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Dessins d'Enfants on Riemann Surfaces

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To Ingrid and Mary

Preface

Dessin d'Annie

The term *dessin d'enfant* was introduced by Alexander Grothendieck, who died as we were completing this manuscript. It appears in his *Esquisse d'un Programme*, a set of notes written and circulated in 1984 but not published until 1997. Graphs embedded in surfaces, or, more precisely, in oriented compact 2-manifolds, can indeed look as simple as children's drawings, especially if they are drawn on the Riemann sphere. However, this does not explain why Grothendieck—and the authors of this book—were so attracted by these simple objects of geometric topology. The reason why dessins have received so much attention from the mathematical community during the last 25 years is probably the fact that they open up a rich world of unexpected links between apparently rather distant mathematical ideas.

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Many people have contributed to the content of this book through their discussions and their constant interest. In the first instance, we must mention David Singerman, but also Ernesto Girondo, Gabino González-Diez, Manfred Streit, the late Claude Itzykson and in more recent times David Torres-Teigell. Special hints for some details have been given by Friedrich Berg, Marston Conder, Philipp Habegger, Rubén Hidalgo, Bernhard Koeck, Roman Nedela, Martin Škoviera and Alexander Zvonkin.

The authors have given graduate courses on material contained in this book in Southampton, Frankfurt, Lancaster and finally in METU Ankara (2011, on invitation by Ayberk Zeytin and Hursit Onsiper), but the strongest roots of this book were Summer School courses in Jyväskylä 2006 (organised by Tapani Kuusalo) and Würzburg 2009 (organised by Jörn Steuding and Peter Müller). Almost one quarter of this volume is based on the perfect notes of our Jyväskylä courses written by Tuomas Puurtinen (directly typed in LaTeX from the blackboard during our lectures!). Some of the diagrams in Part I come from his original notes, while for some others we thank David Torres-Teigell and Martin Fluch. We are also grateful to Annie Jones (age 3), who contributed the dessin which accompanies the Preface.

Finally we thank David Singerman and Mary Tyrer-Jones for their careful reading of the manuscript and for suggesting numerous improvements.

Contents

Part I
Basic Material

The first part of this book is an introduction to the basic ideas of the theory of dessins d'enfants. We give three definitions of a dessin. The simplest is as a bipartite graph embedded in a compact oriented surface; this can be redefined as a pair of permutations (of the edges of the graph) generating a transitive group, called the monodromy group. This group is a quotient of a triangle group, a group of isometries of the hyperbolic plane (or occasionally the complex plane or Riemann sphere), and the complex structure on this surface imposes a complex structure on the underlying surface of the dessin, making it a compact Riemann surface equipped with a Belyĭ function (a meromorphic function with no critical values outside $\{0, 1, \infty\}$). This gives us a third definition of a dessin (though the first we will introduce), as the pre-image of the unit interval under a Belyĭ function. In order to prove that these definitions are mutually equivalent we use ideas from function theory, group theory, hyperbolic geometry and combinatorics, outlining a number of classical concepts required or giving references to standard sources.

Compact Riemann surfaces are equivalent to complex algebraic curves, defined by polynomial equations, an idea which we illustrate in some detail in the case of elliptic curves (Riemann surfaces of genus 1). The most fundamental result about dessins is Belyĭ's Theorem, that the algebraic curves obtained as above from dessins are those for which the coefficients of the defining polynomials can be chosen to be algebraic numbers. This remarkable result leads us into the Galois theory of algebraic number fields, and in particular the action of the absolute Galois group, the automorphism group of the field of all algebraic numbers, on dessins and on their underlying curves.

In order to make individual chapters more self-contained, we have included some sections, called Appendices, which give important background information on the existence of suitable meromorphic functions, on the finite simple groups, and on group presentations; these are all important topics, used here and later in the book, but readers who are familiar with them can safely omit these sections.

Chapter 1
Historical and Introductory Background

Abstract This chapter begins with a brief historical introduction to the theory of dessins d'enfants, from the early discovery of the platonic solids, through nineteenth-century work on Riemann surfaces, algebraic curves and holomorphic functions, and twentieth-century research on regular maps, to the fundamental and far-reaching ideas circulated by Grothendieck in the 1980s, and subsequent efforts to implement his programme. After this we summarise the background knowledge we will assume, together with suggestions for further reading.

