh generation methods for implicitly defined domains (Chapter 16) and other mesh generation techniques (Chapter 8) not covered by the previous cases. Chapter 9 deals with Delaunay-admissible curve or surface meshes and then discusses medial axis construction along with the various applications that can be envisaged based on this entity. Prior to a series of five chapters on lines, curves and surfaces, a short chapter concerns the metric aspects that are encountered in mesh generation activities (Chapter 10). As previously mentioned, Chapters 12 to 16 discuss curves and surfaces while Chapter 11 recalls the basic notions regarding differential geometry for curves and surfaces. One chapter presents various aspects about mesh modification tools (Chapter 17), then, two chapters focus on optimization issues (Chapter 18 for planar or volumic meshes and Chapter 19 for surface meshes). Basic notions about the finite element method are recalled in Chapter 20 before looking at a more advanced mesh generation problems, namely how to construct adapted, mobile or deformable meshes (Chapters 21, 22 and 23). Parallel aspects are discussed in Chapter 24. To conclude, an index is provided to the readers.

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The first edition of this book is a translation of “Maillages. Applications aux éléments finis”, published by Hermès Science Publications, Paris, 1999. The translation was carried out by the two authors with the valuable help of Richard James whom we would like to thank here.

This second edition includes various corrections, improvements, a fully updated index together with a new chapter about mobile and deformable meshes due to Pascal Frey, currently Full Professor at the Université Pierre et Marie Curie.

Finally, let us mention, *at this time*, two websites devoted to meshing technologies and a website offering thousands of surface meshes for downloading:

`http://www-users.informatik.rwth-aachen.de/~roberts/meshgeneration.html`
`http://www.andrew.cmu.edu/user/sowen/mesh.html`
`http://www-c.inria.fr/gamma/download/`

However, to find more information or to find interesting sites, simply use your favorite browser.

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Symbols and Notations

Notations

d	refers to the spatial dimension
$\mathbb{N}, \mathbb{R}$	set of integers, set of reals
Ω	refers to a closed geometric domain of $\mathbb{R}^d$
$\partial\Omega$	refers to the (discretized) boundary of Ω
$\Gamma(\Omega)$	refers to the boundary of Ω
Γ, Σ	refers to a curve, a surface
γ, σ	refers to the parametrization of a curve, a surface
$\mathcal{T}, \mathcal{T}_h, \mathcal{T}_r$	refers to a triangulation or a mesh
$\mathcal{V}$ or $\mathcal{S}$	refers to a set of vertices
$Const$	refers to a constraint (a set of entities)
$Conv(\mathcal{V})$	refers to the convex hull of $\mathcal{V}$
(Δ, H)	refers to a control space
K	refers to a mesh element
S_K, V_K	refers to the surface area, the volume of element K
$\mathcal{Q}_K$	shape quality of mesh element K
d_{AB}, $d(A,B)$	(Euclidean) distance between A and B
$\Vert \overrightarrow{PQ} \Vert$	Euclidean length of segment PQ
l_{AB}	(normalized) length of edge AB

Symbols

∇	gradient operator
$\mathcal{H}$	Hessian tensor
$\vert a \vert$	absolute value
$\lfloor . \rfloor$	integer part or restriction
$\Vert . \Vert$	Euclidean length of a vector
$[a,b]$	a closed interval
$\langle u, v \rangle$	dot product of two vectors
$(. \wedge .)$	cross product of two vectors
${}^t u$	u transposed (also u^t)

Abbreviations

ALE	Arbitrary Lagrangian Eulerian
BRep, F-Rep	Boundary Representation, Function Representation
CAD	Computer Aided Design
CSG	Constructive Solid Geometry
MAT	Medial Axis Transform
FEM	Finite Element Method
PDE	Partial Derivative Equation
NURBS	Non Uniform Rational B-Splines
LIFO	Last In First Out
FIFO	First In First Out
BST	Binary Search Tree
AVL	Adelson, Velskii and Landis tree

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Chapter 1

General Definitions

Before going further, it seems important to clarify the terminology and to provide some basic definitions together with some notions of general interest. First, we define the *covering-up* of a bounded domain, then we present the notion of a *triangulation* before introducing a particular triangulation, namely the well-known *Delaunay triangulation.*

A domain covering-up simply corresponds to the naive meaning of this word and the term may be taken at face value. On the other hand, a triangulation is a specific covering-up that has certain specific properties. Triangulation problems concern the construction, of a covering-up of the convex hull of a given set of points. In general, a triangulation is a set of simplices, triangles in two dimensions, tetrahedra in three dimensions, with certain properties. If, in addition to a set of vertices, the boundary of a domain (more precisely a discretization of this boundary whose vertices are in the above set) is specified or, simply if any set of required edges (faces) is provided, we encounter a problem of *constrained triangulation.* In this case, the expected triangulation of the convex hull must contain these required items.

