A Unified Statistical Methodology for Modeling Fatigue Damage

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DEDICATION

Enrique: To my parents, as a recognition of deep gratitude.

Alfonso: To my wife María Pilar for her support.

Preface

This book is an attempt to provide a unified methodology to derive models for fatigue life. This includes S-N, ε-N and crack propagation models. This is not a conventional book aimed at describing the fatigue fundamentals, but rather a book in which the basic models of the three main fatigue approaches, the stress-based, the strain-based and the fracture mechanics approaches, are contemplated from a novel and integrated point of view. On the other hand, as an alternative to the preferential attention paid to deterministic models based on the physical, phenomenological and empirical description of fatigue, their probabilistic nature is emphasized in this book, in which stochastic fatigue and crack growth models are presented.

This book is the result of a long period of close collaboration between its two authors who, although of different backgrounds, mathematical and mechanical, both have a strong sense of engineering with respect to the fatigue problem.

When the authors of this book first approached the fatigue field in 1982 (twenty six years ago), they found the following scenario:

1. Linear, bilinear or trilinear models were frequently proposed by relevant laboratories and academic centers to reproduce the Wöhler field. This was the case of well known institutions, which justified these models based on client requirements or preferences. This led to the inclusion of such models and methods as, for example, the up-and-down, in standards and official practical directives (ASTM, Euronorm, etc.), which have proved to be unfortunate.

2. The evaluation of the S-N field lacked models not arising from arbitrary hypotheses. At that time the ASTM (1963) suggested a non explicit certain relation between the statistical distributions of $\Delta\sigma$ for fixed N and N for fixed $\Delta\sigma$.

3. The up-and-down method, clearly inefficient from the cost, reliability and extrapolation to other conditions point of view, was commonly used.

4. The existence of proposals for taking into account the length effect was based on families of distributions, such as the normal, log-normal, etc., that are not stable with respect to minimum operations.

5. The existence of models, which did not contemplate the compatibility condition, led to contradictions and inconsistencies in the cumulated damage evaluations.

6. Models based on micro-mechanical considerations combined with speculative assumptions, but such that they satisfied the compatibility requirements though without an explicit formulation of this very important condition, were unfortunately considered as excessively theoretical and useless for practical application.

7. The ε-N field was treated based on the Morrow linear elastic-plastic model, which apart from depending on a relatively large number of parameters (four) for the reduced information it supplies, only provides the mean curve, thus requiring additional methods to deal with percentile curves.

8. Crack growth appeared as a completely different and unrelated problem to the S-N and the ε-N approaches. While the first was considered as a fracture mechanics based problem, the last two were treated as phenomenological approaches to fatigue of a second order scientific level, and this occurred in spite of the fact that the three problems are different ways of contemplating the same fatigue phenomena.

This book presents a new methodology to build-up fatigue models based first on a practical knowledge of the fatigue problem, combined later with common sense, functional equations and statistical knowledge.

The first chapter provides an overview of the book, and as such, it is a summary of the general ideas present in the book. Reading this chapter should provide some type of reaction from the reader and be a good motivation to continue with the remaining chapters.

In Chapter 2 the S-N or Wöhler field models are discussed and built. To this aim, identification of all variables involved and the Buckingham theorem provide the first and unavoidable steps. Next, the models are sequentially extended, starting from (a) the case of constant stress range and level, continuing with (b) varying stress range and fixed stress level, and finally, ending with (c) varying stress range and level. The main ingredients to cook the models are: the weakest link principle combined with extreme value distribution theory, which leads to Weibull and Gumbel models for case (a), compatibility conditions of the random variables lifetime given stress range, and stress range given lifetime, which leads to straight line and hyperbolas for the percentile curves in case (b), and, finally, compatibility of the S-N fields for constant minimum stress and for constant maximum stress in case (c), leading to a model able to deal with any load. The important result is that the functional form of the models is not arbitrarily assumed, but the result of the conditions to be fulfilled.

In Chapter 3 the length (size) effect is dealt with. Since real structures are much bigger or longer than the laboratory specimens, which necessarily imply reduced sizes, engineers must design them making an important extrapolation. This requires the use of models able to make such a size extrapolation possible.

