Markov Processes

Characterization and Convergence

STEWART N. ETHIER
THOMAS G. KURTZ

WILEY-
INTERSCIENCE

A JOHN WILEY & SONS, INC., PUBLICATION

This Page Intentionally Left Blank

Markov Processes

This Page Intentionally Left Blank

Markov Processes

Characterization and Convergence

STEWART N. ETHIER
THOMAS G. KURTZ

WILEY-
INTERSCIENCE

A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication is available.

ISBN-13 978-0-471-76986-6
ISBN-10 0-471-76986-X

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

PREFACE

The original aim of this book was a discussion of weak approximation results for Markov processes. The scope has widened with the recognition that each technique for verifying weak convergence is closely tied to a method of characterizing the limiting process. The result is a book with perhaps more pages devoted to characterization than to convergence.

The Introduction illustrates the three main techniques for proving convergence theorems applied to a single problem. The first technique is based on operator semigroup convergence theorems. Convergence of generators (in an appropriate sense) implies convergence of the corresponding semigroups, which in turn implies convergence of the Markov processes. Trotter's original work in this area was motivated in part by diffusion approximations. The second technique, which is more probabilistic in nature, is based on the martingale characterization of Markov processes as developed by Stroock and Varadhan. Here again one must verify convergence of generators, but weak compactness arguments and the martingale characterization of the limit are used to complete the proof. The third technique depends on the representation of the processes as solutions of stochastic equations, and is more in the spirit of classical analysis. If the equations "converge," then (one hopes) the solutions converge.

Although the book is intended primarily as a reference, problems are included in the hope that it will also be useful as a text in a graduate course on stochastic processes. Such a course might include basic material on stochastic processes and martingales (Chapter 2, Sections 1–6), an introduction to weak convergence (Chapter 3, Sections 1–9, omitting some of the more technical results and proofs), a development of Markov processes and martingale problems (Chapter 4, Sections 1–4 and 8), and the martingale central limit theorem (Chapter 7, Section 1). A selection of applications to particular processes could complete the course.

As an aid to the instructor of such a course, we include a flowchart for all proofs in the book. Thus, if one's goal is to cover a particular section, the chart indicates which of the earlier results can be skipped with impunity. (It also reveals that the course outline suggested above is not entirely self-contained.)

Results contained in standard probability texts such as Billingsley (1979) or Breiman (1968) are assumed and used without reference, as are results from measure theory and elementary functional analysis. Our standard reference here is Rudin (1974). Beyond this, our intent has been to make the book self-contained (an exception being Chapter 8). At points where this has not seemed feasible, we have included complete references, frequently discussing the needed material in appendixes.

Many people contributed toward the completion of this project. Cristina Costantini, Eimear Goggin, S. J. Sheu, and Richard Stockbridge read large portions of the manuscript and helped to eliminate a number of errors. Carolyn Birr, Dee Frana, Diane Reppert, and Marci Kurtz typed the manuscript. The National Science Foundation and the University of Wisconsin, through a Romnes Fellowship, provided support for much of the research in the book.

We are particularly grateful to our editor, Beatrice Shube, for her patience and constant encouragement. Finally, we must acknowledge our teachers, colleagues, and friends at Wisconsin and Michigan State, who have provided the stimulating environment in which ideas germinate and flourish. They contributed to this work in many uncredited ways. We hope they approve of the result.

STEWART N. ETHIER
THOMAS G. KURTZ

Salt Lake City, Utah
Madison, Wisconsin
August 1985

CONTENTS

INTRODUCTION

The development of any stochastic model involves the identification of properties and parameters that, one hopes, uniquely characterize a stochastic process. Questions concerning continuous dependence on parameters and robustness under perturbation arise naturally out of any such characterization. In fact the model may well be derived by some sort of limiting or approximation argument. The interplay between characterization and approximation or convergence problems for Markov processes is the central theme of this book. Operator semigroups, martingale problems, and stochastic equations provide approaches to the characterization of Markov processes, and to each of these approaches correspond methods for proving convergence results.

The processes of interest to us here always have values in a complete, separable metric space E, and almost always have sample paths in $D_E[0, \infty)$, the space of right continuous E-valued functions on $[0, \infty)$ having left limits. We give $D_E[0, \infty)$ the Skorohod topology (Chapter 3), under which it also becomes a complete, separable metric space. The type of convergence we are usually concerned with is convergence in distribution; that is, for a sequence of processes $\{X_n\}$ we are interested in conditions under which $\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$ for every $f \in \bar{C}(D_E[0, \infty))$. (For a metric space S, $\bar{C}(S)$ denotes the space of bounded continuous functions on S. Convergence in distribution is denoted by $X_n \Rightarrow X$.) As an introduction to the methods presented in this book we consider a simple but (we hope) illuminating example.

For each $n \geq 1$, define

(1) $$\lambda_n(x) = 1 + 3x\left(x - \frac{1}{n}\right), \qquad \mu_n(x) = 3x + x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right),$$

and let Y_n be a birth-and-death process in $\mathbb{Z}_+$ with transition probabilities satisfying

(2) $$P\{Y_n(t+h) = j+1 \mid Y_n(t) = j\} = n\lambda_n\left(\frac{j}{n}\right)h + o(h)$$

and

(3) $$P\{Y_n(t+h) = j-1 \mid Y_n(t) = j\} = n\mu_n\left(\frac{j}{n}\right)h + o(h)$$

as $h \to 0+$. In this process, known as the Schlögl model, $Y_n(t)$ represents the number of molecules at time t of a substance R in a volume n undergoing the chemical reactions

(4) $$R_0 \underset{3}{\overset{1}{\rightleftharpoons}} R, \qquad R_2 + 2R \underset{1}{\overset{3}{\rightleftharpoons}} 3R,$$

with the indicated rates. (See Chapter 11, Section 1.)

We rescale and renormalize letting

(5) $$X_n(t) = n^{1/4}(n^{-1}Y_n(n^{1/2}t) - 1), \qquad t \geq 0.$$

The problem is to show that X_n converges in distribution to a Markov process X to be characterized below.

