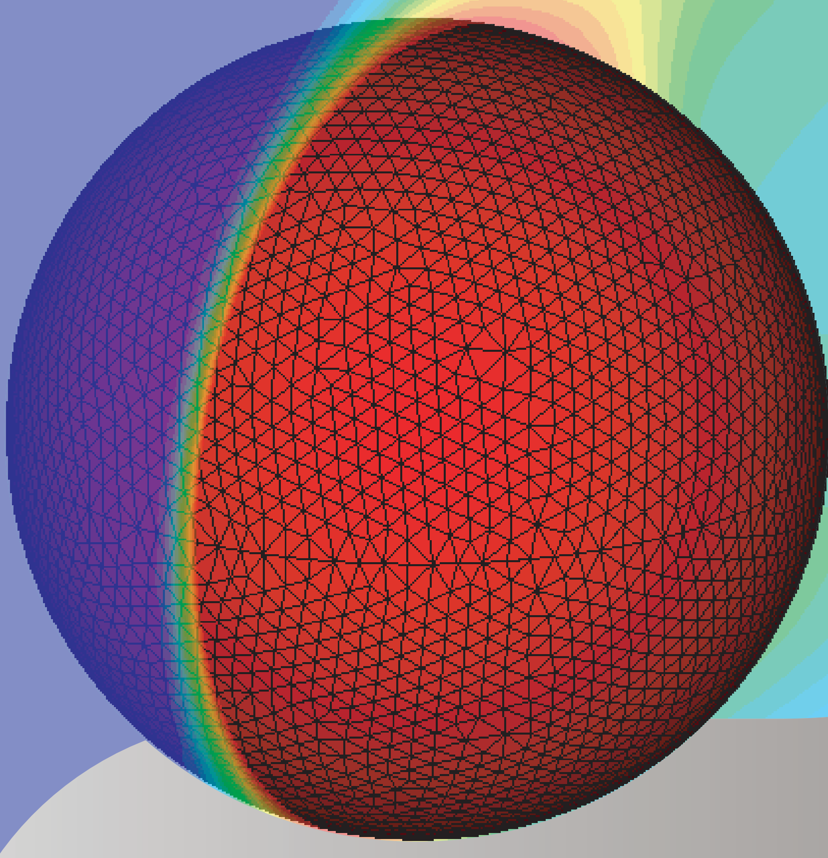
WILEY SERIES IN COMPUTATIONAL MECHANICS

Fundamentals of the Finite Element Method for Heat and Mass Transfer

Second Edition

P. Nithiarasu, R. W. Lewis,
and K. N. Seetharamu



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Preface to the Second Edition

In this second and enhanced edition of the book, we provide the readers with a detailed step-by-step application of the finite element method to heat and mass transfer problems. In addition to the fundamentals of the finite element method and heat and mass transfer, we have attempted to take the readers through some advanced topics of heat and mass transfer. The first edition of the book covered only the application of the finite element method to heat conduction and flow aided laminar heat convection. The second edition of the book has been enhanced further with turbulent flow and heat transfer, and mass transfer, in addition to advanced topics such as fuel cells.We believe that the second edition provides a comprehensive text for students, engineers and scientists who would like to pursue a finite element based heat transfer analysis. This textbook is suitable for beginners, senior undergraduate students, postgraduate students, engineers and early career researchers.

The first three chapters of the book deal with the essential fundamentals of both the heat conduction and the finite element method. In the first chapter, the fundamentals of energy balance and the standard derivations of relevant equations for the heat conduction analysis are discussed. Chapter 2 deals with the basic discrete systems which provide a basis for the finite element method formulations in the following chapters. The discrete system analysis is demonstrated through a variety of simple heat transfer and fluid flow problems. The third chapter gives a comprehensive account of the finite element method formulations and relevant history. Several examples and exercises included in Chapter 3 give the readers a complete overview of the theory and practice associated with the finite element method.

The application of the finite element method to heat conduction problems are discussed in detail in Chapters 4, 5 and 6. The conduction analysis starts with a simple one-dimensional steady-state heat conduction in Chapter 4 and is extended to multi-dimensions in Chapter 5. Chapter 6 gives the transient solution procedures for heat conduction problems.