In contrast, the notion of a *mesh* may now be specified. Given a domain, namely defined by a discretization of its boundary, the problem comes down to constructing a "triangulation" that accurately matches this specific domain. In a way, we are dealing with a constrained triangulation but, now, we no longer face a convex hull problem and, moreover, the mesh elements are not necessarily simplices.

After having established triangulation and mesh definitions, some other aspects are discussed, including a suitable element definition (as an element is the basic component of both a triangulation and a mesh), finite element definition as well as mesh data structure definition which are the fundamental ingredients of any further processing (such as using a finite element method). In addition, we introduce some definitions related to certain data structures which are widely used in mesh construction and mesh optimization processes. To conclude, we propose measures of mesh quality and of mesh optimality.

Obviously this chapter cannot claim to be exhaustive. In fact, more specific ideas will be introduced and discussed as required throughout the book.

1.1 Covering-up and triangulation

If $\mathcal{S}$ is a finite set of points in $\mathbb{R}^d$ ($d = 2$ or $d = 3$), the convex hull of $\mathcal{S}$, denoted as $Conv(\mathcal{S})$, defines a domain Ω in $\mathbb{R}^d$. Let K be a simplex[1] (triangle or tetrahedron according to d, always considered as a connected and closed set). Then a covering-up $\mathcal{T}_r$ of Ω by means of simplices corresponds to the following definition:

Definition 1.1 *$\mathcal{T}_r$ is a simplicial covering-up of Ω if the following conditions hold*

- $(H0)$ *The set of element vertices in $\mathcal{T}_r$ is exactly S.*
- $(H1)$ $\Omega = \overline{\bigcup_{K \in \mathcal{T}_r}}^{\circ} K$, *where K is a simplex.*
- $(H2)$ *The interior of every element K in $\mathcal{T}_r$ is non empty.*
- $(H3)$ *The intersection of the interior of two elements is an empty set.*

Here is a "natural" definition. With respect to condition $(H1)$ (where while not strictly necessary, we restrict ourselves to simplicial elements), one can see that Ω is the open set corresponding to the domain that means, in particular, that $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_r} K$. Condition $(H2)$ is not strictly necessary to define a covering-up, but it is nevertheless practical with respect to the context and, thus, will be assumed. Condition $(H3)$ means that element overlapping is proscribed.

Similarly, we will consider conforming coverings-up, referred to as triangulations.

Definition 1.2 *$\mathcal{T}_r$ is a conforming triangulation or simply a triangulation of Ω if $\mathcal{T}_r$ is a covering-up following Definition (1.1) and if, in addition, the following condition holds:*

- $(H4)$ *the intersection of two elements in $\mathcal{T}_r$ is either reduced to*
  - *the empty set or to*

[1]Let us briefly recall the definition of a d-simplex: we consider $d+1$ points $a_j = (a_{ij})_{i=1}^d \in \mathbb{R}^d, 1 \leq j \leq d+1$, not all in the same hyper-plane, i.e., such that the matrix of order $d+1$:

$$\mathcal{A} = \begin{pmatrix} a_{11} & \dots & a_{1,d+1} \\ \dots & \dots & \dots \\ a_{d1} & \dots & a_{d,d+1} \\ 1 & 1 & 1 \end{pmatrix},$$

is invertible. D-simplex K whose vertices are the a_j is the convex hull of these points a_j. Every point x in $\mathbb{R}^d$, with Cartesian coordinates x_i is fully specified by the data of $d+1$ scalar values $\lambda_j = \lambda_j(x)$ that are solutions of the linear system:

$$\left\{ \sum_{j=1}^{d+1} a_{ij}\lambda_j = x_i \text{ with } \sum_{j=1}^{d+1} \lambda_j = 1 \right.,$$

whose matrix is $\mathcal{A}$. The $\lambda_j(x)$ are the *barycentric coordinates* of point x with respect to the points a_j.