The second section gives a brief introduction to compact Riemann surfaces, including the Riemann-Hurwitz formula for the genus of a surface, and the equivalence of the categories of *compact Riemann surfaces* and of *smooth complex projective algebraic curves*. Elliptic curves (Riemann surfaces of genus 1) are treated in detail, as simple examples of subtler phenomena encountered later. The third section contains technical results on the existence of meromorphic functions with specific properties.

In the final section we define Belyĭ functions and prove one direction of Belyĭ's theorem, that such functions characterise algebraic curves defined over number fields, by using an algorithm which constructs a Belyĭ function on such a curve. We give a first definition of dessins d'enfants as the pre-images of the unit real interval [0, 1] under Belyĭ functions, and we discuss several simple examples of dessins.

Keywords Algebraic curve • Belyĭ function • Belyĭ's theorem • Dessin d'enfant • Elliptic curve • Meromorphic function • Riemann-Hurwitz formula • Riemann surface

1.1 Introduction

1.1.1 History: Topics of the Book

The oldest of the topics in this book is the theory of maps. The regular maps on the sphere, those with the greatest degree of symmetry, are named after Plato, but they were certainly known in times much earlier than his. For us they are the prototypes of regular dessins, a finite number of which exist in every genus greater than 1. Their classification in higher genera began with work of Brahana [6] in 1927 for genus 2.

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After several handmade generalisations to genera $g \leq 6$ by Threlfall [45], Sherk [42] and Garbe [14], this classification is nowadays the object of powerful algorithms of computational group theory (Conder [7]), currently covering all genera up to 301. Research on these higher genera maps would have been impossible without the understanding of hyperbolic tessellations and of Fuchsian groups developed in the late nineteenth century by Fricke and Klein [13] and Poincaré [35].

This line of research leads us to the link between maps and Riemann surfaces. Grothendieck [18] observed that dessins can be defined by purely topological means and that they induce on the underlying surface a unique conformal structure; he attributed the proof to Malgoire and Voisin [47], and in the meantime there have been further proofs of this important fact, for example one by Voevodsky and Shabat [46]. However, the first proof goes back to a paper by Singerman from pre-dessin times, see [23, 43].

Already from Riemann's work [38] one might have guessed that—in modern language—the category of compact Riemann surfaces and the category of smooth projective algebraic curves are equivalent. It took a long time to make this equivalence precise through work of Poincaré and Koebe [28–30, 36, 37] on the uniformisation of Riemann surfaces; for the historical background and details the reader may consult Scholz's book [41]. However, apart from some very exceptional examples with many symmetries, such as Klein's quartic [27], the Fricke-Macbeath curve [12] or the Bolza curves [5], this equivalence was far from explicit: until 1979, there was no function-theoretic criterion giving a necessary and sufficient condition for a compact Riemann surface X of genus $g \geq 1$ to be defined (as an algebraic curve) over a number field, that is, given in suitable coordinates by polynomial equations with coefficients in the field $\overline{\mathbb{Q}}$ of algebraic numbers.

This criterion was provided by Belyĭ's theorem [3]: *X can be defined over a number field if and only if there is a non-constant meromorphic function β on X ramified over at most three points*. Nowadays such functions β are called *Belyĭ functions*, and Grothendieck's "Esquisse d'un programme" [18] showed that Belyĭ functions can be characterized in a simple way by maps on their surfaces X. Later on, it turned out that the slight generalisation to *hypermaps*, introduced by Cori [8] in 1975 with motivation from computer graphics, was an even better adapted tool to treat Belyĭ functions and dessins.

Belyĭ's own work [3] was done in the framework of inverse Galois theory, that is, the question of whether and how it is possible to construct Galois extensions of number fields (or function fields) with a given Galois group, or more generally, to get as much information as possible about the absolute Galois group (the automorphism group of the field $\overline{\mathbb{Q}}$). It has long been known that this group acts faithfully on dessins; an important recent development has been the proof by González-Diez and Jaikin-Zapirain [17] that it acts faithfully on regular dessins, so that in a sense one can see the entire Galois theory of algebraic number fields through these simple and highly symmetric combinatorial objects.