The weakest link principle together with the concepts of statistical dependence or independence of random variables (lifetimes) allows us to extend the models developed in Chapter 2 to varying lengths. Experimental data permit us to discover that not always is the independence assumption valid.

In Chapter 4 we deal with the ε-N model. Contrary to other approaches that consider this case apart from that of S-N models, and separate the elastic and the plastic components, we integrate both cases and show that the same models developed in Chapter 2 are not only valid for this case, but much more convenient and simpler. A derivation of the S-N curves from ε-N curves is also given in this chapter.

Chapter 5 deals with crack growth models. In contrast to other approaches in which some arbitrary mathematical structure is assumed for the crack growth curves, we derive this structure from fracture mechanics, statistical and common sense considerations, which lead to functional equations providing non-arbitrary models. Two different approaches are given leading to two classes of models, the intersection class of which is derived through compatibility analysis. Finally, the compatibility of the S-N curves model derived in Chapter 2 and the crack growth model derived in this chapter, which are two faces of the same fatigue problem, are used to obtain a model, which allows both approaches to be connected in a beautiful way. To our knowledge this is the first time this connection has been made.

In Chapter 6 we deal with the problem of selecting damage measures. We start by discussing the properties a good damage measure must have. Next, we analyze some alternatives for damage measures and discuss whether or not they satisfy this set of good properties, and conclude with two measures: probability of failure and a normalized measure related to the percentile curves, which are shown to be very simple and useful for measuring damage in fatigue analysis.

Chapter 7 is devoted to the damage accumulation problem. Since the damage measures in Chapter 6 were adequately selected, and the models in Chapters 2 and 5 were designed with damage accumulation in mind, a very simple rule for damage accumulation is provided, together with some illustrative examples.

In Appendix A we have included a short description of classical and some more recent fatigue models selected from those existing in the literature. However, the reader must be aware that the book is devoted to describing a new methodology for building fatigue models, and should have no further expectations in order not to be disappointed.

The proposed cumulative damage models are not intended to reproduce any kind of complex real effects occurring at the crack front, which consideration would be possible only by considering sophisticated numerical calculations and using micro-mechanical and fracture mechanics knowledge. In contrast, the proposed models allow simple approaches to be implemented in a practical fatigue design, similar to those contained in current engineering standards.

Finally, we are grateful for all we have learned during these 26 years from colleagues, students, papers and books. The book is hopefully our way of sharing with the readers all we have learnt from other people, an obligation for all

those who were fortunate enough to have access to universities, libraries, journals, books, and the work of others.

Santander and Gijón
August 6th, 2008

Enrique Castillo
Alfonso Fernández Canteli

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Part I

Introduction and Motivation of the Fatigue Problem

Chapter 1

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1.1 Introduction

Though the present book deals with fatigue related models, it does not intend to introduce or revise the classical concepts in fatigue, which have already been extensively treated in the existing literature such as the works of Haibach (1989), Conway and Sjodahl (1991), Zahavi and Torbilo (1996), Suresh (1989) and Schijve (2001). This is not a conventional book aimed at describing the fatigue fundamentals, but rather a book in which the three main fatigue approaches, the stress-based, the strain-based and the fracture mechanics approaches, are contemplated from a novel and integrated point of view. In spite of the common belief that the three approaches are mutually independent and require different models in their application, it will be shown that they are closely connected, and the strong links between them will be discovered.

On the other hand, as an alternative to the preferential attention paid to deterministic models based on the physical, phenomenological and empirical description of fatigue, their probabilistic nature is emphasized as one of the main contributions of this book.

In this chapter a general presentation of the book is made, the aim of which is to motivate the reader to address the following chapters, where a detailed explanation of the proposed methodologies and models is given.

1.2 Models with dimensionless variables

In this book the models presented are written in general in terms of dimensionless variables. The reason for doing this lies in the fact that considering models including variables with dimensions helps neither to discard wrong models, nor identify simple relations nor to get the simplest models. To this end, the Buckingham theorem (see Buckingham (1915)) becomes an essential tool, which is systematically used throughout the book.