The first method we consider is based on a semigroup characterization of X. Let $E_n = \{n^{1/4}(n^{-1}y - 1) : y \in \mathbb{Z}_+\}$, and note that

(6) $$T_n(t)f(x) \equiv E[f(X_n(t)) \mid X_n(0) = x]$$

defines a semigroup $\{T_n(t)\}$ on $B(E_n)$ with generator of the form

(7) $$\begin{aligned} G_n f(x) = {} & n^{3/2}\lambda_n(1 + n^{-1/4}x)\{f(x + n^{-3/4}) - f(x)\} \\ & + n^{3/2}\mu_n(1 + n^{-1/4}x)\{f(x - n^{-3/4}) - f(x)\}. \end{aligned}$$

(See Chapter 1.) Letting $\lambda(x) \equiv 1 + 3x^2$, $\mu(x) \equiv 3x + x^3$, and

(8) $$Gf(x) = 4f''(x) - x^3 f'(x),$$

a Taylor expansion shows that

(9) $$\begin{aligned} G_n f(x) = {} & Gf(x) + n^{3/2}\{\lambda_n(1 + n^{-1/4}x) - \lambda(1 + n^{-1/4}x)\}\{f(x + n^{-3/4}) - f(x)\} \\ & + n^{3/2}\{\mu_n(1 + n^{-1/4}x) - \mu(1 + n^{-1/4}x)\}\{f(x - n^{-3/4}) - f(x)\} \\ & + \lambda(1 + n^{-1/4}x)\int_0^1 (1-u)\{f''(x + un^{-3/4}) - f''(x)\}\,du \\ & + \mu(1 + n^{-1/4}x)\int_0^1 (1-u)\{f''(x - un^{-3/4}) - f''(x)\}\,du \\ & + \{(\lambda + \mu)(1 + n^{-1/4}x) - (\lambda + \mu)(1)\}\tfrac{1}{2}f''(x), \end{aligned}$$

for all $f \in C^2(\mathbb{R})$ with $f' \in C_c(\mathbb{R})$ and all $x \in E_n$. Consequently, for such f,

(10) $$\lim_{n\to\infty} \sup_{x \in E_n} |G_n f(x) - Gf(x)| = 0.$$

Now by Theorem 1.1 of Chapter 8,

(11) $$A \equiv \{(f, Gf): f \in C[-\infty, \infty] \cap C^2(\mathbb{R}), Gf \in C[-\infty, \infty]\}$$

is the generator of a Feller semigroup $\{T(t)\}$ on $C[-\infty, \infty]$. By Theorem 2.7 of Chapter 4 and Theorem 1.1 of Chapter 8, there exists a diffusion process X corresponding to $\{T(t)\}$, that is, a strong Markov process X with continuous sample paths such that

(12) $$E[f(X(t)) \mid \mathscr{F}_s^X] = T(t-s)f(X(s))$$

for all $f \in C[-\infty, \infty]$ and $t \geq s \geq 0$. ($\mathscr{F}_s^X = \sigma(X(u): u \leq s)$.)

To prove that $X_n \Rightarrow X$ (assuming convergence of initial distributions), it suffices by Corollary 8.7 of Chapter 4 to show that (10) holds for all f in a core D for the generator A, that is, for all f in a subspace D of $\mathscr{D}(A)$ such that A is the closure of the restriction of A to D. We claim that

(13) $$D \equiv \{f + g: f, g \in C^2(\mathbb{R}), f' \in C_c(\mathbb{R}), (x^2 g)' \in C_c(\mathbb{R})\}$$

is a core, and that (10) holds for all $f \in D$. To see that D is a core, first check that

(14) $$\mathscr{D}(A) = \{f \in C[-\infty, \infty] \cap C^2(\mathbb{R}): f'' \in \hat{C}(\mathbb{R}), x^3 f' \in C[-\infty, \infty]\}.$$

Then let $h \in C_c^2(\mathbb{R})$ satisfy $\chi_{[-1,1]} \leq h \leq \chi_{[-2,2]}$ and put $h_m(x) = h(x/m)$. Given $f \in \mathscr{D}(A)$, choose $g \in D$ with $(x^2 g)' \in C_c(\mathbb{R})$ and $x^3(f-g)' \in \hat{C}(\mathbb{R})$ and define

(15) $$f_m(x) = f(0) - g(0) + \int_0^x (f-g)'(y) h_m(y)\, dy.$$

Then $f_m + g \in D$ for each m, $f_m + g \to f$, and $G(f_m + g) \to Gf$.

The second method is based on the characterization of X as the solution of a martingale problem. Observe that

(16) $$f(X_n(t)) - \int_0^t G_n f(X_n(s))\, ds$$

is an $\{\mathscr{F}_t^{X_n}\}$-martingale for each $f \in B(E_n)$ with compact support. Consequently, if some subsequence $\{X_{n'}\}$ converges in distribution to X, then, by the continuous mapping theorem (Corollary 1.9 of Chapter 3) and Problem 7 of Chapter 7,

(17) $$f(X(t)) - \int_0^t Gf(X(s))\, ds$$

is an $\{\mathscr{F}_t^X\}$-martingale for each $f \in C_c^2(\mathbb{R})$, or in other words, X is a solution of the martingale problem for $\{(f, Gf): f \in C_c^2(\mathbb{R})\}$. But by Theorem 2.3 of Chapter 8, this property characterizes the distribution on $D_{\mathbb{R}}[0, \infty)$ of X. Therefore, Corollary 8.16 of Chapter 4 gives $X_n \Rightarrow X$ (assuming convergence of initial distributions), provided we can show that

$$\lim_{\alpha \to \infty} \overline{\lim_{n \to \infty}} P\left\{ \sup_{0 \le t \le T} |X_n(t)| \ge \alpha \right\} = 0, \qquad T > 0. \tag{18}$$

Let $\varphi(x) \equiv e^x + e^{-x}$, and check that there exist constants $C_{n,\alpha} > 0$ such that $G_n \varphi \le C_{n,\alpha} \varphi$ on $[-\alpha, \alpha]$ for each $n \ge 1$ and $\alpha > 0$, and $\overline{\lim}_{\alpha \to \infty} \overline{\lim}_{n \to \infty} C_{n,\alpha} < \infty$. Letting $\tau_{n,\alpha} = \inf\{t \ge 0: |X_n(t)| \ge \alpha\}$, we have

$$\begin{aligned} e^{-C_{n,\alpha} T} \inf_{|y| \ge \alpha} \varphi(y) P\left\{ \sup_{0 \le t \le T} |X_n(t)| \ge \alpha \right\} \tag{19} \\ \le E[\exp\{-C_{n,\alpha}(\tau_{n,\alpha} \wedge T)\} \varphi(X_n(\tau_{n,\alpha} \wedge T))] \\ \le E[\varphi(X_n(0))] \end{aligned}$$

by Lemma 3.2 of Chapter 4 and the optional sampling theorem. An additional (mild) assumption on the initial distributions therefore guarantees (18).

Actually we can avoid having to verify (18) by observing that the uniform convergence of $G_n f$ to Gf for $f \in C_c^2(\mathbb{R})$ and the uniqueness for the limiting martingale problem imply (again by Corollary 8.16 of Chapter 4) that $X_n \Rightarrow X$ in $D_{\mathbb{R}^\Delta}[0, \infty)$ where $\mathbb{R}^\Delta$ denotes the one-point compactification of $\mathbb{R}$. Convergence in $D_{\mathbb{R}}[0, \infty)$ then follows from the fact that X_n and X have sample paths in $D_{\mathbb{R}}[0, \infty)$.