– *a vertex, an edge or a face (for $d = 3$).*

More generally, in d dimensions, such an intersection must be a k-face[2], for $k = -1, ..., d-1$, d being the spatial dimension.

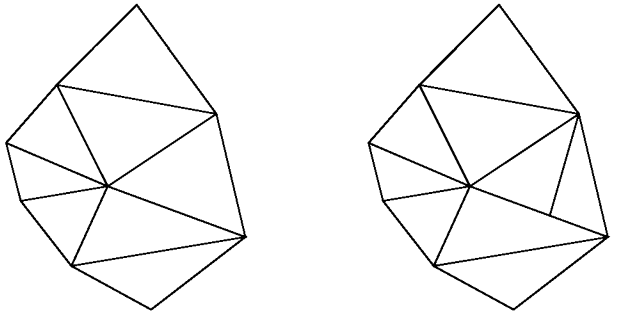

Figure 1.1: *Conformal triangles (left-hand side) and non-conformal triangles (right-hand side). Note the vertex located on one edge in this case.*

Remark 1.1 *For the moment, we are not concerned with the existence and possibly uniqueness of such a triangulation for a given set of points. Nevertheless, a theorem of existence will be provided below and, based on some specific assumptions, the particular case of a Delaunay triangulation will be described.*

Euler characteristics. The Euler formula, and its extensions, the Dehn-Sommerville relationships, relate the number of k-faces ($k = 0, ..., d-1$) in a triangulation of Ω. Such formula can be used to check the topological validity of a given mesh or also for other purposes, such as the determination of the genus of a surface.

Definition 1.3 *The Euler characteristics of a triangulation $\mathcal{T}_r$, is the alterned summation:*

$$\chi = \sum_{k=0}^{d} (-1)^k n_k \,, \tag{1.1}$$

where $n_k, k = 0, .., d$ denotes the number of the k-faces in the triangulation.

When the triangulation is homotopic to the topological ball, its characteristic is 1. If the triangulation is homeomorphic to the topological sphere, its Euler characteristic is $1 + (-1)^d$. In two dimensions, the following relation holds:

$$nv - ne + nt = 2 - c \,,$$

where nv, ne and nt are respectively the number of vertices, edges and triangles in the triangulation, c corresponds to the number of connected components of

[2] A (-1)-face is the empty set, a 0-face is a vertex, a 1-face is an edge, a k-face is in fact a k-simplex with $k < d$, d being the spatial dimension.

the boundary of Ω. More precisely, if the triangulation includes no hole, then $nv - ne + nt = 1$. In three dimensions, the above formula becomes:

$$nv - ne + nf - nt = 2 - 2g\,,$$

where nf is the number of faces, nt the number of tets and g stands for the *genus* of the surface (i.e., the number of holes) of the triangulation. Thus, a triangulation of a closed surface is such that $nv - ne + nf = 2$.

Delaunay triangulation. Among the different possible types of triangulations, the Delaunay triangulation is of great interest. Let us recall that $\mathcal{S}$ is a set (a cloud) of points (sites) and that Ω is $Conv(\mathcal{S})$, the convex hull of $\mathcal{S}$.

Definition 1.4 *$\mathcal{T}_r$ is the* Delaunay triangulation *of Ω if the open discs (balls) circumscribed to any of its elements does not contain any vertex of $\mathcal{S}$.*

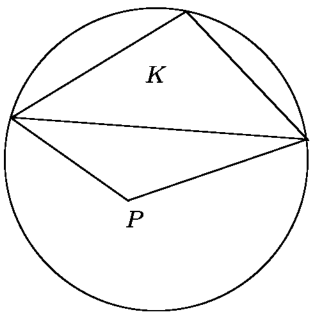

Figure 1.2: *The empty sphere criterion is violated, the disc of K encloses the point P. Similarly, the circumdisc of the triangle with vertex P includes the vertex of triangle K opposite the common edge (the criterion is symmetric for any pair of adjacent elements).*

This criterion, the so-called *empty sphere criterion* or *Delaunay criterion*, means that all open balls associated with all elements do not contain any vertex, a closed ball containing the vertices of the element under consideration only. This is the main characterization of the Delaunay triangulation. The Delaunay criterion leads to several other characteristics of any Delaunay triangulation. Figure 1.2 shows an example of an element K which does not meet the Delaunay criterion.