Chapters 7, 8 and 9 deal with heat transfer by convection. In Chapter 7, heat transfer aided by the laminar motion of a single phase flow is discussed in detail. All the relevant differential equations are derived from first principles. All the three types of convection modes; forced, mixed and natural convection, are discussed in detail. Several examples and comparisons are provided to support the accuracy and flexibility of the finite element procedures discussed. In Chapter 8 the turbulent flow and heat transfer are discussed in some detail. Some examples and comparisons provide the readers a chance to assess the accuracy of the methods employed. Chapter 9 utilizes the finite element method developed in Chapters 1, 7 and 8 to provide a solution approach to flow and heat transfer in compact heat exchangers. Chapter 10 provides an introduction to the application of the finite element to problems of mass transfer. A detailed description of heat and mass transfer in porous media is then provide in Chapter 11. Two important applications of the finite element method for heat and mass transfer are explained in Chapters 12 and 13. Chapter 12 briefly introduces solidification problems using both heat conduction and convection approaches. Simple examples of solidification in this chapter may serve as a reference for students and researchers working in the area of solidification. In Chapter 13, we introduced a finite element solution approach to studying heat and mass transfer in fuel cells. Although the approach is only explained for solid oxide fuel cells, the method can be easily generalized to other types of fuel cells. Chapter 14 gives the reader sufficient information to understand the process of mesh generation. The main focus of this chapter is automatic and unstructured mesh generation. Some aspects of the adaptive mesh generation are also covered in this chapter. Finally, Chapter 15 briefly introduces the topic of computer implementation. The readers will be able to download the two-dimensional source codes and documentations from the website: **www.zetacomp.com**

Many people have assisted the authors either directly or indirectly during the preparation of this textbook. In particular, the authors wish to thank Dr Alessandro Mauro, Universitá degli Studi di Napoli Parthenope, for proofreading Chapter 13 and Dr Igor Sazonov, Swansea University, for helping the authors to put together part of Chapter 14. We would also like thank all our students, postdoctoral researchers and colleagues for providing help and support.

P. Nithiarasu, Swansea
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Series Editor's Preface

It is known that heat transfer provides a good context for teaching finite element methods and other computational mechanics topics. Fundamental concepts can be explained with such simple examples as heat conduction in 1D, then in 2D and 3D, and convective terms can be added to describe the special methods needed to deal with that class of partial differential equations. This book in our series does that, and with its distinguished, experienced authors, does it well. It not only teaches how to solve heat and mass transfer problems with finite element methods, but it also serves the purpose of teaching many different concepts in finite element methods. Readers from very diverse backgrounds will be able to benefit from this book. The book can be used by engineering undergraduate students to learn the fundamentals of heat and mass transfer and numerical methods, by graduate students in engineering and sciences to learn the advanced topics they need to know, and by practicing engineers and scientists as a good source and guide for research and development work in heat and mass transfer.

1

Introduction

1.1 Importance of Heat and Mass Transfer

The subject of heat and mass transfer is of fundamental importance in many branches of engineering. A *mechanical engineer* may be interested to know the mechanisms of heat transfer involved in the operation of equipment, for example, boilers, condensers, air pre-heaters, economizers etc., in a thermal power plant in order to improve their performance. Nuclear power plants require precise information on heat transfer as safe operation is an important factor in their design. Refrigeration and air-conditioning systems also involve heat-exchanging devices, which need careful design. *Electrical engineers* are keen to avoid material damage in electric motors, generators and transformers due to hot spots, developed by improper heat transfer design. An *electronic engineer* is interested in knowing efficient methods of heat dissipation from chips and semi-conductor devices so that they function within safe operating temperatures. A *computer hardware engineer* is interested to know the cooling requirements of circuit-boards, as the miniaturization of computing devices is advancing at a rapid rate. *Chemical engineers* are interested in heat and mass transfer processes in various chemical reactions. A *metallurgical engineer* would be interested in knowing the rate of heat transfer required for a particular heat treatment process, e.g. the rate of cooling in a casting process has a profound influence on the quality of the final product. *Aeronautical engineers* are interested in knowing the heat transfer rate in rocket nozzles and in heat shields used in re-entry vehicles. An *agricultural engineer* would be interested in the drying of food grains, food processing and preservation. A *civil engineer* would need to be aware of the thermal stresses developed in quick setting concrete, the influence of heat and mass transfer on building and building materials as well as the effect of heat on nuclear containment and buildings etc. An *environmental engineer* is concerned with the effect of heat on dispersion of pollutants in air, transport of pollutants in soils, lakes and seas and their impact on life. A *bioengineer* is often interested in the heat and

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mass transfer processes, such as hypothermia and hyperthermia associated with the human body.

The above-mentioned applications are only a sample of heat and mass transfer applications. The solar system and the associated energy transfer from the sun are the principal factors for existence of life on Earth. It is not untrue to say that it is extremely difficult, often impossible, to avoid some form of heat transfer in any process on Earth.

The study of heat and mass transfer provides economical and efficient solutions for many critical problems encountered in diverse engineering items of equipment. For example, we can consider the development of heat pipes which can transport heat at a much greater rate than that of copper or silver rods of the same dimensions and even at almost isothermal conditions. The development of present-day gas turbine blades, where the gas temperature exceeds the melting point of the blade material, is possible by providing efficient cooling systems. This is another example of the success of heat transfer design methods. The design of computer chips, which encounter heat flux of the order occurring in re-entry vehicles, especially when the surface temperature of the chips is limited to less than 100 °C, is again a success story of heat transfer design.