Grothendieck broadened this viewpoint by linking dessins to questions about moduli spaces and motives. This so-called Grothendieck-Teichmüller theory is beyond the scope of this book. For the reader who wants to learn more about this branch of dessin theory we refer to the volumes edited by Schneps and Lochak [39, 40] and to the surveys by Guillot [19] and Oesterlé [34]. Concerning Galois theory, we restrict our coverage to elementary matters such as defining Galois actions on dessins, their invariants and their interpretation as map and hypermap operations, with the emphasis on concrete examples rather than abstraction and generalisation.

Another important link between dessins and the rest of the mathematical world is given by explicit uniformisation. The classical uniformisation theorem says that each Riemann surface X—for simplicity, say compact and of genus greater than 1—can be written as a quotient $\Gamma\backslash\mathbb{H}$ of the upper half plane $\mathbb{H}$ by some Fuchsian group Γ. However in general it is impossible to determine generators of Γ from the equations for X or to determine the explicit equations for X from group theoretic properties of Γ. With dessins, we are now in a much more favourable situation: there is a Belyĭ function on X if and only if we can choose Γ as a subgroup of a certain triangle group [1], so we have a kind of explicit uniformisation theory for curves defined over number fields. However this does not mean that it is always easy to determine explicit curve equations or coefficients of Belyĭ functions from dessins. In particular, we leave aside all hard questions concerning these computational aspects.

We also leave aside the possible connections with Physics (see the short account in [1]), but in Chap. 10 we briefly indicate links with the abc-theorem for function fields [32, 53], and with complex multiplication [2, 49, 51]. At the moment, apart from algebraic curves, maps and Galois theory, the most important application of (regular) dessins seems to be the construction and the properties of Beauville surfaces, the subject of the last chapter.

1.1.2 Prerequisities: Suggestions for the Reader

The reader of this volume should have a sufficient basic knowledge of complex functions (including Möbius transformations, the monodromy theorem and Schwarz's reflection principle) and group theory. It would also be useful to have some familiarity with covering spaces and the basics about hyperbolic geometry in the Poincaré model. Several less common concepts and results from function theory and group theory are presented in the appendices to the respective sections. Most topics needed about Riemann surfaces and Galois theory are briefly explained in Chaps. 1 and 4, but for the inexperienced reader these explanations may be rather short. All other chapters of Part I contain important results about Belyĭ functions and dessins developed during the last 40 years and essential for the understanding of everything in the later parts of the book.

Much of the material presented in Part I and many examples can also be found in the excellent and much more detailed introduction [15] by Girondo and González-Diez. Other sources of information about these basic questions are the survey articles [19, 23, 25, 50] or the dessin sections of the books by Lando and Zvonkin [32] or Degtyarev [9].

Part II deals with regular dessins (roughly speaking, those with the largest possible symmetry groups) and their underlying 'quasiplatonic' surfaces. Chapters 5 and 8 contain the most important basic results; the other chapters describe how regular dessins can be constructed and classified. There we discuss examples of families of regular dessins and quasiplatonic surfaces for which the Galois action is completely understood.

Part III contains two chapters, one about the abc theorem and complex multiplication, which can be read without the results of Part II. The last chapter about Beauville surfaces depends on Chap. 5 from Part II.

1.2 Compact Riemann Surfaces and Algebraic Curves

This section does not replace a book about Riemann surfaces. Here we simply collect some important facts, arguments, and examples serving as a guideline for what follows. For a more detailed account we refer to the many books about the topic, such as [10, 11, 24, 26].

Riemann surfaces are Hausdorff spaces which have a countable base for their topology. They are manifolds whose chart maps (also called local coordinates) take their values in the complex plane $\mathbb{C}$, and are defined with biholomorphic transition functions where their domains overlap. Here we will restrict our attention to connected Riemann surfaces.

1.2.1 Examples and Some Basic Facts

Example 1.1 Our first example is the *Riemann sphere*, or *complex projective line*,

$$\hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}.$$

We take two open subsets, for example $U_1 = \mathbb{C}$ and $U_2 = (\mathbb{C}\backslash\{0\})\cup\{\infty\}$, with chart maps $U_1 \to \mathbb{C},\ z \mapsto z$ and $U_2 \to \mathbb{C},\ z \mapsto 1/z$ (where, by convention, we interpret $1/\infty$ as 0). Then $z \mapsto 1/z$ is a biholomorphic transition function between local coordinates on the intersection $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$. We therefore have a Riemann surface. It can be identified, by stereographic projection, with the unit sphere in euclidean 3-space, so it is compact.