The Buckingham Π theorem states that if we have a physically meaningful equation involving a certain number, n, of physical variables, and these variables are expressible in terms of k independent fundamental physical quantities (in our case 3), then the original expression is equivalent to an equation involving only a set of $p = n - k$ dimensionless variables constructed from the original variables. This theorem states not only whether or not an initial set of variables is sufficient to describe a physical problem, but that a reduced set of the initial variables is sufficient to analyze the problem under consideration, and that dimensionless variables can be used. One of the main consequences is that all the resulting parameters are dimensionless. Ignoring dimensional analysis techniques in building models is an important error and can lead to models being much more complex than is strictly required, and even worse, to invalid models. In particular, it is fairly frequent to find models in the fatigue area, which depend on parameters with dimensions, and that such dimensions cannot be determined before other parameters are known. In this book we want to emphasize this problem and provide methods to avoid these fundamental errors.

Before selecting a model to solve a fatigue problem, the relevant variables involved have to be identified. For example, assume that from previous experience, accumulated in the study of the fatigue phenomenon, we know that the 12 variables initially involved in a fatigue problem of a given piece are those in the set:

$$\mathcal{V} \equiv \{\Delta\sigma, \Delta\sigma_0, \sigma_\ell, N, N_0, \Delta K, \Delta K_{th}, L, L_0, a, a_0, p\},$$

where p is the probability of fatigue failure of a piece when subject to N cycles at a stress range $\Delta\sigma$ and stress level σ_ℓ, N_0 is the threshold value for N, i.e. the minimum possible lifetime, $\Delta\sigma_0$ is the fatigue limit,[1] which is defined in this book as the $\Delta\sigma$ value leading to a lifetime of 1×10^6 cycles for the median ($p = 0.5$) S-N curve,[2] a is the crack size after N cycles, a_0 is the initial crack size, ΔK is the stress intensity factor range, ΔK_{th} is a threshold stress intensity factor range, and L and L_0 are the lengths of the specimen and a reference specimen length, respectively. Instead of the pair $(\Delta\sigma, \sigma_\ell)$, we could use the pair (σ_m, σ_M), where σ_m and σ_M are the minimum and maximum stresses, respectively. In fact, in some parts of the book these alternative variables are used for convenience.

Table 1.1: Dimensional analysis of the initial set of variables involved in the fatigue problem, and exponents to build these variables in terms of the basic magnitudes M (mass), L (length) and T (time).

	N	N_0	$\Delta\sigma$	$\Delta\sigma_0$	σ_ℓ	ΔK	ΔK_{th}	L	L_0	a	a_0	p
M	0	0	1	1	1	1	1	0	0	0	0	0
L	0	0	-1	-1	-1	$-1/2$	$-1/2$	1	1	1	1	0
T	1	1	-2	-2	2	-2	-2	0	0	0	0	0

These 12 variables can be written in terms of the basic magnitudes M (mass), L (length) and T (time), as indicated in Table 1.1. Since the rank of the matrix in this table is 3, we know from Buckingham's theorem that any relation involving these 12 variables can be written in terms of $n - k = 12 - 3 = 9$ dimensionless variables, for example, those in the set[3]

$$\{N^*, \Delta\sigma^*, \sigma_\ell^*, \Delta K^*, \Delta K_{th}^*, L^*, a^*, a_0^*, p\}, \tag{1.1}$$

where

$$N^* = \log(N/N_0); \quad \Delta\sigma^* = \Delta\sigma/\Delta\sigma_0; \quad \sigma_\ell^* = \sigma_\ell/\Delta\sigma_0; \quad a^* = a/L_0$$
$$\Delta K_{th}^* = \Delta K_{th}/(\Delta\sigma_0\sqrt{L_0}); \quad L^* = L/L_0; \quad \Delta K^* = \Delta K/(\Delta\sigma_0\sqrt{L_0}),$$

that is, any valid physical relation involving the initial 12 variables is of the form

$$\eta(N^*, \Delta\sigma^*, \sigma_\ell^*, \Delta K^*, \Delta K_{th}^*, L^*, a^*, a_0^*, p) = 0.$$

[1] We use the fatigue limit instead of the endurance limit, because this can be null, and then it is not valid as a normalizing variable.