Both of the approaches considered so far have involved characterizations in terms of generators. We now consider methods based on stochastic equations. First, by Theorems 3.7 and 3.10 of Chapter 5, we can characterize X as the unique solution of the stochastic integral equation

$$X(t) = X(0) + 2\sqrt{2}W(t) - \int_0^t X(s)^3 \, ds, \tag{20}$$

where W is a standard, one-dimensional, Brownian motion. (In the present example, the term $2\sqrt{2}W(t)$ corresponds to the stochastic integral term.) A convergence theory can be developed using this characterization of X, but we do not do so here. The interested reader is referred to Kushner (1974).

The final approach we discuss is based on a characterization of X involving random time changes. We observe first that Y_n satisfies

$$Y_n(t) = Y_n(0) + N_+\left(n \int_0^t \lambda_n(n^{-1} Y_n(s)) \, ds\right) - N_-\left(n \int_0^t \mu_n(n^{-1} Y_n(s)) \, ds\right), \tag{21}$$

where N_+ and N_- are independent, standard (parameter 1), Poisson processes. Consequently, X_n satisfies

$$(22) \qquad X_n(t) = X_n(0) + n^{-3/4}\tilde{N}_+\left(n^{3/2}\int_0^t \lambda_n(1 + n^{-1/4}X_n(s))\,ds\right) - n^{-3/4}\tilde{N}_-\left(n^{3/2}\int_0^t \mu_n(1 + n^{-1/4}X_n(s))\,ds\right) + n^{3/4}\int_0^t (\lambda_n - \mu_n)(1 + n^{-1/4}X_n(s))\,ds,$$

where $\tilde{N}_+(u) = N_+(u) - u$ and $\tilde{N}_-(u) = N_-(u) - u$ are independent, centered, standard, Poisson processes. Now it is easy to see that

$$(23) \qquad (n^{-3/4}\tilde{N}_+(n^{3/2}\,\cdot\,), n^{-3/4}\tilde{N}_-(n^{3/2}\,\cdot\,)) \Rightarrow (W_+, W_-),$$

where W_+ and W_- are independent, standard, one-dimensional Brownian motions. Consequently, if some subsequence $\{X_{n'}\}$ converges in distribution to X, one might expect that

$$(24) \qquad X(t) = X(0) + W_+(4t) + W_-(4t) - \int_0^t X(s)^3\,ds.$$

(In this simple example, (20) and (24) are equivalent, but they will not be so in general.) Clearly, (24) characterizes X, and using the estimate (18) we conclude $X_n \Rightarrow X$ (assuming convergence of initial distributions) from Theorem 5.4 of Chapter 6.

For a further discussion of the Schlögl model and related models see Schlögl (1972) and Malek-Mansour et al. (1981). The martingale proof of convergence is from Costantini and Nappo (1982), and the time change proof is from Kurtz (1981c).

Chapters 4–7 contain the main characterization and convergence results (with the emphasis in Chapters 5 and 7 on diffusion processes). Chapters 1–3 contain preliminary material on operator semigroups, martingales, and weak convergence, and Chapters 8–12 are concerned with applications.

1 OPERATOR SEMIGROUPS

Operator semigroups provide a primary tool in the study of Markov processes. In this chapter we develop the basic background for their study and the existence and approximation results that are used later as the basis for existence and approximation theorems for Markov processes. Section 1 gives the basic definitions, and Section 2 the Hille–Yosida theorem, which characterizes the operators that are generators of semigroups. Section 3 concerns the problem of verifying the hypotheses of this theorem, and Sections 4 and 5 are devoted to generalizations of the concept of the generator. Sections 6 and 7 present the approximation and perturbation results.

Throughout the chapter, L denotes a real Banach space with norm $\|\cdot\|$.

1. DEFINITIONS AND BASIC PROPERTIES

A one-parameter family $\{T(t): t \geq 0\}$ of bounded linear operators on a Banach space L is called a *semigroup* if $T(0) = I$ and $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$. A semigroup $\{T(t)\}$ on L is said to be *strongly continuous* if $\lim_{t \to 0} T(t)f = f$ for every $f \in L$; it is said to be a *contraction* semigroup if $\|T(t)\| \leq 1$ for all $t \geq 0$.

Given a bounded linear operator B on L, define

$$e^{tB} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k B^k, \qquad t \geq 0. \tag{1.1}$$

A simple calculation gives $e^{(s+t)B} = e^{sB}e^{tB}$ for all $s, t \geq 0$, and hence $\{e^{tB}\}$ is a semigroup, which can easily be seen to be strongly continuous. Furthermore we have

$$\|e^{tB}\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} t^k \|B^k\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} t^k \|B\|^k = e^{t\|B\|}, \qquad t \geq 0. \tag{1.2}$$

An inequality of this type holds in general for strongly continuous semigroups.

1.1 Proposition Let $\{T(t)\}$ be a strongly continuous semigroup on L. Then there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t}, \qquad t \geq 0. \tag{1.3}$$

Proof. Note first that there exist constants $M \geq 1$ and $t_0 > 0$ such that $\|T(t)\| \leq M$ for $0 \leq t \leq t_0$. For if not, we could find a sequence $\{t_n\}$ of positive numbers tending to zero such that $\|T(t_n)\| \to \infty$, but then the uniform boundedness principle would imply that $\sup_n \|T(t_n)f\| = \infty$ for some $f \in L$, contradicting the assumption of strong continuity. Now let $\omega = t_0^{-1} \log M$. Given $t \geq 0$, write $t = kt_0 + s$, where k is a nonnegative integer and $0 \leq s < t_0$; then

$$\|T(t)\| = \|T(s)T(t_0)^k\| \leq MM^k \leq MM^{t/t_0} = Me^{\omega t}. \tag{1.4}$$

□

1.2 Corollary Let $\{T(t)\}$ be a strongly continuous semigroup on L. Then, for each $f \in L$, $t \to T(t)f$ is a continuous function from $[0, \infty)$ into L.

Proof. Let $f \in L$. By Proposition 1.1, if $t \geq 0$ and $h \geq 0$, then

$$\begin{aligned} \|T(t+h)f - T(t)f\| &= \|T(t)[T(h)f - f]\| \\ &\leq Me^{\omega t}\|T(h)f - f\|, \end{aligned} \tag{1.5}$$

and if $0 \leq h \leq t$, then

$$\begin{aligned} \|T(t-h)f - T(t)f\| &= \|T(t-h)[T(h)f - f]\| \\ &\leq Me^{\omega t}\|T(h)f - f\|. \end{aligned} \tag{1.6}$$

□

1.3 Remark Let $\{T(t)\}$ be a strongly continuous semigroup on L such that (1.3) holds, and put $S(t) = e^{-\omega t}T(t)$ for each $t \geq 0$. Then $\{S(t)\}$ is a strongly continuous semigroup on L such that

$$\|S(t)\| \leq M, \qquad t \geq 0. \tag{1.7}$$

In particular, if $M = 1$, then $\{S(t)\}$ is a strongly continuous contraction semigroup on L.