Example 1.2 The *Fermat curve* of degree $n > 1$ (as an affine curve) is defined to be

$$F_n^{\text{aff}} := \{\, (x, y) \in \mathbb{C}^2 \mid x^n + y^n = 1 \,\}.$$

We can take as chart maps $(x, y) \mapsto y$, which is a homeomorphism on suitable neighbourhoods of all points except those where $x = 0$ and $y^n = 1$, and $(x, y) \mapsto x$ which behaves similarly except where $y = 0$ and $x^n = 1$. As transition functions we take holomorphic branches of $x = \sqrt[n]{1 - y^n}$ and $y = \sqrt[n]{1 - x^n}$. Unlike $\hat{\mathbb{C}}$, this Riemann surface is not compact: for instance, the continuous real-valued function $(x, y) \mapsto |x|$ is unbounded on F_n^{aff}.

Example 1.3 More generally we can take any *smooth* affine algebraic curve

$$X^{\text{aff}} := \{ (x, y) \in \mathbb{C}^2 \,|\, f(x, y) = 0 \}$$

where f is a polynomial such that at each point $p \in X^{\text{aff}}$ either

$$\frac{\partial f}{\partial x}(p) \neq 0 \quad \text{or} \quad \frac{\partial f}{\partial y}(p) \neq 0.$$

The implicit function theorem implies that locally around p all solutions of $f(x, y) = 0$ are of the form $(h(y), y)$ or $(x, g(x))$ respectively where h and g are holomorphic. Then the projections onto the coordinates y or x can be used as chart maps.

Example 1.4 As a special case of this, an affine *hyperelliptic curve* is given by an equation

$$y^2 = (x - a_1) \dots (x - a_n)$$

with pairwise distinct $a_1, \dots, a_n \in \mathbb{C}$. Taking $f(x, y) = y^2 - \prod_j (x - a_j)$ we have

$$\frac{\partial f}{\partial y}(p) = 2y = 0$$

only at the points $p = p_j = (a_j, 0)$, so away from these points we can use x as a local coordinate. At the points $p = p_j$ we have

$$\frac{\partial f}{\partial x}(p) \neq 0,$$

so near them we can use y as a local coordinate instead.

The chart maps allow us to define *holomorphic* and *meromorphic functions* on Riemann surfaces and *holomorphic mappings* between Riemann surfaces by tracing back all these properties locally to the usual definitions in domains of the complex plane. Thus holomorphic and meromorphic functions on Riemann surfaces inherit the usual properties from holomorphic and meromorphic functions in the plane.

Exercise 1.1 Prove that there are no non-constant holomorphic functions on compact Riemann surfaces.

Exercise 1.2 Prove that the meromorphic functions on a Riemann surface X, with the usual addition and multiplication of meromorphic functions, form a field $\mathbb{C}(X)$.

Exercise 1.3 Prove that the field of meromorphic functions on the Riemann sphere $\hat{\mathbb{C}}$ is isomorphic to the rational function field $\mathbb{C}(z)$.

Proposition 1.1 *Let $f : X \to Y$ be a non-constant holomorphic mapping between connected Riemann surfaces X and Y, let $p \in X$, and let $p' = f(p)$. Then there exist chart maps $z : U(p) \to V \subset \mathbb{C}$ and $w : U'(p') \to V' \subset \mathbb{C}$ with $z(p) = 0$ and $w(p') = 0$, and an integer $n \in \mathbb{N}$ such that the diagram*

$$\begin{array}{ccc} U & \stackrel{f}{\longrightarrow} & U' \\ z\downarrow & & \downarrow w \\ \mathbb{C} \ni z & \longmapsto & w = z^n \in \mathbb{C} \end{array}$$

is commutative. This integer n, which is independent of the choice of the charts, is called the 'multiplicity' $\operatorname{mult}_p f$ *of f at p.*

If $n = 1$ then f is locally biholomorphic (*unramified at p*); otherwise it is *ramified* with order $n > 1$. Here are some consequences.