[2] Alternatively, 2×10^6 or 10^7 cycles could be used instead, but a smaller number seems to be better in order to facilitate its practical obtention in laboratory tests.

[3] There are many alternative selections for these dimensionless variables. Here we have chosen $N_0, \Delta\sigma_0$ and L_0 as normalizing variables, but other combinations are possible too.

Note that to refer to dimensionless variables we use an asterisk, and we have selected $\log(N/N_0)$ instead of N/N_0 because it is commonly used in fatigue analysis.[4]

The Buckingham theorem mainly tells us three things:

1. Whether there are enough variables for a valid physical relation to be established. If the rank of the matrix in Table 1.1 were null or $n-k=0$, such a relation would not be possible.

2. The minimum number of dimensionless variables required to reproduce the physical relation.

3. What sets of dimensionless variables can be used.

For example, Paris et al. (1961) proposed the well-known crack growth rate law

$$\frac{da}{dN} = C\Delta K^m, \tag{1.2}$$

where C is a constant, which relates linearly the crack growth rate da/dN and the stress intensity factor range ΔK, when plotted on logarithmic scales. First, this formula involves only variables a, N and ΔK, which are not sufficient for a physical relation to be possible.[5] In addition, since the left hand side $\frac{da}{dN}$ of this equation, and the term ΔK on the right hand side have different dimensions, constant C has dimensions too; however, they cannot be determined before knowing the value of the m parameter. In addition, for different materials or even for the same material under different loading conditions, the dimensions of constant C may become different.

Unfortunately, most of the modified versions of Paris' law present dimensional problems too. All these matters are discussed in Chap. 5 of this book, where alternative physically valid laws are given and identified.

1.3 S-N or Wöhler curves

In this section we derive a probabilistic model for the S-N, also known as Wöhler curves, and consider a structural element subject to a cyclic load ranging from a minimum σ_m to a maximum σ_M (see the upper right corner in Fig. 1.1). In his pioneer research, August Wöhler (1860, 1870) recognized the stress range $\Delta\sigma = \sigma_M - \sigma_m$ and the stress level σ_ℓ, as the main and secondary parameters governing the fatigue lifetime, respectively. The stress level σ_ℓ can be expressed as $\sigma_M, \sigma_m, \sigma_{mean} = (\sigma_m + \sigma_M)/2$ or $R = \sigma_m/\sigma_M$, depending on the case, but the important thing is that giving $\Delta\sigma$ and σ_ℓ is equivalent to giving σ_m and σ_M.

[4] Some authors define $\Delta\sigma^* = \log(\Delta\sigma/\Delta\sigma_0)$ instead of $\Delta\sigma/\Delta\sigma_0$, i.e., they use a logarithmic scale for the stress range too.

[5] The resulting matrix rank is 3, but then $n-k=0$. So, one would need to consider C as yet another variable

Since then, the Wöhler curves have been a reference for analyzing and solving the fatigue problem, providing the mean lifetime N as a function of the stress range $\Delta\sigma$ for a constant stress level σ_ℓ.

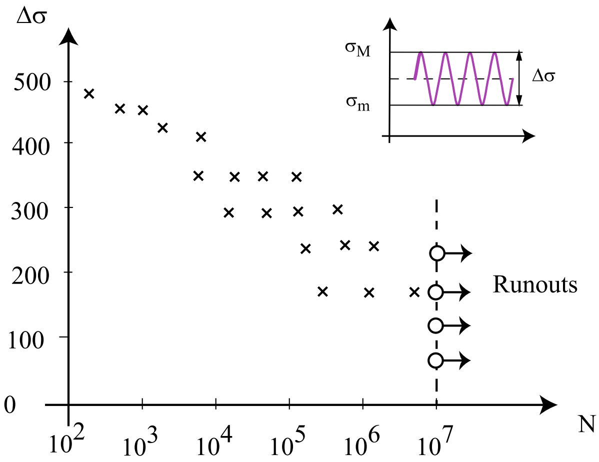

Figure 1.1: A typical example of stress-lifetime fatigue data for constant stress level.