A basic theoretical issue follows.

Theorem 1.1 *There exists a unique Delaunay triangulation of a set of points.*

The proof is evident by involving the duality with the Voronoï diagram associated with the set of points (cf. Chapter 7). The existence is then immediate and the uniqueness is achieved as the points are assumed in general position[3] if one wishes to have a simplicial triangulation. Otherwise, the following remark holds.

[3] A set of points is said to be in general position if there is no configuration of more than three points that are co-circular (more than four co-spherical points) such that the corresponding open disk (ball) is empty.

Remark 1.2 *In the case of more than three co-circular (resp. four co-spherical) points, a circle (resp. sphere) exists enclosing these points. If the related disk (resp. ball) is empty, the Delaunay triangulation exists but contains non-simplicial elements such as polygons (resp. polyhedra).*

Hence, the uniqueness holds if non-simplicial elements are allowed while if the latter are subdivided by means of simplices, several solutions can be found. Nevertheless, while it may be excessive, we will continue to speak of **the** *Delaunay triangulation by observing that all any partitions of a non-simplicial element are equivalent after swapping*[4] *a k-face.*

A brief digression. The notion of a Voronoï diagram (though it had yet to be called as such!) first appeared in the work of the French philosopher R. Descartes (1596-1650) who introduced this notion in 1644 in his *Principia Philosophiae*, which aimed to give a mathematical description of the arrangement of matter in the solar system. In 1850, G. Dirichlet (1805-1859) studied this idea in two and three dimensions and this diagram came to be called the *Dirichlet tessellation* [Dirichlet-1850]. However, its definitive name came after M.G. Voronoï (1868-1908), who generalized these results in d dimensions [Voronoï-1908].

Nature provides numerous examples of arrangements and quasi-regular paving which bear a strange resemblance to Voronoï diagrams. Figure 1.3 illustrates some of these typical arrangements[5].

Constrained triangulation. Provided a set of points and, in addition, a set of edges (resp. edges and faces in three dimensions), an important problem is to ensure the existence of these edges (resp. these edges and faces) in a triangulation. In the following, *Const* denotes a set of such entities.

Definition 1.5 *$\mathcal{T}_r$ is a constrained triangulation of Ω for Const if all and any element of Const is an entity of $\mathcal{T}_r$.*

In particular, a constrained triangulation[6] can satisfy the Delaunay criterion locally, except in some neighborhood of the constraints.

Remark 1.3 *As above, provided a set of points and a constraint, we are not concerned here with the existence of a solution triangulation.*

[4] A 2-face swap (flip) consists of replacing the diagonal of the convex quadrilateral made up of two adjacent triangles by the alternate configuration, see Chapter 18 for the precise definition.

[5] Given a set of geometric objects, an arrangement is a covering-up of the space by means of the regions (cells) formed by the given objects and their (potential) intersections.

[6] Whereas a *constrained Delaunay triangulation* in two dimensions is a triangulation which satisfies the empty sphere criterion, where a open ball can contain a vertex in the case where the latter is not seen, due to a constrained edge, by all the vertices of the considered element. In other words, a constrained entity exists which separates the above vertices and the others.