1. If $f : X \to \hat{\mathbb{C}}$ is meromorphic and non-constant, then its zeros and poles form a discrete subset of X.
2. The ramification points of f form a discrete subset of X .
3. The identity theorem, the maximum principle, and the open mapping theorem, familiar from complex function theory for domains in the complex plane, are also valid on Riemann surfaces.
4. If X is a compact Riemann surface then a non-constant meromorphic function $f : X \to \hat{\mathbb{C}}$ has only a finite number of zeros or poles, and also only a finite number of ramification points. A holomorphic function $f : X \to \mathbb{C}$ must be constant (see Exercise 1.1).
5. If X is a compact Riemann surface then any non-constant holomorphic function $f : X \to Y$ is surjective, and Y is also compact.
6. Under the same hypotheses the *degree*

$$\deg f := \sum_{p \in f^{-1}(y)} \operatorname{mult}_p f$$

of f is independent of the choice of $y \in Y$.

Before continuing with basic facts we give some more examples.

Example 1.5 Fermat curves again: the projective version of the Fermat curve of degree n is

$$F_n := \{\, [x, y, z] \in \mathbb{P}^2(\mathbb{C}) \mid x^n + y^n = z^n \,\}.$$

(It is sometimes more useful, because of the greater symmetry, to define F_n by the equation

$$x^n + y^n + z^n = 0,$$

obtained by replacing z with ζz where $\zeta^n = -1$.) By taking $z = 1$, $y = 1$ and $x = 1$ respectively, we see that F_n is covered by three copies of the affine curve F_n^{aff}, which omit the points of F_n where z, y and x are zero. For the first affine curve we can take chart maps of the form $[x, y, z] \mapsto x/z$ or y/z, and similarly for the other two. This is a typical example of a *smooth projective algebraic curve*. Of course, because $\mathbb{P}^2(\mathbb{C})$ is compact, its closed subset F_n is also compact, whereas affine Fermat curves are not.

The great advantage of using projective algebraic curves is that they are compact. There are a number of very useful theorems about Riemann surfaces which have compactness among their hypotheses; these include various numerical results such as the Riemann-Hurwitz formula, which we shall state shortly. However, there is a disadvantage in passing from an affine model of a Riemann surface to a projective model: we need an extra coordinate (generally three, rather than two, when we use plane models), and each point no longer determines its coordinates uniquely, but rather their ratios, which can be less convenient (in defining chart maps, for instance). A good compromise is to represent a projective algebraic curve as the union of two or more affine curves, and then to work with whichever model is most convenient.

Example 1.6 Let us try to compactify the hyperelliptic curve X^{aff}, given by

$$y^2 = \prod_{j=1}^{n} (x - a_j)\,,$$

which we considered in Example 1.4. The corresponding projective curve X^{proj} is given by the equation

$$y^2 z^{n-2} = \prod_{j=1}^{n} (x - a_j z).$$

As in the case of the Fermat curves, this is compact. If $z \neq 0$ then since these are projective coordinates we can take $z = 1$, giving the original affine curve $X^{\mathrm{aff}} \subset X^{\mathrm{proj}}$; on the other hand, if $z = 0$ then $x^n = 0$ (provided $n \geq 3$) and so $x = 0$, giving a single point $p_\infty = [0, 1, 0]$ as the complement of X^{aff} in X^{proj}. As we saw in Example 1.4, we can use either x or y as a local coordinate on X^{aff}, as we are away from or near a point $p_j = [a_j, 0, 1]$.

Similarly, if $y \neq 0$ we can take $y = 1$, giving an affine curve $Y^{\mathrm{aff}} \subset X^{\mathrm{proj}}$ with equation

$$z^{n-2} = \prod_{j=1}^{n} (x - a_j z)\,;$$

its complement consists of the n points p_j, so X^{proj} is the union of these two affine curves.

In order to define local coordinates near p_∞, which is in Y^{aff} but not in X^{aff}, we would like to apply the implicit function theorem to the polynomial

$$h(x,z) = z^{n-2} - \prod_{j=1}^{n} (x - a_j z)\,.$$

Unfortunately, logarithmic differentiation shows that if $n > 3$ then $\partial h/\partial x = \partial h/\partial z = 0$ at p_∞, so the theorem does not apply. Instead, let us go back to X^{aff}, and write its defining equation as

$$y^2 = q(x) := \prod_{j=1}^{n} (x - a_j)\,.$$

We may assume that each $a_j \neq 0$, by replacing x with $x - a$ for some constant a if necessary. Let us define new variables s and t by

$$t := \frac{1}{x} \quad \text{and} \quad s := \frac{y}{x^{g+1}},$$

where $g := \lfloor (n-1)/2 \rfloor$, so that

$$\deg q = n = \begin{cases} 2g+1 & (n \text{ odd}), \\ 2g+2 & (n \text{ even}). \end{cases} \tag{1.1}$$