We remind the reader that the first step to modeling is observation of the reality or a set of experiments. A typical example of stress-lifetime fatigue data for constant stress range resulting from laboratory experimentation is provided in Fig. 1.1 and an estimate of the S-N curve (the regression of stress range $\Delta\sigma$ on line of lifetime N) together with some percentiles have been added in Fig. 1.2, from which the following features can be outlined:

1. Fatigue lifetime increases with decreasing stress range.

2. In its upper part the data exhibit a negative curvature (concave from below).

3. In its central and lower part the data exhibit a positive curvature (concave from above).

4. In its lower part the data seem to have a horizontal asymptotic behavior.

5. Below a given stress range no fatigue failure exists. Laboratory constant amplitude[6] tests indicate that fatigue lifetime becomes very large (possibly infinite) below a certain threshold value $\Delta\sigma_0$, of the stress range $\Delta\sigma$, called the endurance limit.

6. Data also suggest that the scatter of fatigue lifetime increases with decreasing stress range.

[6] Amplitude is also used in mechanical fatigue as half of the stress range.

7. The Wöhler curve is upper limited by a given $\Delta\sigma_M$, related to the ultimate stress.

8. Fatigue lifetime is a random variable. Thus, instead of a single S-N curve (the mean), a family of percentile S-N curves, associated with the corresponding percentiles (see Fig. 1.2), is rather suggested. For simplicity's sake, we will refer to the set of percentile curves in this book as S-N curves.[7]

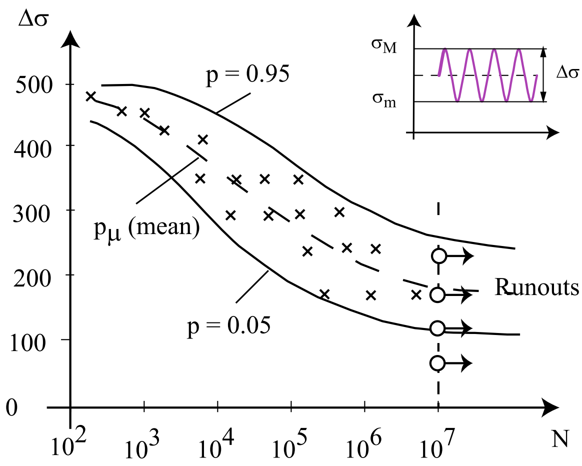

Figure 1.2: A typical example of stress-lifetime fatigue data for constant stress level with superimposed mean and percentile curves.

Since the set of variables $\{\Delta K^*, \Delta K^*_{th}, a^*, a^*_0\}$ is not involved in this problem, only the remaining 5 variables $\{N^*, \Delta\sigma^*, \sigma^*_\ell, L^*, p\}$ in the initial set (1.1) are considered. Figure 1.2 and the discussion above suggest the following relation between the dimensionless variables N^* and $\Delta\sigma^*$ for a given dimensionless length L^*, stress level σ^*_ℓ, and percentile probability p:

$$N^* = f^*(\Delta\sigma^*; p \mid \sigma^*_\ell, L^*), \tag{1.3}$$ [8]

where f^* is a monotonically decreasing function. Alternatively, we can use the inverse relation

$$\Delta\sigma^* = g^*(N^*; p \mid \sigma^*_\ell, L^*), \tag{1.4}$$

where g^* is a monotonically decreasing function too, or since the percentile curves define the probability of failure, and because in this book our focus is on

[7] Note that many references do not usually provide random information of the S-N field.

[8] In this book we use the comma to separate variables in a list, semicolon to separate variables from parameters, and the symbol | for given values of the parameters or variables.

stochastic models, we can also work with the relation

$$p = h^*(N^*, \Delta\sigma^* \mid \sigma_\ell^*, L^*), \tag{1.5}$$

where h^* is a cumulative distribution function (cdf) of N^* for given $\Delta\sigma^*$, and also a cdf for $\Delta\sigma^*$ given N^*, which is equivalent to (1.3) or (1.4), and can be obtained from them. In fact, Eqs. (1.3), (1.4) and (1.5) are three different forms of expressing the same relation.

In this phenomenological approach to fatigue, obtaining the analytical form of one of these families of functions (because the other is the corresponding inverse) is of crucial importance in fatigue analysis.