Let $\{S(t)\}$ be a strongly continuous semigroup on L such that (1.7) holds, and define the norm $|||\cdot|||$ on L by

$$|||f||| = \sup_{t \ge 0} \|S(t)f\|. \tag{1.8}$$

Then $\|f\| \le |||f||| \le M\|f\|$ for each $f \in L$, so the new norm is equivalent to the original norm; also, with respect to $|||\cdot|||$, $\{S(t)\}$ is a strongly continuous contraction semigroup on L.

Most of the results in the subsequent sections of this chapter are stated in terms of strongly continuous contraction semigroups. Using these reductions, however, many of them can be reformulated in terms of noncontraction semigroups. □

A (possibly unbounded) *linear operator* A on L is a linear mapping whose domain $\mathscr{D}(A)$ is a subspace of L and whose range $\mathscr{R}(A)$ lies in L. The *graph* of A is given by

$$\mathscr{G}(A) = \{(f, Af): f \in \mathscr{D}(A)\} \subset L \times L. \tag{1.9}$$

Note that $L \times L$ is itself a Banach space with componentwise addition and scalar multiplication and norm $\|(f, g)\| = \|f\| + \|g\|$. A is said to be *closed* if $\mathscr{G}(A)$ is a closed subspace of $L \times L$.

The *(infinitesimal) generator* of a semigroup $\{T(t)\}$ on L is the linear operator A defined by

$$Af = \lim_{t \to 0} \frac{1}{t}\{T(t)f - f\}. \tag{1.10}$$

The domain $\mathscr{D}(A)$ of A is the subspace of all $f \in L$ for which this limit exists.

Before indicating some of the properties of generators, we briefly discuss the calculus of Banach space-valued functions.

Let Δ be a closed interval in $(-\infty, \infty)$, and denote by $C_L(\Delta)$ the space of continuous functions $u: \Delta \to L$. Let $C_L^1(\Delta)$ be the space of continuously differentiable functions $u: \Delta \to L$.

If Δ is the finite interval $[a, b]$, $u: \Delta \to L$ is said to be *(Riemann) integrable* over Δ if $\lim_{\delta \to 0} \sum_{k=1}^{n} u(s_k)(t_k - t_{k-1})$ exists, where $a = t_0 \le s_1 \le t_1 \le \cdots \le t_{n-1} \le s_n \le t_n = b$ and $\delta = \max(t_k - t_{k-1})$; the limit is denoted by $\int_\Delta u(t)\,dt$ or $\int_a^b u(t)\,dt$. If $\Delta = [a, \infty)$, $u: \Delta \to L$ is said to be *integrable* over Δ if $u|_{[a,b]}$ is integrable over $[a, b]$ for each $b \ge a$ and $\lim_{b \to \infty} \int_a^b u(t)\,dt$ exists; again, the limit is denoted by $\int_\Delta u(t)\,dt$ or $\int_a^\infty u(t)\,dt$.

We leave the proof of the following lemma to the reader (Problem 3).

1.4 Lemma **(a)** If $u \in C_L(\Delta)$ and $\int_\Delta \|u(t)\| \, dt < \infty$, then u is integrable over Δ and

$$\left\| \int_\Delta u(t) \, dt \right\| \le \int_\Delta \|u(t)\| \, dt. \tag{1.11}$$

In particular, if Δ is the finite interval $[a, b]$, then every function in $C_L(\Delta)$ is integrable over Δ.

(b) Let B be a closed linear operator on L. Suppose that $u \in C_L(\Delta)$, $u(t) \in \mathscr{D}(B)$ for all $t \in \Delta$, $Bu \in C_L(\Delta)$, and both u and Bu are integrable over Δ. Then $\int_\Delta u(t) \, dt \in \mathscr{D}(B)$ and

$$B \int_\Delta u(t) \, dt = \int_\Delta Bu(t) \, dt. \tag{1.12}$$

(c) If $u \in C_L^1[a, b]$, then

$$\int_a^b \frac{d}{dt} u(t) \, dt = u(b) - u(a). \tag{1.13}$$

1.5 Proposition Let $\{T(t)\}$ be a strongly continuous semigroup on L with generator A.

(a) If $f \in L$ and $t \ge 0$, then $\int_0^t T(s)f \, ds \in \mathscr{D}(A)$ and

$$T(t)f - f = A \int_0^t T(s)f \, ds. \tag{1.14}$$

(b) If $f \in \mathscr{D}(A)$ and $t \ge 0$, then $T(t)f \in \mathscr{D}(A)$ and

$$\frac{d}{dt} T(t)f = AT(t)f = T(t)Af. \tag{1.15}$$

(c) If $f \in \mathscr{D}(A)$ and $t \ge 0$, then

$$T(t)f - f = \int_0^t AT(s)f \, ds = \int_0^t T(s)Af \, ds. \tag{1.16}$$

Proof. **(a)** Observe that

$$\begin{aligned} \frac{1}{h}[T(h) - I] \int_0^t T(s)f \, ds &= \frac{1}{h} \int_0^t [T(s + h)f - T(s)f] \, ds \\ &= \frac{1}{h} \left\{ \int_h^{t+h} T(s)f \, ds - \int_0^t T(s)f \, ds \right\} \\ &= \frac{1}{h} \int_t^{t+h} T(s)f \, ds - \frac{1}{h} \int_0^h T(s)f \, ds \end{aligned} \tag{1.17}$$

for all $h > 0$, and as $h \to 0$ the right side of (1.17) converges to $T(t)f - f$.

(b) Since

$$\frac{1}{h}[T(t+h)f - T(t)f] = A_h T(t)f = T(t)A_h f \tag{1.18}$$

for all $h > 0$, where $A_h = h^{-1}[T(h) - I]$, it follows that $T(t)f \in \mathscr{D}(A)$ and $(d/dt)^+ T(t)f = AT(t)f = T(t)Af$. Thus, it suffices to check that $(d/dt)^- T(t)f = T(t)Af$ (assuming $t > 0$). But this follows from the identity

$$\frac{1}{-h}[T(t-h)f - T(t)f] - T(t)Af \tag{1.19}$$
$$= T(t-h)[A_h - A]f + [T(t-h) - T(t)]Af,$$

valid for $0 < h \leq t$.