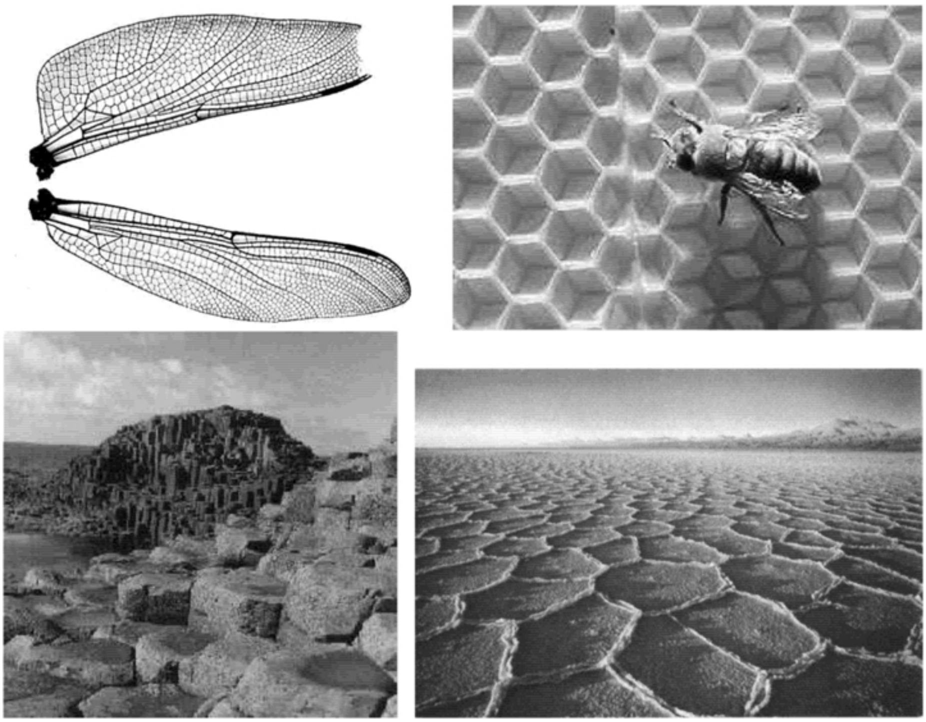

Figure 1.3: *Top, the wings of a dragonfly (doc. A. LeBéon) show an alveolar structure apparently close to a Voronoï diagram (left-hand side) and one of the more representative examples of regular paving (consisting of hexagonal cells) is that of a bee's nest (right-hand side). Bottom, two examples of natural arrangements. Left-hand side: the basaltic rock site of the Giant's Causeway, Co Antrim, Northern Ireland (photo credit: John Hinde Ltd.). Right-hand side, desert region of Atacama (Chile), the drying earth forms patterns close to Voronoï cells.*

1.2 Mesh, mesh element, finite element mesh

Now we turn to a different problem. Let Ω be a closed bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. The question is how to construct a conforming triangulation of this domain. Such a triangulation will be referred to as a *mesh* of Ω and will be denoted by $\mathcal{T}_r$ or $\mathcal{T}_h$ for reasons that will be made clear in the following. Thus,

Definition 1.6 *$\mathcal{T}_h$ is a mesh of Ω if*

- $(H1)$ $\quad \Omega = \overset{\circ}{\overline{\bigcup_{K \in \mathcal{T}_h} K}}\,.$

- $(H2)$ *The interior of every element K in $\mathcal{T}_h$ is non-empty.*

- $(H3)$ *The intersection of the interior of two elements is empty.*

Condition $(H2)$ is clearly not verified for a beam element for instance. Condition $(H3)$ avoids element overlapping. In contrast to the definition of a triangulation, Condition $(H0)$ is no longer assumed, which means that the vertices are not, in general, given *a priori* (see hereafter) and, in $(H1)$, the K's are not necessarily simplices.

Most computational schemes using a mesh as a spatial support assume that this mesh is conforming (although, this property is not strictly necessary for some solution methods).

Definition 1.7 *$\mathcal{T}_h$ is a conformal mesh of Ω if Definition (1.6) holds and*

- $(H4)$ *the intersection of two elements in $\mathcal{T}_h$ is either the empty set, a vertex, an edge or a face ($d = 3$).*

Clearly, the set of definitions related to a triangulation is again met. There is a fundamental difference between a triangulation and a mesh. A triangulation is a covering-up of the convex hull of a given set of points which, in general, is composed of simplicial elements. A mesh is a covering-up of a given domain defined, in most of the applications, via a given discretization of its boundary, this covering-up being composed of possibly non simplicial elements. On the other hand, at least two new problems occur, namely:

- the respect or *enforcement*, in some sense, of the boundary of the domain so that the triangulation is a constrained triangulation,
- the necessity of *constructing* the set of points which will define the vertices of the mesh. Usually the boundary points of the given boundary discretization are given as sole input and field points must be explicitly created.