Note that deterministic models simply supply the mean curve as the dashed line shown in Fig. 1.2, that is

$$N^* = f^*(\Delta\sigma^*; p_\mu \mid \sigma_\ell^*, L^*) \quad \text{or} \quad \Delta\sigma^* = g^*(N^*; p_\mu \mid \sigma_\ell^*, L^*),$$

where p_μ is the percentile associated with the mean values, whereas the probabilistic models deal with the whole set of percentiles, i.e., with the p-family of functions (1.3), (1.4) and/or (1.5).

In the existing literature different models of both groups, deterministic and probabilistic, have been proposed. Unfortunately, most of them, such as piecewise linear or S-shaped models, are supported by an arbitrary selection of the functional form of the f^* and/or g^* families of functions, and base this selection on convenience reasons (simplicity, easiness of mathematical representation or treatment). In contrast, in Chap. 2 of this book, we show how to:

1. Derive the analytical form of a parametric family of functions for f^*, g^* and/or h^*, based on common sense, physical, statistical and compatibility considerations, freeing the model from arbitrary assumptions.

2. Normalize the densities, that is, reducing any of them to the one corresponding to a given stress range (the normalization stress range, which is any arbitrary range) or even to a given distribution, not necessarily associated with a given range, by using the fact that the percentile curves represent lifetimes associated with different stress ranges for a given specimen.

Next, we use a compatibility condition which allows us to derive the S-N field model without arbitrary assumptions. This condition is stated as a functional equation which, when solved, provides all possible S-N field models compatible with this constraint, thus, leading to a net and elegant modeling method.

1.3.1 Compatibility condition of $N^*|\Delta\sigma$ and $\Delta\sigma|N^*$

Figure 1.3[8] shows the density of the fatigue lifetime N^* for a given stress level $\Delta\sigma^*$, and the density of $\Delta\sigma^*$ for a given lifetime N^*. Since these two densities

[8]Note that in the following, instead of plotting the mean, we use the median ($p = 0.5$).

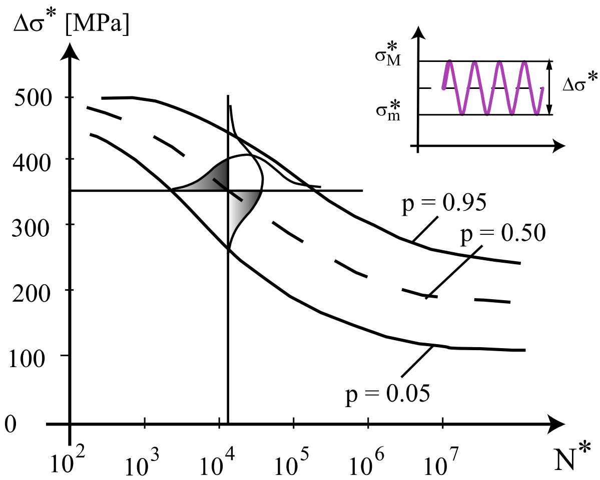

Figure 1.3: Illustration of the compatibility condition in the S-N field.

come from the same percentile curves, they are not independent and must satisfy a compatibility condition. This compatibility condition will play a fundamental role in obtaining the analytical form of functions f^*, g^* and h^*.

Since, as illustrated in Fig. 1.3, the mean and variance of N^* depend on $\Delta\sigma^*$ and the mean and variance of $\Delta\sigma^*$ depend on N^*, assuming a location and scale family of distributions for h^*,[9] Eq. (1.5) can be written as

$$p = h^* \left(\frac{N^* - \mu_1^*(\Delta\sigma^*)}{\sigma_1^*(\Delta\sigma^*)} \middle| \sigma_\ell^*, L^* \right) \tag{1.6}$$

and also as

$$p = h^* \left(\frac{\Delta\sigma^* - \mu_2^*(N^*)}{\sigma_2^*(N^*)} \middle| \sigma_\ell^*, L^* \right), \tag{1.7}$$