(c) This is a consequence of (b) and Lemma 1.4(c). □

1.6 Corollary If A is the generator of a strongly continuous semigroup $\{T(t)\}$ on L, *then* $\mathscr{D}(A)$ *is dense in* L *and* A *is closed.*

Proof. Since $\lim_{t \to 0+} t^{-1} \int_0^t T(s)f\,ds = f$ for every $f \in L$, Proposition 1.5(a) implies that $\mathscr{D}(A)$ is dense in L. To show that A is closed, let $\{f_n\} \subset \mathscr{D}(A)$ satisfy $f_n \to f$ and $Af_n \to g$. Then $T(t)f_n - f_n = \int_0^t T(s)Af_n\,ds$ for each $t > 0$, so, letting $n \to \infty$, we find that $T(t)f - f = \int_0^t T(s)g\,ds$. Dividing by t and letting $t \to 0$, we conclude that $f \in \mathscr{D}(A)$ and $Af = g$. □

2. THE HILLE–YOSIDA THEOREM

Let A be a closed linear operator on L. If, for some real λ, $\lambda - A$ ($\equiv \lambda I - A$) is one-to-one, $\mathscr{R}(\lambda - A) = L$, and $(\lambda - A)^{-1}$ is a bounded linear operator on L, then λ is said to belong to the *resolvent set* $\rho(A)$ of A, and $R_\lambda = (\lambda - A)^{-1}$ is called the *resolvent* (at λ) of A.

2.1 Proposition Let $\{T(t)\}$ be a strongly continuous contraction semigroup on L with generator A. Then $(0, \infty) \subset \rho(A)$ and

$$(\lambda - A)^{-1}g = \int_0^\infty e^{-\lambda t}T(t)g\,dt \tag{2.1}$$

for all $g \in L$ and $\lambda > 0$.

Proof. Let $\lambda > 0$ be arbitrary. Define U_λ on L by $U_\lambda g = \int_0^\infty e^{-\lambda t}T(t)g\,dt$. Since

$$\|U_\lambda g\| \leq \int_0^\infty e^{-\lambda t}\|T(t)g\|\,dt \leq \lambda^{-1}\|g\| \tag{2.2}$$

for each $g \in L$, U_λ is a bounded linear operator on L. Now given $g \in L$,

$$\text{(2.3)} \qquad \frac{1}{h}[T(h) - I]U_\lambda g = \frac{1}{h}\int_0^\infty e^{-\lambda t}[T(t+h)g - T(t)g]\,dt$$

$$= \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda t}T(t)g\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}T(t)g\,dt$$

for every $h > 0$, so, letting $h \to 0$, we find that $U_\lambda g \in \mathscr{D}(A)$ and $AU_\lambda g = \lambda U_\lambda g - g$, that is,

$$\text{(2.4)} \qquad (\lambda - A)U_\lambda g = g, \qquad g \in L.$$

In addition, if $g \in \mathscr{D}(A)$, then (using Lemma 1.4(b))

$$\text{(2.5)} \qquad U_\lambda Ag = \int_0^\infty e^{-\lambda t}T(t)Ag\,dt = \int_0^\infty A(e^{-\lambda t}T(t)g)\,dt$$

$$= A\int_0^\infty e^{-\lambda t}T(t)g\,dt = AU_\lambda g,$$

so

$$\text{(2.6)} \qquad U_\lambda(\lambda - A)g = g, \qquad g \in \mathscr{D}(A).$$

By (2.6), $\lambda - A$ is one-to-one, and by (2.4), $\mathscr{R}(\lambda - A) = L$. Also, $(\lambda - A)^{-1} = U_\lambda$ by (2.4) and (2.6), so $\lambda \in \rho(A)$. Since $\lambda > 0$ was arbitrary, the proof is complete. □

Let A be a closed linear operator on L. Since $(\lambda - A)(\mu - A) = (\mu - A)(\lambda - A)$ for all $\lambda, \mu \in \rho(A)$, we have $(\mu - A)^{-1}(\lambda - A)^{-1} = (\lambda - A)^{-1}(\mu - A)^{-1}$, and a simple calculation gives the *resolvent identity*

$$\text{(2.7)} \qquad R_\lambda R_\mu = R_\mu R_\lambda = (\lambda - \mu)^{-1}(R_\mu - R_\lambda), \qquad \lambda, \mu \in \rho(A).$$

If $\lambda \in \rho(A)$ and $|\lambda - \mu| < \|R_\lambda\|^{-1}$, then

$$\text{(2.8)} \qquad \sum_{n=0}^\infty (\lambda - \mu)^n R_\lambda^{n+1}$$

defines a bounded linear operator that is in fact $(\mu - A)^{-1}$. In particular, this implies that $\rho(A)$ is open in $\mathbb{R}$.

A linear operator A on L is said to be *dissipative* if $\|\lambda f - Af\| \geq \lambda\|f\|$ for every $f \in \mathscr{D}(A)$ and $\lambda > 0$.

2.2 Lemma Let A be a dissipative linear operator on L and let $\lambda > 0$. Then A is closed if and only if $\mathscr{R}(\lambda - A)$ is closed.

Proof. Suppose A is closed. If $\{f_n\} \subset \mathscr{D}(A)$ and $(\lambda - A)f_n \to h$, then the dissipativity of A implies that $\{f_n\}$ is Cauchy. Thus, there exists $f \in L$ such that

$f_n \to f$, and hence $Af_n \to \lambda f - h$. Since A is closed, $f \in \mathscr{D}(A)$ and $h = (\lambda - A)f$. It follows that $\mathscr{R}(\lambda - A)$ is closed.

Suppose $\mathscr{R}(\lambda - A)$ is closed. If $\{f_n\} \subset \mathscr{D}(A)$, $f_n \to f$, and $Af_n \to g$, then $(\lambda - A)f_n \to \lambda f - g$, which equals $(\lambda - A)f_0$ for some $f_0 \in \mathscr{D}(A)$. By the dissipativity of A, $f_n \to f_0$, and hence $f = f_0 \in \mathscr{D}(A)$ and $Af = g$. Thus, A is closed. □

2.3 Lemma Let A be a dissipative closed linear operator on L, and put $\rho^+(A) = \rho(A) \cap (0, \infty)$. If $\rho^+(A)$ is nonempty, then $\rho^+(A) = (0, \infty)$.

Proof. It suffices to show that $\rho^+(A)$ is both open and closed in $(0, \infty)$. Since $\rho(A)$ is necessarily open in $\mathbb{R}$, $\rho^+(A)$ is open in $(0, \infty)$. Suppose that $\{\lambda_n\} \subset \rho^+(A)$ and $\lambda_n \to \lambda > 0$. Given $g \in L$, let $g_n = (\lambda - A)(\lambda_n - A)^{-1}g$ for each n, and note that, because A is dissipative,

$$(2.9)\quad \lim_{n\to\infty} \|g_n - g\| = \lim_{n\to\infty} \|(\lambda - \lambda_n)(\lambda_n - A)^{-1}g\| \le \lim_{n\to\infty} \frac{|\lambda - \lambda_n|}{\lambda_n} \|g\| = 0.$$

Hence $\mathscr{R}(\lambda - A)$ is dense in L, but because A is closed and dissipative, $\mathscr{R}(\lambda - A)$ is closed by Lemma 2.2, and therefore $\mathscr{R}(\lambda - A) = L$. Using the dissipativity of A once again, we conclude that $\lambda - A$ is one-to-one and $\|(\lambda - A)^{-1}\| \le \lambda^{-1}$. It follows that $\lambda \in \rho^+(A)$, so $\rho^+(A)$ is closed in $(0, \infty)$, as required. □

2.4 Lemma Let A be a dissipative closed linear operator on L, and suppose that $\mathscr{D}(A)$ is dense in L and $(0, \infty) \subset \rho(A)$. Then the *Yosida approximation* A_λ of A, defined for each $\lambda > 0$ by $A_\lambda = \lambda A(\lambda - A)^{-1}$, has the following properties:

(a) For each $\lambda > 0$, A_λ is a bounded linear operator on L and $\{e^{tA_\lambda}\}$ is a strongly continuous contraction semigroup on L.

(b) $A_\lambda A_\mu = A_\mu A_\lambda$ for all $\lambda, \mu > 0$.

(c) $\lim_{\lambda\to\infty} A_\lambda f = Af$ for every $f \in \mathscr{D}(A)$.

Proof. For each $\lambda > 0$, let $R_\lambda = (\lambda - A)^{-1}$ and note that $\|R_\lambda\| \le \lambda^{-1}$. Since $(\lambda - A)R_\lambda = I$ on L and $R_\lambda(\lambda - A) = I$ on $\mathscr{D}(A)$, it follows that

$$(2.10)\qquad A_\lambda = \lambda^2 R_\lambda - \lambda I \quad \text{on} \quad L, \qquad \lambda > 0,$$

and

$$(2.11)\qquad A_\lambda = \lambda R_\lambda A \quad \text{on} \quad \mathscr{D}(A), \qquad \lambda > 0.$$

By (2.10), we find that, for each $\lambda > 0$, A_λ is bounded and

$$(2.12)\qquad \|e^{tA_\lambda}\| = e^{-t\lambda}\|e^{t\lambda^2 R_\lambda}\| \le e^{-t\lambda}e^{t\lambda^2\|R_\lambda\|} \le 1$$

for all $t \geq 0$, proving (a). Conclusion (b) is a consequence of (2.10) and (2.7). As for (c), we claim first that

$$\lim_{\lambda \to \infty} \lambda R_\lambda f = f, \qquad f \in L. \tag{2.13}$$

Noting that $\|\lambda R_\lambda f - f\| = \|R_\lambda A f\| \leq \lambda^{-1}\|Af\| \to 0$ as $\lambda \to \infty$ for each $f \in \mathscr{D}(A)$, (2.13) follows from the facts that $\mathscr{D}(A)$ is dense in L and $\|\lambda R_\lambda - I\| \leq 2$ for all $\lambda > 0$. Finally, (c) is a consequence of (2.11) and (2.13). □

2.5 Lemma If B and C are bounded linear operators on L such that $BC = CB$ and $\|e^{tB}\| \leq 1$ and $\|e^{tC}\| \leq 1$ for all $t \geq 0$, then

$$\|e^{tB}f - e^{tC}f\| \leq t\|Bf - Cf\| \tag{2.14}$$

for every $f \in L$ and $t \geq 0$.

Proof. The result follows from the identity

$$\begin{aligned} e^{tB}f - e^{tC}f &= \int_0^t \frac{d}{ds}[e^{sB}e^{(t-s)C}]f\,ds = \int_0^t e^{sB}(B - C)e^{(t-s)C}f\,ds \\ &= \int_0^t e^{sB}e^{(t-s)C}(B - C)f\,ds. \end{aligned} \tag{2.15}$$

(Note that the last equality uses the commutivity of B and C.) □

We are now ready to prove the Hille–Yosida theorem.

2.6 Theorem A linear operator A on L is the generator of a strongly continuous contraction semigroup on L if and only if:

- **(a)** $\mathscr{D}(A)$ is dense in L.
- **(b)** A is dissipative.
- **(c)** $\mathscr{R}(\lambda - A) = L$ for some $\lambda > 0$.

Proof. The necessity of the conditions (a)–(c) follows from Corollary 1.6 and Proposition 2.1. We therefore turn to the proof of sufficiency.

By (b), (c), and Lemma 2.2, A is closed and $\rho(A) \cap (0, \infty)$ is nonempty, so by Lemma 2.3, $(0, \infty) \subset \rho(A)$. Using the notation of Lemma 2.4, we define for each $\lambda > 0$ the strongly continuous contraction semigroup $\{T_\lambda(t)\}$ on L by $T_\lambda(t) = e^{tA_\lambda}$. By Lemmas 2.4(b) and 2.5,

$$\|T_\lambda(t)f - T_\mu(t)f\| \leq t\|A_\lambda f - A_\mu f\| \tag{2.16}$$

for all $f \in L$, $t \geq 0$, and λ, $\mu > 0$. Thus, by Lemma 2.4(c), $\lim_{\lambda \to \infty} T_\lambda(t)f$ exists for all $t \geq 0$, uniformly on bounded intervals, for all $f \in \mathscr{D}(A)$, hence for every $f \in \overline{\mathscr{D}(A)} = L$. Denoting the limit by $T(t)f$ and using the identity

$$(2.17) \quad T(s+t)f - T(s)T(t)f = [T(s+t) - T_\lambda(s+t)]f + T_\lambda(s)[T_\lambda(t) - T(t)]f + [T_\lambda(s) - T(s)]T(t)f,$$

we conclude that $\{T(t)\}$ is a strongly continuous contraction semigroup on L.

It remains only to show that A is the generator of $\{T(t)\}$. By Proposition 1.5(c),

$$(2.18) \qquad T_\lambda(t)f - f = \int_0^t T_\lambda(s)A_\lambda f\, ds$$

for all $f \in L$, $t \geq 0$, and $\lambda > 0$. For each $f \in \mathscr{D}(A)$ and $t \geq 0$, the identity

$$(2.19) \qquad T_\lambda(s)A_\lambda f - T(s)Af = T_\lambda(s)(A_\lambda f - Af) + [T_\lambda(s) - T(s)]\, Af,$$

together with Lemma 2.4(c), implies that $T_\lambda(s)A_\lambda f \to T(s)Af$ as $\lambda \to \infty$, uniformly in $0 \leq s \leq t$. Consequently, (2.18) yields

$$(2.20) \qquad T(t)f - f = \int_0^t T(s)Af\, ds$$

for all $f \in \mathscr{D}(A)$ and $t \geq 0$. From this we find that the generator B of $\{T(t)\}$ is an extension of A. But, for each $\lambda > 0$, $\lambda - B$ is one-to-one by the necessity of (b), and $\mathscr{R}(\lambda - A) = L$ since $\lambda \in \rho(A)$. We conclude that $B = A$, completing the proof. □

The above proof and Proposition 2.9 below yield the following result as a by-product.

2.7 Proposition Let $\{T(t)\}$ be a strongly continuous contraction semigroup *on L with generator A, and let A_λ be the Yosida approximation of A (defined in* Lemma 2.4). Then

$$(2.21) \qquad \|e^{tA_\lambda}f - T(t)f\| \leq t\|A_\lambda f - Af\|, \qquad f \in \mathscr{D}(A),\ t \geq 0,\ \lambda > 0,$$

so, for each $f \in L$, $\lim_{\lambda \to \infty} e^{tA_\lambda}f = T(t)f$ for all $t \geq 0$, uniformly on bounded intervals.

2.8 Corollary Let $\{T(t)\}$ be a strongly continuous contraction semigroup on L with generator A. For $M \subset L$, let

$$(2.22) \qquad \Lambda_M = \{\lambda > 0 \colon \lambda(\lambda - A)^{-1} \colon M \to M\}.$$

If either (a) M is a closed convex subset of L and Λ_M is unbounded, or (b) M is a closed subspace of L and Λ_M is nonempty, then

$$(2.23) \qquad T(t) \colon M \to M, \qquad t \geq 0.$$

Proof. If $\lambda, \mu > 0$ and $|1 - \mu/\lambda| < 1$, then (cf. (2.8))

$$\mu(\mu - A)^{-1} = \sum_{n=0}^{\infty} \frac{\mu}{\lambda}\left(1 - \frac{\mu}{\lambda}\right)^n [\lambda(\lambda - A)^{-1}]^{n+1}. \tag{2.24}$$

Consequently, if M is a closed convex subset of L, then $\lambda \in \Lambda_M$ implies $(0, \lambda] \subset \Lambda_M$, and if M is a closed subspace of L, then $\lambda \in \Lambda_M$ implies $(0, 2\lambda) \subset \Lambda_M$. Therefore, under either (a) or (b), we have $\Lambda_M = (0, \infty)$. Finally, by (2.10),

$$\begin{aligned} \exp\{tA_\lambda\} &= \exp\{-t\lambda\}\exp\{t\lambda[\lambda(\lambda - A)^{-1}]\} \\ &= e^{-t\lambda}\sum_{n=0}^{\infty}\frac{(t\lambda)^n}{n!}[\lambda(\lambda - A)^{-1}]^n \end{aligned} \tag{2.25}$$

for all $t \geq 0$ and $\lambda > 0$, so the conclusion follows from Proposition 2.7. □

2.9 Proposition Let $\{T(t)\}$ and $\{S(t)\}$ be strongly continuous contraction semigroups on L with generators A and B, respectively. If $A = B$, then $T(t) = S(t)$ for all $t \geq 0$.

Proof. This result is a consequence of the next proposition. □

2.10 Proposition Let A be a dissipative linear operator on L. Suppose that $u: [0, \infty) \to L$ is continuous, $u(t) \in \mathscr{D}(A)$ for all $t > 0$, $Au: (0, \infty) \to L$ is continuous, and

$$u(t) = u(\varepsilon) + \int_\varepsilon^t Au(s)\, ds, \tag{2.26}$$

for all $t > \varepsilon > 0$. Then $\|u(t)\| \leq \|u(0)\|$ for all $t \geq 0$.

Proof. Let $0 < \varepsilon = t_0 < t_1 < \cdots < t_n = t$. Then

(2.27)

$$\begin{aligned} \|u(t)\| &= \|u(\varepsilon)\| + \sum_{i=1}^{n}[\|u(t_i)\| - \|u(t_{i-1})\|] \\ &= \|u(\varepsilon)\| + \sum_{i=1}^{n}[\|u(t_i)\| - \|u(t_i) - (t_i - t_{i-1})Au(t_i)\|] \\ &\quad + \sum_{i=1}^{n}[\|u(t_i) - (t_i - t_{i-1})Au(t_i)\| - \|u(t_i) - (u(t_i) - u(t_{i-1}))\|] \\ &\leq \|u(\varepsilon)\| + \sum_{i=1}^{n}\left[\|u(t_i) - (t_i - t_{i-1})Au(t_i)\| - \left\|u(t_i) - \int_{t_{i-1}}^{t_i} Au(s)\, ds\right\|\right] \\ &\leq \|u(\varepsilon)\| + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|Au(t_i) - Au(s)\|\, ds, \end{aligned}$$

where the first inequality is due to the dissipativity of A. The result follows from the continuity of Au and u by first letting $\max(t_i - t_{i-1}) \to 0$ and then letting $\varepsilon \to 0$. □

In many applications, an alternative form of the Hille–Yosida theorem is more useful. To state it, we need two definitions and a lemma.

A linear operator A on L is said to be *closable* if it has a closed linear extension. If A is closable, then the *closure* $\bar{A}$ of A is the minimal closed linear extension of A; more specifically, it is the closed linear operator B whose graph is the closure (in $L \times L$) of the graph of A.

2.11 Lemma Let A be a dissipative linear operator on L with $\mathscr{D}(A)$ dense in L. Then A is closable and $\overline{\mathscr{R}(\lambda - A)} = \mathscr{R}(\lambda - \bar{A})$ for every $\lambda > 0$.

Proof. For the first assertion, it suffices to show that if $\{f_n\} \subset \mathscr{D}(A)$, $f_n \to 0$, and $Af_n \to g \in L$, then $g = 0$. Choose $\{g_m\} \subset \mathscr{D}(A)$ such that $g_m \to g$. By the dissipativity of A,

$$(2.28)\quad \|(\lambda - A)g_m - \lambda g\| = \lim_{n \to \infty} \|(\lambda - A)(g_m + \lambda f_n)\| \geq \lim_{n \to \infty} \lambda \|g_m + \lambda f_n\| = \lambda \|g_m\|$$

for every $\lambda > 0$ and each m. Dividing by λ and letting $\lambda \to \infty$, we find that $\|g_m - g\| \geq \|g_m\|$ for each m. Letting $m \to \infty$, we conclude that $g = 0$.

Let $\lambda > 0$. The inclusion $\overline{\mathscr{R}(\lambda - A)} \supset \mathscr{R}(\lambda - \bar{A})$ is obvious, so to prove equality, we need only show that $\mathscr{R}(\lambda - \bar{A})$ is closed. But this is an immediate consequence of Lemma 2.2. □

2.12 Theorem A linear operator A on L is closable and its closure $\bar{A}$ is the generator of a strongly continuous contraction semigroup on L if and only if:

(a) $\mathscr{D}(A)$ is dense in L.

(b) A is dissipative.

(c) $\mathscr{R}(\lambda - A)$ is dense in L for some $\lambda > 0$.

Proof. By Lemma 2.11, A satisfies (a)–(c) above if and only if A is closable and $\bar{A}$ satisfies (a)–(c) of Theorem 2.6. □

3. CORES

In this section we introduce a concept that is of considerable importance in Sections 6 and 7.

Let A be a closed linear operator on L. A subspace D of $\mathscr{D}(A)$ is said to be a *core* for A if the closure of the restriction of A to D is equal to A (i.e., if $\overline{A|_D} = A$).

3.1 Proposition Let A be the generator of a strongly continuous contraction semigroup on L. Then a subspace D of $\mathscr{D}(A)$ is a core for A if and only if D is dense in L and $\mathscr{R}(\lambda - A|_D)$ is dense in L for some $\lambda > 0$.

3.2 Remark A subspace of L is dense in L if and only if it is weakly dense (Rudin (1973), Theorem 3.12). □

Proof. The sufficiency follows from Theorem 2.12 and from the observation that, if A and B generate strongly continuous contraction semigroups on L and if A is an extension of B, then $A = B$. The necessity depends on Lemma 2.11. □

3.3 Proposition Let A be the generator of a strongly continuous contraction semigroup $\{T(t)\}$ on L. Let D_0 and D be dense subspaces of L with $D_0 \subset D \subset \mathscr{D}(A)$. (Usually, $D_0 = D$.) If $T(t)$: $D_0 \to D$ for all $t \geq 0$, then D is a core for A.

Proof. Given $f \in D_0$ and $\lambda > 0$,

$$f_n \equiv \frac{1}{n} \sum_{k=0}^{n^2} e^{-\lambda k/n} T\left(\frac{k}{n}\right) f \in D \tag{3.1}$$

for $n = 1, 2, \ldots$. By the strong continuity of $\{T(t)\}$ and Proposition 2.1,

$$\begin{aligned} \lim_{n \to \infty} (\lambda - A) f_n &= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n^2} e^{-\lambda k/n} T\left(\frac{k}{n}\right)(\lambda - A) f \\ &= \int_0^\infty e^{-\lambda t} T(t)(\lambda - A) f \, dt \\ &= (\lambda - A)^{-1}(\lambda - A) f = f, \end{aligned} \tag{3.2}$$

so $\overline{\mathscr{R}(\lambda - A|_D)} \supset D_0$. This suffices by Proposition 3.1 since D_0 is dense in L. □

Given a dissipative linear operator A with $\mathscr{D}(A)$ dense in L, one often wants to show that $\bar{A}$ generates a strongly continuous contraction semigroup on L. By Theorem 2.12, a necessary and sufficient condition is that $\mathscr{R}(\lambda - A)$ be dense in L for some $\lambda > 0$. We can view this problem as one of characterizing a core (namely, $\mathscr{D}(A)$) for the generator of a strongly continuous contraction semigroup, except that, unlike the situation in Propositions 3.1 and 3.3, the generator is not provided in advance. Thus, the remainder of this section is primarily concerned with verifying the range condition (condition (c)) of Theorem 2.12.

Observe that the following result generalizes Proposition 3.3.

3.4 Proposition Let A be a dissipative linear operator on L, and D_0 a subspace of $\mathscr{D}(A)$ that is dense in L. Suppose that, for each $f \in D_0$, there exists a continuous function $u_f: [0, \infty) \to L$ such that $u_f(0) = f$, $u_f(t) \in \mathscr{D}(A)$ for all $t > 0$, $Au_f: (0, \infty) \to L$ is continuous, and

$$u_f(t) - u_f(\varepsilon) = \int_\varepsilon^t Au_f(s)\, ds \tag{3.3}$$

for all $t > \varepsilon > 0$. Then A is closable, the closure of A generates a strongly continuous contraction semigroup $\{T(t)\}$ on L, and $T(t)f = u_f(t)$ for all $f \in D_0$ and $t \geq 0$.

Proof. By Lemma 2.11, A is closable. Fix $f \in D_0$ and denote u_f by u. Let $t_0 > \varepsilon > 0$, and note that $\int_\varepsilon^{t_0} e^{-t}u(t)\, dt \in \mathscr{D}(\bar{A})$ and

$$\bar{A} \int_\varepsilon^{t_0} e^{-t}u(t)\, dt = \int_\varepsilon^{t_0} e^{-t}Au(t)\, dt. \tag{3.4}$$

Consequently,

$$\begin{aligned} \int_\varepsilon^{t_0} e^{-t}u(t)\, dt &= (e^{-\varepsilon} - e^{-t_0})u(\varepsilon) + \int_\varepsilon^{t_0} e^{-t} \int_\varepsilon^t Au(s)\, ds\, dt \\ &= (e^{-\varepsilon} - e^{-t_0})u(\varepsilon) + \int_\varepsilon^{t_0} (e^{-s} - e^{-t_0})Au(s)\, ds \\ &= \bar{A} \int_\varepsilon^{t_0} e^{-t}u(t)\, dt + e^{-\varepsilon}u(\varepsilon) - e^{-t_0}u(t_0). \end{aligned} \tag{3.5}$$

Since $\|u(t)\| \leq \|f\|$ for all $t \geq 0$ by Proposition 2.10, we can let $\varepsilon \to 0$ and $t_0 \to \infty$ in (3.5) to obtain $\int_0^\infty e^{-t}u(t)\, dt \in \mathscr{D}(\bar{A})$ and

$$(1 - \bar{A}) \int_0^\infty e^{-t}u(t)\, dt = f. \tag{3.6}$$

We conclude that $\mathscr{R}(1 - \bar{A}) \supset D_0$, which by Theorem 2.6 proves that $\bar{A}$ generates a strongly continuous contraction semigroup $\{T(t)\}$ on L. Now for each $f \in D_0$,

$$T(t)f - T(\varepsilon)f = \int_\varepsilon^t \bar{A}T(s)f\, ds \tag{3.7}$$

for all $t > \varepsilon > 0$. Subtracting (3.3) from this and applying Proposition 2.10 once again, we obtain the second conclusion of the proposition. □

The next result shows that a sufficient condition for $\bar{A}$ to generate is that A be triangulizable. Of course, this is a very restrictive assumption, but it is occasionally satisfied.