Remark 1.4 *For a boundary discretization defining a domain, the existence of a mesh conforming to this discretization holds in two dimensions but is still, at least from a computer point of view, a delicate question in three dimensions.*

Remark 1.5 *In the finite element method, the meshes*[7] *are generally denoted by $\mathcal{T}_h$, where the index h of the notation refers to the diameters of the elements in the mesh, these quantities being used in error bound theorems.*

As previously mentioned, a mesh can be composed of elements of different geometric natures. A mesh consists of a finite number of segments in one dimension, segments, triangles and quadrilaterals (quads for short) in two dimensions and the above elements, tetrahedra (tets), pentahedra and hexahedra (hexes) in three dimensions. The mesh elements must generally satisfy some specific properties depending on the application involved.

Meshes can be classified into three main classes according to their *connectivity*.

[7] It should be noted that people with a finite element background use the term triangulation and use the term mesh synonymously.

Definition 1.8 *The connectivity of a mesh is the definition of the connection between its vertices.*

Then, following this definition

Definition 1.9 *A mesh is called structured (resp. unstructured) if its connectivity is of the finite difference type (resp. any other type).*

A structured mesh can be termed as a *grid*[8]. In two dimensions, a grid element is a quadrilateral while, in three dimensions, a grid consists of hexahedra. The connectivity between nodes is of the type (i, j, k), i.e., assuming the indices of a given node, the node with indices (i, j, k) has the node with indices $((i-1), j, k)$ as its "left" neighbor and that with indices $((i+1), j, k)$ as its "right" neighbor; this kind of mesh is convenient for geometries for which such properties are suitable, i.e., for generalized quadrilateral or hexahedral configurations.

Remark 1.6 *Peculiar meshes other than quad or hex meshes could have a structured connectivity. For instance, one can consider a classical grid of quads where each of them are subdivided into two triangles using the same subdivision pattern.*

Such a mesh is usually composed of triangles (tetrahedra) but can also be a set of quadrilaterals (hexahedra) or, more generally, a combination of elements of a different geometric nature. Note that quad or hex unstructured meshes are such that the internal vertices may be shared by more than 4 (8) elements (unlike the case of structured meshes).

For completeness, we introduce two more definitions.

Definition 1.10 *A mesh is said to be* mixed *if it includes some elements of a different geometric nature.*

Definition 1.11 *A mesh is said to be* hybrid *if it includes some elements with a different spatial dimension.*

A mixed mesh, in two dimensions, is composed of triangles and quads. A hybrid mesh, again in two dimensions, is clearly a mixed mesh but, for instance, includes some triangles together with some segments.

To complete this classification, a mesh may be *manifold* or not. This point concerns only surface meshes.

Definition 1.12 *A (conformal) surface mesh is called manifold if its internal edges are shared by exactly two elements or only one element in the case of a boundary edge for an open surface.*

Otherwise, the surface mesh is said to be *non-manifold.* This is the case of surface meshes which include stiffeners or which have two or more connected components.

[8]Note that some authors use the term "grid" to refer to any kind of mesh whatever its connectivity.

Mesh element

The elements are the basic components of a mesh. An element is defined by its geometric nature (triangle, quadrilateral, etc.) and a list of vertices. This list, enriched with some conventions (see hereafter), allows the complete definition of an element, including the definition of its edges and faces (in three dimensions).

Definition 1.13 *The connectivity of a mesh element is the definition of the connections between the vertices at the element level.*

This connectivity, the local equivalent of the mesh connectivity, makes the description of the *topology* of the element possible.

Definition 1.14 *The topology of a mesh element is a definition of the relationships between its faces, edges and vertices.*

Triangle connectivity and topology. For convenient purposes, the (local) numbering of vertices and edges is pre-defined in such a way that some properties are implicitly induced[9]. This definition is only a convention leading to implicit properties. In particular, a ordered numbering of the vertices enables us to compute the surface area of a triangle with a positive, or directional, sense. It also allows us to evaluate directional normals for each edge.

In the case of a triangle with connectivity $[1,2,3]$, the first vertex (1) having been chosen, the numbering of the others is deduced counterclockwise. Then the topology can be well defined by means of the edge definition:

- edge [1] runs from vertex (1) to vertex (2),
- edge [2] : (2) $\rightarrow$ (3),
- edge [3] : (3) $\rightarrow$ (1),

or alternatively,

- edge [1] is opposite vertex (1), it runs from vertex (2) to vertex (3),
- edge [2] : (3) $\rightarrow$ (1),
- edge [3] : (1) $\rightarrow$ (2).