where $\mu_1^*(\Delta\sigma^*), \sigma_1^*(\Delta\sigma^*)$ and $\mu_2^*(N^*), \sigma_2^*(N^*)$ are the location and scale parameters of N^* given $\Delta\sigma^*$ and of $\Delta\sigma^*$ given N^*, respectively, leading to the compatibility condition

$$h^* \left(\frac{N^* - \mu_1^*(\Delta\sigma^*)}{\sigma_1^*(\Delta\sigma^*)} \middle| \sigma_\ell^*, L^* \right) = h^* \left(\frac{\Delta\sigma^* - \mu_2^*(N^*)}{\sigma_2^*(N^*)} \middle| \sigma_\ell^*, L^* \right), \tag{1.8}$$

or equivalently,

$$\frac{N^* - \mu_1^*(\Delta\sigma^*)}{\sigma_1^*(\Delta\sigma^*)} = \frac{\Delta\sigma^* - \mu_2^*(N^*)}{\sigma_2^*(N^*)}, \tag{1.9}$$

[9] Assuming a location and scale family of distributions for h^* is a natural assumption, because this means that the functional form is not affected by a change of location (origin) and scale (units being used).

which is a functional equation,[10] the solution of which leads to the two following models

$$\textbf{Model 1 :} \qquad p = h^*\left(\frac{(N^*-B^*)(\Delta\sigma^*-C^*)-\lambda^*}{\delta}\middle|\, \sigma_\ell^*, L^*\right), \tag{1.10}$$

$$\textbf{Model 2 :} \qquad p = h^*\left(A^*N^*+B^*\Delta\sigma^*+C^*|\, \sigma_\ell^*, L^*\right), \tag{1.11}$$

where λ^* and δ^* are location and scale parameters, respectively, and A^*, B^* and C^* are parameters too, this occurring no matter what the function h^* is. In other words, the functions μ_1^*, σ_1^*, μ_2^* and σ_2^* cannot be given arbitrarily if the relation (1.9) must hold. We note that the relation (1.10) is completely independent on the distributional assumptions of N^* given $\Delta\sigma^*$ and of $\Delta\sigma^*$ given N^*.

Note that the percentile curves in Model 1 are hyperbolas with common asymptotes $N^* = B^*$ and $\Delta\sigma^* = C^*$, and the percentiles in Model 2 are parallel straight lines.

1.3.2 Statistical considerations

Further considerations such as limit or asymptotic properties, and the weakest link principle, which states that the lifetime of a series system of n elements is the lifetime of its element having the minimum lifetime, lead to a Weibull or Gumbel distribution for h when the size effect is large, and finally to the following Weibull family of S-N curves:

$$\boxed{p = 1-\exp\left[-\left(\frac{(N^*-B^*)(\Delta\sigma^*-C^*)-\lambda^*}{\delta^*}\right)^{\beta^*}\right],} \tag{1.12}$$

where we remind the reader that $N^* = \log(N/N_0)$, or the associated Gumbel model

$$\boxed{p = 1-\exp\left[-\exp\left(\frac{(N^*-B^*)(\Delta\sigma^*-C^*)-\lambda^*}{\delta^*}\right)\right].} \tag{1.13}$$

In summary, the compatibility condition (1.9) together with the weakest link principle and some statistical and physical considerations allow us to obtain the functional form of the S-N curves given in (1.12) and (1.13).

A detailed analysis, together with the list of all the assumptions and the corresponding proofs, is given in Chap. 2 of this book.

[10] A functional equation (see Aczél (1966) or Castillo and Ruiz Cobo (1992); Castillo et al. (2005b)) is an equation in which the unknowns are not variables but functions, and these functions appear as they are, i.e., without any of their derivatives or integrals.

1.4 ε-N curves

When local plastic deformations are present during the fatigue process, for instance at stress raisers (edges, discontinuities, etc.) the so called strain-based must be applied instead of the stress based approach. The strain-based approach demands more material related information, including the ε-N curves, which represent the mean lifetime N as a function of the strain range $\Delta\varepsilon$ for a constant stress or strain level.