Although there are many successful heat transfer designs, further developments on heat and mass transfer studies are necessary in order to increase the life span and efficiency of the many devices discussed previously, which can lead to many more new inventions. Also, if we are to protect our environment, it is essential to understand the many heat and mass transfer processes involved and if necessary to take appropriate action.

1.2 Heat Transfer Modes

Heat transfer is that section of engineering science that studies the energy transport between material bodies due to temperature difference (Bejan 1993; Holman 1989; Incropera and Dewitt 1990; Sukhatme 1992). The three modes of heat transfer are:

(a) conduction

(b) convection and

(c) radiation.

The conduction mode of heat transport occurs either because of an exchange of energy from one molecule to another without actual motion of the molecules, or is due to the motion of free electrons if they are present. Therefore, this form of heat transport depends heavily on the properties of the medium and takes place in solids, liquids and gases if a difference in temperature exists.

Molecules present in liquids and gases have freedom of motion and by moving from a hot to a cold region, they carry energy with them. The transfer of heat from one region to another due to such macroscopic motion in a liquid or gas, added to the energy transfer by conduction within the fluid, is called heat transfer by convection. Convection may be either free, forced or mixed. When fluid motion occurs due to a density variation caused by temperature differences, the situation is said to be a free or natural convection. When the fluid motion is caused by an external force, such as pumping or blowing, the state is defined as being forced convection.

A mixed convection state is one in which both natural and forced convection are present. Convection heat transfer also occurs in boiling and condensation processes.

All bodies emit thermal radiation at all temperatures. This is the only mode which does not require a material medium for heat transfer to occur. The nature of thermal radiation is such that a propagation of energy, carried by *electromagnetic waves*, is emitted from the surface of the body. When these electromagnetic waves strike other body surfaces, a part is reflected, a part transmitted and the remaining part is absorbed.

All modes of heat transfer are generally present in varying degrees in a real physical problem. The important aspects in solving heat transfer problems are to identify the significant modes and to decide whether the heat transferred by other modes can be neglected.

1.3 The Laws of Heat Transfer

It is important to quantify the amount of energy being transferred per unit time and for that we require the use of rate equations. For heat conduction, the rate equation is known as *Fourier's law* (Fourier 1955) which is expressed for one dimension, as

$$q_x = -k\frac{dT}{dx}, \tag{1.1}$$

where q_x is the heat flux in the x direction (W/m^2); k is the thermal conductivity (W/mK, a property of the material, see Table 1.1) and dT/dx the temperature gradient (K/m).

Table 1.1 Typical values of thermal conductivity of some materials in W/mK at 20 °C.

Material	Thermal conductivity, k
Metals:	
Pure silver	410
Pure copper	385
Pure aluminium	200
Pure iron	73
Alloys:	
Stainless steel (18% Cr, 8% Ni)	16
Aluminium alloy (4.5% Cr)	168
Non metals:	
Plastics	0.6
Wood	0.2
Liquid:	
Water	0.6
Gasses:	
Dry air	0.025 (at atmospheric pressure)

Table 1.2 Typical values of heat transfer coefficient in W/m^2K

Gases (stagnant)	15
Gases (flowing)	15–250
Liquids (stagnant)	100
Liquids (flowing)	100–2000
Boiling liquids	2000–35 000
Condensing vapors	2000–25 000

For convective heat transfer, the rate equation is given by *Newton's law* of cooling (Whewell 1866) as

$$q = h(T_w - T_a), \tag{1.2}$$

where q is the convective heat flux; (W/m^2); $(T_w - T_a)$ the temperature difference between the wall and the fluid and h is the convection heat transfer coefficient (W/m^2K) (or film coefficient, see Table 1.2).

The convection heat transfer coefficient frequently appears as a boundary condition in the solution of heat conduction through solids, where h is often known (Table 1.2).

The maximum flux that can be emitted by radiation from a black surface is given by the *Stefan-Boltzmann Law* (Boltzmann 1884; Stefan 1879), that is,

$$q = \sigma {T_w}^4, \tag{1.3}$$

where q is the radiative heat flux (W/m^2); σ is the Stefan-Boltzmann constant (5.669×10^{-8}), in W/m^2K^4 and T_w is the surface temperature (K).

The heat flux emitted by a real surface is less than that of a black surface and is given by

$$q = \epsilon \sigma {T_w}^4, \tag{1.4}$$

where ϵ is the radiative property of the surface and is referred to as the emissivity. The net radiant energy exchange between any two surfaces 1 and 2 is given by

$$Q = F_\epsilon F_G \sigma A_1 (T_1^4 - T_2^4), \tag{1.5}$$

where F_ϵ is a factor which takes into account the nature of the two radiating surfaces; F_G a factor which takes into account the geometric orientation of the two radiating surfaces and A_1 is the area of surface 1.