Once a topology has been chosen, all mesh elements must conform to this rule. Such an implicit definition will be a source of simplicity hereafter, avoiding explicit definitions at the element level during the computational step, as mentioned earlier.

Usual element connectivities and topologies. Elements other than triangles are now defined in terms of the two above definitions.

- The segment: $[1,2]$, $(1) \rightarrow (2)$.
- The quadrilateral: $[1,2,3,4]$ with a numbering as for the triangle,

$$\begin{array}{ll} \text{edge } [1] : (1) \rightarrow (2) & \text{edge } [2] : (2) \rightarrow (3) \\ \text{edge } [3] : (3) \rightarrow (4) & \text{edge } [4] : (4) \rightarrow (1) \end{array}$$

[9] Given a vertex numbering (index) based on an implicit definition results in implicit definitions for both the edges and the faces, thus avoiding an explicit definition of these entities at the element level, which would be not unique and memory consuming.

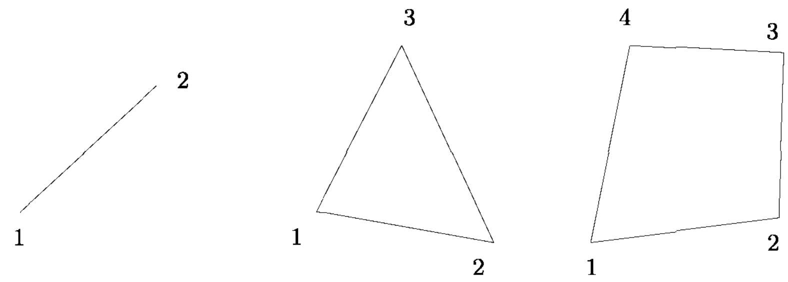

Figure 1.4: *Local vertex numbering of segment, triangle and quadrilateral, given the first vertex index.*

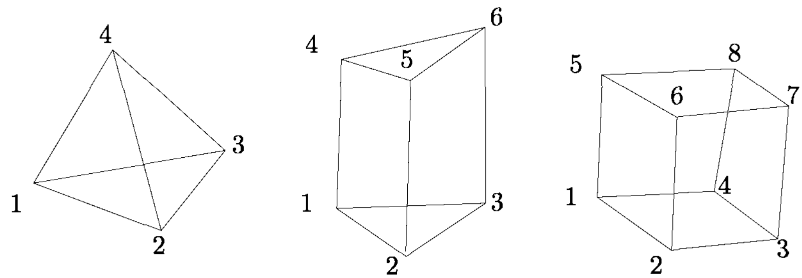

Figure 1.5: *Tetrahedron, pentahedron and hexahedron.*

- The tetrahedron[10]: $[1,2,3,4]$ with $(\vec{12},\vec{13},\vec{14})$ assumed to be positive with, for the edges:

edge [1] : (1) → (2) edge [2] : (2) → (3) edge [3] : (3) → (1)
edge [4] : (1) → (4) edge [5] : (2) → (4) edge [6] : (3) → (4)

and, for the faces:

face [1] : (1) (3) (2) face [2] : (1) (4) (3)
face [3] : (1) (2) (4) face [4] : (2) (3) (4)

- The pentahedron: $[1,2,3,4,5,6]$ with $(\vec{12},\vec{13},\vec{14})$ assumed to be positive, with, for the edges:

edge [1] : (1) → (2) edge [2] : (2) → (3) edge [3] : (3) → (1)
edge [4] : (1) → (4) edge [5] : (2) → (5) edge [6] : (3) → (6)
edge [7] : (4) → (5) edge [8] : (5) → (6) edge [9] : (6) → (4)

and, for the faces:

face [1] : (1) (3) (2) face [2] : (1) (4) (6) (3)
face [3] : (1) (2) (5) (4) face [4] : (4) (5) (6)
face [5] : (2) (3) (5) (6)

[10]Similarly to the triangle, an alternative definition also suits well where face [i] is opposite vertex (i). Actually, the latter convention leads to greater simplicity.