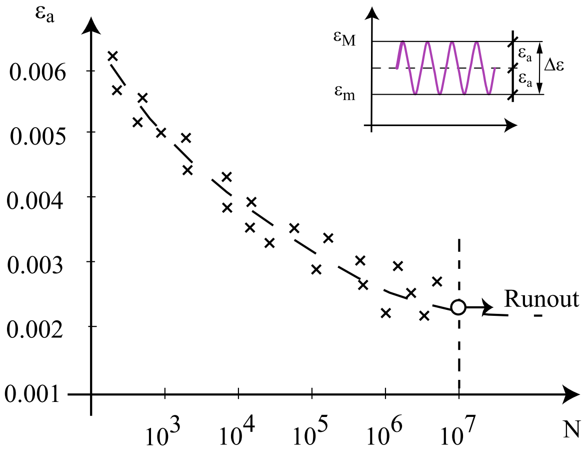

Figure 1.4: A typical example of strain-lifetime fatigue data with superimposed regression line (the ε-N curve).

A typical example of fatigue data for constant strain range is provided in Fig. 1.4, and estimates of the ε-N mean and some percentile curves have been added in Fig. 1.5 where some percentiles are shown. A detailed inspection of these figures shows the following facts:

1. Fatigue lifetime increases with decreasing strain range.
2. The data exhibit all over the lifetime range a positive curvature (concave from above).
3. Data also suggest that the scatter of fatigue lifetime increases with decreasing strain range.
4. Below a certain strain range no fatigue failure is expected.
5. The fatigue lifetime has a random character. Thus, instead of a single ε-N curve (the mean) it seems better to use a family of ε-N curves, associated with the corresponding percentiles (see Fig. 1.5). So, once again for simplicity's sake we will refer to the associated percentile curves as ε-N curves.

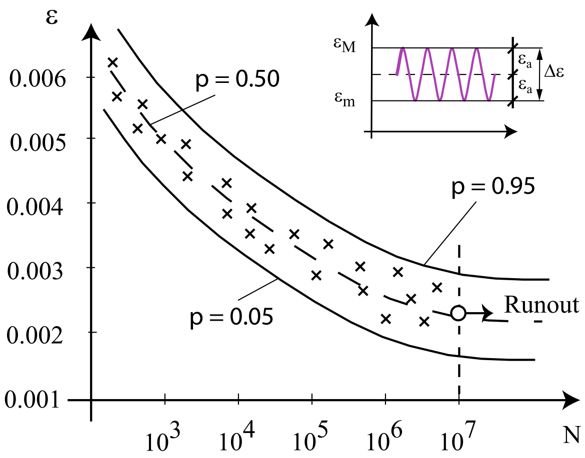

Figure 1.5: A typical example of strain-lifetime fatigue data with superimposed ε-N mean and percentile curves.

Note that the conclusions drawn from the ε-N curves are similar to those resulting from the S-N curves, with the exception of the concavity of the curves. Thus, Figs. 1.4 and 1.5 and the discussion above suggest the following relation between N^* and $\Delta\varepsilon^*$, stress σ_ℓ^* or strain level ε_ℓ, and p

$$N^* = f^*(\Delta\varepsilon^*; p \mid \sigma_\ell^*, L^*), \tag{1.14}$$

where f^* is a monotonically decreasing function. Alternatively, one can use the inverse relation

$$\Delta\varepsilon = g^*(N^*; p \mid \sigma_\ell^*, L^*), \tag{1.15}$$

where g^* is a monotonically decreasing function too, or

$$p = h^*(N^*, \Delta\varepsilon^* \mid \sigma_\ell^*, L^*). \tag{1.16}$$

Similar to the case of S-N curves, we can consider deterministic, i.e., only the mean curve, and probabilistic models, i.e., all the percentiles, and they can be obtained based on similar considerations.

In Chap. 4 some models similar to those in (1.12) and (1.13) are derived.

Finally, it is interesting to point out that it is possible to relate the ε-curves in Fig. 1.5 to the S-N curves in Fig. 1.2, with the help of the cyclic stress-strain curve. Chapter 4 provides a detailed analysis and description of ε-N curves and explains how this is done.

1.5 Stress level effect

As already indicated in the introduction, the stress level effect (sometimes colloquially denoted as mean stress effect) is, after the stress range, the most important parameter governing fatigue lifetime.