When a heat transfer surface, at temperature T_1, is completely enclosed by a much larger surface at temperature T_2, the net radiant exchange can be calculated by

$$Q = qA_1 = \epsilon_1 \sigma A_1 (T_1^4 - T_2^4). \tag{1.6}$$

With respect to the laws of thermodynamics, only the first law (Clausius 1850) is of interest in heat transfer problems. The increase of energy in a system is equal to the difference between the energy transfer by heat to the system and the energy transfer by work done on the surroundings by the system, that is,

$$dE = dQ - dW, \tag{1.7}$$

where Q is the total heat entering the system and W is the work done by the system on the surroundings. Since we are interested in the rate of energy transfer in heat transfer processes, we can restate the first law of thermodynamics as:

"The rate of increase of the energy of the system is equal to the difference between the rate at which energy enters the system and the rate at which the system does work on the surroundings," that is,

$$\frac{dE}{dt} = \frac{dQ}{dt} - \frac{dW}{dt}, \tag{1.8}$$

where t is the time.

1.4 Mathematical Formulation of Some Heat Transfer Problems

In analyzing a thermal system, the engineer should be able to identify the relevant heat transfer processes and only then can the system behavior be quantified properly. In this section, some typical heat transfer problems are formulated by identifying the appropriate heat transfer mechanisms.

1.4.1 Heat Transfer from a Plate Exposed to Solar Heat Flux

Consider a plate of size L x B x d exposed to the solar flux of intensity q_s as shown in Figure 1.1. In many solar applications, such as a solar water heater, solar cooker etc., the temperature of the plate is a function of time. The plate loses heat by convection and radiation to the ambient air, which is at temperature T_a. Some heat flows through the plate and is convected

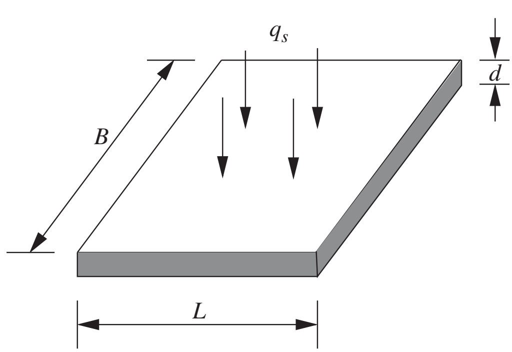

Figure 1.1 Heat transfer from a plate subjected to solar heat flux.

to the atmosphere from the bottom side. We shall apply the law of conservation of energy to derive an appropriate equation, the solution of which gives the temperature of the plate with respect to time.

Heat entering the top surface of the plate:

$$q_s A_T. \tag{1.9}$$

Heat loss from the plate to the surroundings:
Top surface:

$$hA_T(T - T_a) + \epsilon\sigma A_T(T^4 - T_a^4), \tag{1.10}$$

Side surface:

$$hA_S(T - T_a) + \epsilon\sigma A_S(T^4 - T_a^4), \tag{1.11}$$

Bottom surface:

$$hA_B(T - T_a) + \epsilon\sigma A_B(T^4 - T_a^4), \tag{1.12}$$

where the subscripts T, S and B refer respectively to the top, side and bottom surface areas. The topic of radiation exchange between a gas and a solid surface is not simple. Readers are referred to appropriate texts for details (Holman 1989; Siegel and Howell 1992). Under steady-state conditions, the heat received by the plate is lost to the surroundings, thus

$$\begin{aligned} q_s A_T = hA_T\left(T - T_a\right) + \epsilon\sigma A_T\left(T^4 - T_a^4\right) + hA_S\left(T - T_a\right) \\ +\epsilon\sigma A_S\left(T^4 - T_a^4\right) + hA_B\left(T - T_a\right) + \epsilon\sigma A_B\left(T^4 - T_a^4\right). \end{aligned} \tag{1.13}$$

This is a nonlinear algebraic equation because of the presence of the T^4 term. The solution of this equation results in the steady-state temperature of the plate. If we want to calculate the temperature of the plate as a function of time, t, then we have to consider the rate of rise in the internal energy of the plate. Substituting $E = volume \times \rho \times c_p \times T$ into the LHS of the Equation (1.8) gives

$$(volume) \times \rho c_p \frac{dT}{dt} = (LBd)\rho c_p \frac{dT}{dt}, \tag{1.14}$$

where ρ is the density and c_p is the specific heat of the plate. Thus, at any instant of time, the difference between the heat received and lost (work done on the surroundings) by the plate will be equal to the rate of change in internal energy heat stored (Equation (1.8)). Thus,

$$\begin{aligned} (LBd)\rho c_p \frac{dT}{dt} = q_s A_T - \left[hA_T(T - T_a) + \epsilon\sigma A_T\left(T^4 - T_a^4\right) + \right. \\ \left. \epsilon\sigma A_S\left(T^4 - T_a^4\right) + hA_B(T - T_a) + \epsilon\sigma A_B\left(T^4 - T_a^4\right)\right]. \end{aligned} \tag{1.15}$$

This is a first-order nonlinear differential equation, which requires an initial condition, viz.,

$$\text{at} \quad t = 0, T = T_a. \tag{1.16}$$

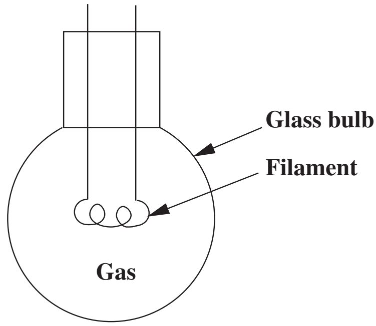

Figure 1.2 Energy balance in an incandescent light source.