The equation $y^2 = q(x)$ is then equivalent to $s^2 = k(t)$ at all points with $x \neq 0$, where

$$k(t) := t^{2g+2} q\left(\frac{1}{t}\right) = \frac{q(x)}{x^{2g+2}}$$

is a polynomial of degree $2g+2$ in $\mathbb{C}[t]$ with simple zeros at the points $t = a_j^{-1}$, and also at $t = 0$ if n is odd. Note that

$$x = \infty \Leftrightarrow t = 0 \Rightarrow s = \sqrt{k(0)} = \begin{cases} 0 & (n \text{ odd}), \\ \pm 1 & (n \text{ even}). \end{cases}$$

We can now apply the implicit function theorem to the polynomial $f^*(s,t) = s^2 - k(t)$ since, as in Example 1.4, its two partial derivatives do not simultaneously vanish. Specifically, if n is even then we can use t as a local coordinate since $\partial f^*/\partial s = 2s \neq 0$, while if n is odd, so that $\partial f^*/\partial s = 2s = 0$, we can use s since $\partial f^*/\partial t \neq 0$.

This calculation illustrates an important general point about the Riemann surface X associated with the hyperelliptic curve $y^2 = q(x)$. The projection $(x,y) \mapsto x$ realises it as a 2-sheeted covering of $\hat{\mathbb{C}}$, branched over the roots a_j of q, and also over ∞ if n is odd: in the latter case there is a single point $(s,t) = (0,0)$ where $x = \infty$, whereas if n is even there are two points $(s,t) = (\pm 1, 0)$. This can be explained by writing y as a 2-valued function

$$y = \sqrt{(x-a_1)\dots(x-a_n)}\,,$$

so that each point $x \neq a_j, \infty$ in $\hat{\mathbb{C}}$ is covered by two points $(x, \pm y)$. Now let $x = re^{i\theta}$ for fixed $r > |a_1|, \dots, |a_n|$, and let θ increase by 2π, so that x follows a circular path enclosing all the roots of q, or equivalently, enclosing the point $\infty \in \hat{\mathbb{C}}$; each factor $\sqrt{(x-a_j)}$ is multiplied by $e^{i\pi} = -1$, so y changes sign, that is, (x,y) passes from one sheet to the other, if and only if n is odd. (A similar argument explains the branching at roots a_j of q for all n: if x follows a small closed path enclosing a_j but no other roots, then just one factor $\sqrt{(x-a_j)}$ is multiplied by -1, while the rest are unchanged.)

The projective model X^{proj} of X discussed earlier obscures this distinction between odd and even values of n: in either case, it has a single point $p_\infty = [0,1,0]$ at infinity. When n is odd this is where the covering is branched, but when n is even it represents a singularity in the model, where two sheets of the Riemann surface X intersect.

We can use the 2-sheeted covering $X \to \hat{\mathbb{C}}$ to construct a topological model of X by taking two copies of $\hat{\mathbb{C}}$, one for each branch of $\sqrt{q(x)}$, and joining them across disjoint cuts between $g+1$ pairs of branch-points, namely the roots of q if $n = 2g+2$ is even, together with ∞ if $n = 2g+1$ is odd. A topological model of X^{proj} is then formed by identifying the two points $(s,t) = (\pm 1, 0)$ of X over ∞ if n is even.

Every Riemann surface is orientable! This is because the transition functions are biholomorphic, and therefore preserve the orientation.

Riemann surfaces can also be triangulated. In fact, this is true more generally for all topological surfaces. This may seem intuitively obvious, but in fact the proof, by Radó in 1925, is not straightforward. (The corresponding result for three-dimensional manifolds is also true, but it is false in dimension 4, Freedman's E_8 manifold providing a counterexample.) However, there is a simpler proof for compact Riemann surfaces X if one accepts the existence of a non-constant (and hence surjective) meromorphic function $f : X \to \hat{\mathbb{C}}$ (see Exercise 1.12): construct a triangulation $\mathscr{T}$ of $\hat{\mathbb{C}}$ such that the vertices include the *critical* values, that is, the images of all the ramification points of f, and each face is sufficiently small that

f is injective on each connected component of its inverse image; then $f^{-1}(\mathscr{T})$ is a triangulation of X.