The solution is determined iteratively because of the nonlinearity of the problem. Equation (1.15) can be simplified by substituting relations for the surface areas. It should be noted, however, that this is a general equation, which can be used for similar systems.

It is important to note that the spatial variation of temperature within the plate is neglected here. However, this variation can be included via Fourier's law of heat conduction, Equation (1.1). Such a variation is necessary if the plate is not thin enough to reach equilibrium instantly (Section 1.5).

1.4.2 Incandescent Lamp

Figure 1.2 shows an idealized incandescent lamp. The filament is heated to a temperature T_f by an electric current. Heat is convected to the surrounding gas and is radiated to the wall, which also receives heat from the gas by convection. The wall in turn convects and radiates heat to the ambient at T_a. A formulation of equations, based on energy balance, is necessary in order to determine the temperature of the gas and the wall with respect to time.

1.4.2.1 Gas

Rise in internal energy of the gas:

$$\rho_g c_{pg} \frac{dT_g}{dt}. \tag{1.17}$$

Convection from the filament to the gas:

$$h_f A_f (T_f - T_g). \tag{1.18}$$

Convection from the gas to the wall:

$$h_g A_g (T_g - T_w). \tag{1.19}$$

Radiation from the filament to the gas:

$$\epsilon_f A_f \sigma \left(T_f^4 - T_g^4 \right). \tag{1.20}$$

Now, the energy balance for the gas gives

$$\rho_g c_{pg} \frac{dT_g}{dT} = h_f A_f (T_f - T_g) - h_g A_g (T_g - T_w) + \epsilon_f A_f \sigma \left(T_f^4 - T_g^4 \right). \quad (1.21)$$

1.4.2.2 Wall

Rise in internal energy of the wall:

$$\rho_w c_{pw} \frac{dT_w}{dt}. \quad (1.22)$$

Radiation from the filament to the wall:

$$\epsilon_f \sigma A_f \left(T_f^4 - T_w^4 \right). \quad (1.23)$$

Convection from the wall to ambient:

$$h_w A_w (T_w - T_a). \quad (1.24)$$

Radiation from the wall to ambient:

$$\epsilon_w \sigma A_w \left(T_w^4 - T_a^4 \right). \quad (1.25)$$

Energy balance for the wall gives

$$\rho_w c_{pw} \frac{dT_w}{dt} = h_g A_g (T_g - T_w) + \epsilon_f \sigma A_f \left(T_f^4 - T_w^4 \right) - h_w A_w (T_w - T_a) - \epsilon_w \sigma A_w \left(T_w^4 - T_a^4 \right), \quad (1.26)$$

where ρ_g is the density of the gas in the bulb; c_{pg} the specific heat of the gas; ρ_w the density of the wall of the bulb; c_{pw} the specific heat of the wall; h_f the heat transfer coefficient between filament and gas; h_g the heat transfer coefficient between gas and wall; h_w the heat transfer coefficient between wall and ambient and ϵ the emissivity. The subscripts f, w, g and a respectively indicate the filament, wall, gas and ambient.

Equations (1.21) and (1.26) are first-order nonlinear differential equations. The initial conditions required are

At $t = 0$,

$$T_g = T_a \quad \text{and} \quad T_w = T_a. \quad (1.27)$$

The simultaneous solution of Equations (1.21) and (1.26), along with the above initial condition, results in the temperatures of the gas and the wall as functions of time.

1.4.3 Systems with a Relative Motion and Internal Heat Generation

The extrusion of plastics, drawing of wires and artificial fiber (optical fiber), suspended electrical conductors of various shapes, continuous casting etc. can be treated alike.

In order to derive an energy balance for such a system, we consider a small differential control volume of length, Δx, as shown in Figure 1.3. In this problem, the heat lost to the environment by radiation is assumed to be negligibly small. The energy is conducted, convected

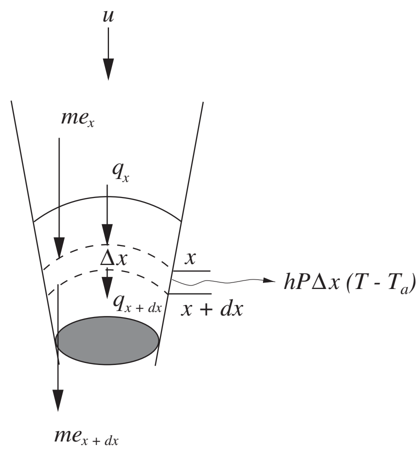

Figure 1.3 Conservation of energy in a moving body.

and transported with the material in motion. With reference to Figure 1.3, we can write the following equations of conservation of energy, that is,

$$Q_x + me_x + GA\Delta x = Q_{x+dx} + me_{x+dx} + hP\Delta x(T - T_a), \tag{1.28}$$

where $Q = Aq$ is the total heat; m is the mass flow ρAu and is assumed to be constant; e_x is the specific energy; ρ the density of the material; A the cross-sectional area; P the perimeter of the control volume; G is the heat generated per unit volume and u is the velocity at which the material is moving. Using the Taylor series of expansion we obtain

$$m(e_x - e_{x+dx}) = -m\frac{de_x}{dx}\Delta x = -mc_p\frac{dT}{dx}\Delta x. \tag{1.29}$$

Note that $de_x = c_p dT$ at constant pressure. Similarly, using Fourier's law (Equation (1.1)),

$$Q_x - Q_{x+dx} = -\frac{dQ_x}{dx} = \frac{d}{dx}\left[kA\frac{dT}{dx}\right]. \tag{1.30}$$

On substituting Equations (1.29) and (1.30) into Equation (1.28), we obtain the following conservation equation,

$$\frac{d}{dx}\left[kA\frac{dT}{dx}\right] - hP(T - T_a) - \rho c_p Au\frac{dT}{dx} + GA = 0. \tag{1.31}$$

In the above equation, the first term is derived from the heat diffusion (conduction) within the material, the second term is due to convection from the material surface to ambient, the third term represents the heat transport due to the motion of the material, and finally the last term is added to account for heat generation within the body.

1.5 Heat Conduction Equation

The determination of temperature distribution in a medium (solid, liquid, gas or combination of phases) is the main objective of a conduction analysis, that is, to know the temperature in the medium as a function of space at steady state and as a function of time during the transient state. Once this temperature distribution is known, the heat flux at any point within the medium, or on its surface, may be computed from Fourier's law, Equation (1.1). A knowledge of the temperature distribution within a solid can be used to determine the structural integrity via a determination of the thermal stresses and distortion. The optimization of the thickness of an insulating material and the compatibility of any special coatings or adhesives used on the material can be studied by knowing the temperature distribution.

We shall now derive the conduction equation in Cartesian coordinates by applying the energy conservation law to a differential control volume as shown in Figure 1.4. The solution of the resulting differential equation, with prescribed boundary conditions, gives the temperature distribution in the medium.

The Taylor series expansion gives:

$$\begin{aligned} Q_{x+dx} &= Q_x + \frac{\partial Q_x}{\partial x}\Delta x \\ Q_{y+dy} &= Q_y + \frac{\partial Q_y}{\partial y}\Delta y \\ Q_{z+dz} &= Q_z + \frac{\partial Q_z}{\partial z}\Delta z. \end{aligned} \tag{1.32}$$

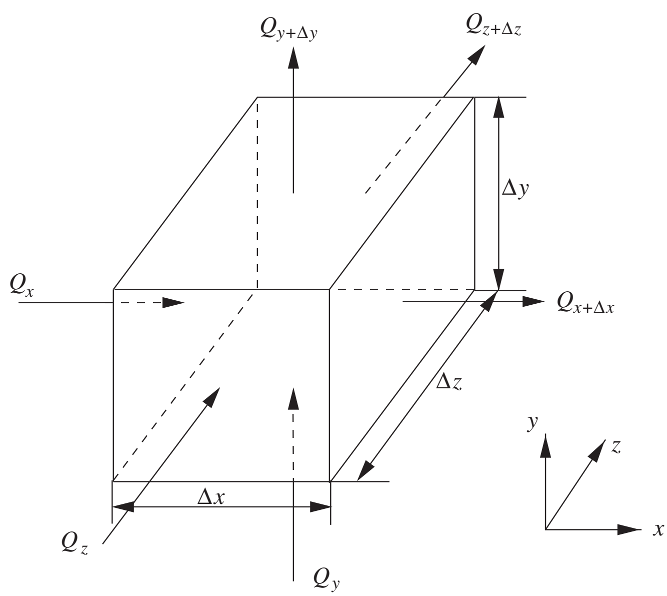

Figure 1.4 A differential control volume for heat conduction analysis.

Note that second and higher order terms are neglected in the above equation.The heat generated in the control volume is $G\Delta x\Delta y\Delta z$ and the rate of change in energy storage is given as

$$\rho c_p(\Delta x\Delta y\Delta z)\frac{\partial T}{\partial t}. \tag{1.33}$$

Now, with reference to Figure 1.4, we can write the energy balance as

```
"energy inlet + energy generated = energy stored + energy exit"
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that is:

$$(Q_x + Q_y + Q_z) + G(\Delta x\Delta y\Delta z) = \rho(\Delta x\Delta y\Delta z)\frac{\partial T}{\partial t} + Q_{x+dx} + Q_{y+dy} + Q_{z+dz}. \tag{1.34}$$

Substituting Equation (1.32) into the previous equation and rearranging results in;

$$-\frac{\partial Q_x}{\partial x}\Delta x - \frac{\partial Q_y}{\partial y}\Delta y - \frac{\partial Q_z}{\partial z}\Delta z + G(\Delta x\Delta y\Delta z) = \rho c_p(\Delta x\Delta y\Delta z)\frac{\partial T}{\partial t}. \tag{1.35}$$

The total heat transfer Q in each direction can be expressed as (area perpendicular to heat flux direction × heat flux):

$$\begin{aligned} Q_x &= (\Delta y\Delta z)q_x = -k_x(\Delta y\Delta z)\frac{\partial T}{\partial x} \\ Q_y &= (\Delta x\Delta z)q_y = -k_y(\Delta x\Delta z)\frac{\partial T}{\partial y} \\ Q_z &= (\Delta x\Delta y)q_z = -k_z(\Delta x\Delta y)\frac{\partial T}{\partial z}. \end{aligned} \tag{1.36}$$

Substituting Equation (1.36) into Equation (1.35) and dividing by the volume, $\Delta x\Delta y\Delta z$, we get

$$\frac{\partial}{\partial x}\left[k_x\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y\frac{\partial T}{\partial y}\right] + \frac{\partial}{\partial z}\left[k_z\frac{\partial T}{\partial z}\right] + G = \rho c_p\frac{\partial T}{\partial t}. \tag{1.37}$$

Equation (1.37) is the transient heat conduction equation for a stationary system expressed in Cartesian coordinates. The thermal conductivity, k, in the above equation is a vector. In its most general form, the thermal conductivity can be expressed as a tensor, that is,

$$\mathbf{k} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix}. \tag{1.38}$$

The preceding Equations (1.37) and (1.38) are valid for solving heat conduction problems in anisotropic materials with directional variation in thermal conductivities. In many situations, however, thermal conductivity can be taken as a nondirectional property, that is, the material is isotropic in nature. In such materials, the heat conduction equation is written as (constant thermal conductivity):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{G}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \tag{1.39}$$

where $\alpha = k/\rho c_p$ is the *thermal diffusivity*, which is an important parameter in transient heat conduction analyses. If the analysis is restricted only to steady-state heat conduction without heat generation, the equation is reduced to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0. \tag{1.40}$$

For a one-dimensional case, the steady-state heat conduction equation is further reduced to

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0. \tag{1.41}$$

The heat conduction equation for a cylindrical coordinate system is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left[k_r r\frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left[k_\phi \frac{\partial T}{\partial \phi}\right] + \frac{\partial}{\partial z}\left[k_z \frac{\partial T}{\partial z}\right] + G = \rho c_p \frac{\partial T}{\partial t}. \tag{1.42}$$

In cylindrical coordinates, the heat fluxes can be expressed as

$$\begin{aligned} q_r &= -k_r \frac{\partial T}{\partial r} \\ q_\phi &= -\frac{k_\phi}{r}\frac{\partial T}{\partial \phi} \\ q_z &= -k_z \frac{\partial T}{\partial z}, \end{aligned} \tag{1.43}$$

where r, ϕ and z are the cylindrical coordinate directions. The heat conduction equation for a spherical coordinate system is given by

$$\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left[k_r r^2\frac{\partial T}{\partial r}\right] + \left(\frac{1}{r^2 sin^2\theta}\right)\frac{\partial}{\partial \phi}\left[k_\phi \frac{\partial T}{\partial \phi}\right] + \\ &\quad \left(\frac{1}{r^2 sin\theta}\right)\frac{\partial}{\partial \theta}\left[k_\theta sin\theta \frac{\partial T}{\partial \theta}\right] + G = \rho c_p \frac{\partial T}{\partial t}. \end{aligned} \tag{1.44}$$

The heat fluxes in a spherical coordinate system can be expressed as

$$\begin{aligned} q_r &= -k_r \frac{\partial T}{\partial r} \\ q_\phi &= -\frac{k_\phi}{r sin\theta}\frac{\partial T}{\partial \phi} \\ q_\theta &= -\frac{k_\theta}{r}\frac{\partial T}{\partial \theta}, \end{aligned} \tag{1.45}$$

where r, ϕ and θ are the spherical coordinate directions. It should be noted that for both cylindrical and spherical coordinate systems (Equations (1.42) and (1.44)) can be derived in a similar fashion as for Cartesian coordinates by considering the appropriate differential control volumes.

1.6 Mass Transfer

When a concentration gradient exists in a fluid mixture, mass transfer takes place from a higher concentration to a lower concentration location. Such mass transport often takes place at the molecular level in the form of mass diffusion. The mass transport at the macroscopic level is referred to as mass convection. Thus, the modes of mass transfer are very similar to the first two modes of heat transfer, that is, conduction (diffusion) and convection. Mass diffusion is often described using Fick's law of mass transport (Fick 1855). This states that the mass flux of a constituent per unit area is proportional to the concentration gradient, that is,

$$J_A = \frac{\dot{m}_A}{A} = -D_{AB}\frac{dC_A}{dx}, \tag{1.46}$$

where $\dot{m}_A$ is the mass flux per unit time, D_{AB} is the diffusion coefficient and C_A is the mass concentration of the component A. As seen, this expression is very similar to Fourier's law of heat conduction (Equation (1.1)). The convective mass flux per unit area may be defined as

$$\frac{\dot{m}_A}{A} = h_A(C_A - C_{A\infty}), \tag{1.47}$$

where h_A is the mass transfer coefficient and $C_A - C_{A\infty}$ is the concentration difference through which mass transfer occurs. Equation (1.47) is analogous to the Newton's law of cooling for heat transfer (Equation (1.2)). Further details on mass transfer are given in Chapter 10.

1.7 Boundary and Initial Conditions

The heat conduction equations discussed in Section 1.5 will be complete for any problem only if the appropriate boundary and initial conditions are stated. With the necessary boundary and initial conditions, a solution to the heat conduction equation is possible. The boundary conditions for the conduction equation can be of two types or a combination of these: the *Dirichlet* condition, in which the temperature on the boundaries is known and/or the *Neumann* condition, in which the heat flux is imposed, that is (see Figure 1.5):

Dirichlet condition:

$$T = T_o \quad \text{on} \quad \Gamma_T. \tag{1.48}$$

Neumann condition:

$$q = -k\frac{\partial T}{\partial n} = \bar{q} \quad \text{on} \quad \Gamma_{qf}. \tag{1.49}$$

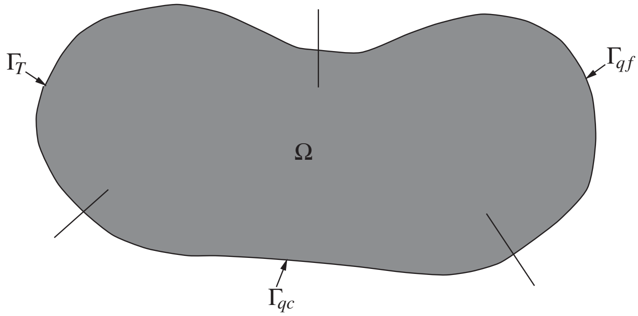

Figure 1.5 Boundary conditions.

In the above equations (Equations (1.48) and (1.49)), T_o is the prescribed temperature; Γ the boundary surface; n is the outward direction normal to the surface and $\bar{q}$ is the constant flux given. The insulated, or adiabatic, condition can be obtained by substituting $\bar{q} = 0$. The convective heat transfer boundary condition also falls into the *Neumann* category and can be expressed as

$$-k\frac{\partial T}{\partial n} = h(T_w - T_a) \quad \text{on} \quad \Gamma_{qc}. \tag{1.50}$$

It should be observed that the heat conduction equation has second-order terms and hence faces two types of boundary conditions. Since the time appears as a first-order term, at least one initial value (i.e., at some instant of time all temperatures must be known) is to be specified for the entire body, that is,

$$T = T_0 \quad \text{all over the domain} \quad \Omega \quad \text{at} \quad t = t_0, \tag{1.51}$$

where t_0 is a reference time.

The constant or variable temperature conditions are generally easy to implement as temperature is a scalar. However, the implementation of surface fluxes is not as straightforward. Equation (1.49) can be rewritten with direction cosines of the outward normals as

$$-\left(k_x\frac{\partial T}{\partial x}\tilde{l} + k_y\frac{\partial T}{\partial y}\tilde{m} + k_z\frac{\partial T}{\partial z}\tilde{n}\right) = \bar{q} \quad \text{on} \quad \Gamma_{qf}. \tag{1.52}$$

Similarly, Equation (1.50) can be rewritten as

$$-\left(k_x\frac{\partial T}{\partial x}\tilde{l} + k_y\frac{\partial T}{\partial y}\tilde{m} + k_z\frac{\partial T}{\partial z}\tilde{n}\right) = h(T - T_a) \quad \text{on} \quad \Gamma_{qc}, \tag{1.53}$$

where $\tilde{l}$, $\tilde{m}$ and $\tilde{n}$ are the direction cosines of the appropriate outward surface normals.

In many industrial applications, for example, wire drawing, crystal growth, continuous casting, etc., the material will have a motion in space and this motion may be restricted to one