Any triangulation of a compact surface X has finite numbers V, E and F of vertices, edges and faces. The *Euler characteristic* of X is defined to be

$$\chi(X) := V - E + F;$$

this can be shown to be independent of the choice of a triangulation. For example $\chi(\hat{\mathbb{C}}) = 2$, since one can triangulate a sphere with three vertices, three edges and two faces. Similarly, $\chi(X) = 0$ if X is a torus (easy exercise!).

For compact orientable surfaces X (including compact Riemann surfaces), the *genus* $g(X)$ is a more commonly-used invariant than the Euler characteristic; this is defined by

$$2 - 2g(X) = \chi(X)\,.$$

In topology, one shows that such a surface X is homeomorphic to a sphere with $g(X)$ handles attached. Thus $g(X)$ is a non-negative integer, so that $\chi(X)$ is an even integer, with $\chi(X) \leq 2$.

The reader may find more information about all these topological aspects in textbooks on surface topology or in [44].

Proposition 1.2 (Riemann-Hurwitz Formula) *If $f : X \to Y$ is a non-constant holomorphic mapping of compact Riemann surfaces, then*

$$2g(X) - 2 = (\deg f)(2g(Y) - 2) + \sum_{p \in X}(\mathrm{mult}_p f - 1).$$

(Note that the sum on the right-hand side is finite, since $\mathrm{mult}_p f = 1$ for all but finitely many points $p \in X$.)

Outline Proof We can choose a triangulation $\mathscr{T}$ of Y such that the vertices include all the finitely many points of Y over which f is ramified. Since there is no ramification away from the vertices, the pre-image of $\mathscr{T}$ is a triangulation $\mathscr{S}$ of X. If $\mathscr{T}$ has v vertices, e edges and f faces, then $\mathscr{S}$ has de edges and df faces, where $d = \deg f$. If there were no ramification then $\mathscr{S}$ would also have dv vertices, but in fact we 'lose' $\mathrm{mult}_p f - 1$ vertices at each ramification point $p \in X$, so $\mathscr{S}$ has $dv - \sum_p(\mathrm{mult}_p f - 1)$ vertices. Thus

$$\chi(X) = d\chi(Y) - \sum_{p \in X}(\mathrm{mult}_p f - 1),$$

giving the required formula. □

Here we give some applications of this important result.

1. We have $g(Y) \leq g(X)$, with equality if and only if either f is an isomorphism (unramified), or $g(X) = g(Y) = 1$ and f is unramified. If $g(X) > g(Y) = 0$ or 1, then f is ramified.
2. The Fermat curve F_n has genus $(n-1)(n-2)/2$. This can be seen by considering the function $f : F_n \to \hat{\mathbb{C}},\ [x, y, z] \mapsto x/z$. On the affine part of the curve given by $z = 1$ and $x^n + y^n = 1$, we have $f : (x, y) \mapsto x$. This shows that $\deg f = n$, since for general x there are n solutions $y \in \mathbb{C}$ of the equation $x^n + y^n = 1$. The exceptions are those points x with $x^n = 1$, where f has only one pre-image, giving us n points $p = (x, 0)$ with $\mathrm{mult}_p f = n$. The points of F_n on the line at infinity $z = 0$ are those of the form $[\zeta, 1, 0]$ where $\zeta^n = -1$, giving n simple (therefore unramified) poles of f. The Riemann-Hurwitz formula now implies that
$$2g(F_n) - 2 = n(-2) + n(n-1) = n^2 - 3n,$$
so
$$g(F_n) = \frac{(n-1)(n-2)}{2}.$$
3. More generally, if $f : X \to \hat{\mathbb{C}}$ is a non-constant meromorphic function on a compact Riemann surface X, then
$$g(X) = 1 - \deg f + \frac{1}{2}\sum_{p \in X}(\mathrm{mult}_p f - 1).$$

Exercise 1.4 Use the Riemann-Hurwitz formula to show that the genus of the hyperelliptic Riemann surface X considered in Example 1.6 is just the integer g given by Eq. (1.1).

1.2.2 Algebraic Curves

For our purposes, the following is a crucial result:

Theorem 1.1 *There is an equivalence between the two categories of compact Riemann surfaces and of smooth complex projective algebraic curves.*

We can represent this schematically as follows: