Frank Stenger · Don Tucker
Gerd Baumann

Navier–Stokes Equations on $\mathbb{R}^3 \times [0, T]$

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Preface

In this monograph we study the properties of solutions of the Navier–Stokes (N–S) partial differential equations (PDE) on $(x, y, z, t) \in \mathbb{R}^3 \times (0, T)$. Initially we convert the PDE to a system of integral equations (IE). We then describe spaces **A** of analytic functions that house solutions of this equation, and we show that these spaces of analytic functions are dense in the spaces **S** of rapidly decreasing and infinitely differentiable functions. These spaces are defined more explicitly later in this monograph. Some reasons for doing this are the following:

1. The functions of **S** are nearly always conceptual rather than explicit, i.e., relatively few such explicit functions are known, and except in concept, they differ from functions of calculus, which are generally analytic.
2. Initial and boundary conditions of solutions of PDE are usually given by scientists of applications, and as such, they are nearly always piecewise analytic, and in this case the solutions have the same properties.
3. When methods of approximation are applied to functions of **A**, they converge at an exponential rate, whereas methods of approximation applied to the functions of **S** converge only at a polynomial rate.
4. The space **A** also provides other conveniences, such as enabling sharper bounds on the solution, enabling easier existence proofs, and enabling a more accurate and more efficient method of solution including accurate error bounds—all of which are included in this monograph.

Following our proofs of denseness, we prove the existence of a solution of the IE in the space of functions $\mathbf{A} \cap \mathbb{R}^3 \times (0, T)$, and we provide an explicit novel algorithm based on Sinc approximation and Picard-like iteration for computing the solution.

We also provide an explicit *Mathematica* program for computing the solution based on our approximation procedure, given the initial divergence-free velocity, and we provide explicit illustrations of our computed solution.

More specifically, the problem which we shall analyze and solve numerically in this monograph is the PDE problem as described by Fefferman [1] for the space **S**, i.e.,

$$\frac{\partial u^j}{\partial t} - \varepsilon\,\Delta\,u^j = -\sum_{k=1}^{3} u^k\,\frac{\partial u^j}{\partial x^k} - \frac{\partial p}{\partial x^j}, \quad (\mathbf{r}, t) \in \mathbb{R}^3 \times \mathbb{R}_+. \tag{1}$$

This problem is to be solved subject to the divergence-free condition

$$\operatorname{div}\mathbf{u} = \sum_{k=1}^{3} \frac{\partial u^j}{\partial x^j} = 0, \quad (\bar{r}, t) \in \mathbb{R}^3 \times \mathbb{R}_+, \tag{2}$$

and subject to initial conditions

$$\mathbf{u}^0(\bar{r}) = \mathbf{u}(\bar{r}, 0) \quad \mathbf{r} \in \mathbb{R}^3. \tag{3}$$

Here, $\mathbf{u} = \left(u^1, u^2, u^3\right)$ denotes a velocity vector of flow, p denotes the pressure, $\mathbf{u}^0 = \left(u^{0,1}, u^{0,2}, u^{0,3}\right)$ is a given divergence-free velocity field on $\mathbb{R}^3$, ε is a positive coefficient (the viscosity), and Δ denotes the Laplacian, $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{(\partial x^i)^2}$.

This monograph deals mainly with the solution of the above PDE using its corresponding integral equation formulation, due to recent developments enabling much more efficient and much more rapidly convergent solutions, making it possible for us to obtain solutions over infinite domains [2]. We thus first derive the following integral equation (IE), which can be written in the operator form

$$\mathbf{u} = \mathbf{v} + \mathbf{N}\,\mathbf{u}, \tag{4}$$

where the terms on the right-hand side are defined as follows:

$$\begin{aligned} \mathbf{v}(\bar{r}, t) &= \int_{\mathbb{R}^3} \mathscr{G}(\bar{r} - \bar{r}', t)\,\mathbf{u}^0(\bar{r})\,d\bar{r}' \\ \mathbf{N}\,\mathbf{u}(\bar{r}, t) &= \int_0^t \int_{\mathbb{R}^3} \left\{\left(\left(\nabla' \mathscr{G}\right) \cdot \mathbf{u}(\bar{r}', t')\right)\,\mathbf{u}(\bar{r}', t') \right. \\ &\quad \left. + \left(\nabla' \mathscr{G}\right)\,p(\bar{r}', t')\right\}\,d\bar{r}'\,dt'. \end{aligned} \tag{5}$$

In (5) ∇' indicates the gradient taken with respect to $\bar{r}'$, and we have written $\mathscr{G}$ for $\mathscr{G}(\bar{r} - \bar{r}', t - t')$.

According to [1], the initial condition $\mathbf{u}^0$ must belong to a class of functions $\mathbf{S}^3$ which are infinitely differentiable with respect to all variables and which are rapidly decreasing with respect to each spacial variable on the real line. This class is described in more detail in Sect. 1.2 below. We also introduce in Chap. 2 a class of analytic functions $\mathbf{A}$ which is a subclass of $\mathbf{S}$, which we prove to be dense in the class $\mathbf{S}$, and to which the solutions of (4) belong to whenever $\mathbf{u}^0 \in \mathbf{A}$.

After proving the denseness of the class $\mathbf{A}$ in $\mathbf{S}$, we prove in Chap. 3 that the integral part of the above IE maps functions of $\mathbf{A}$ back into $\mathbf{A}$; in Chap. 4 we prove that if the initial condition vector $\mathbf{u}^0$ belongs to $\mathbf{A}$ and is divergence-free, then the above (IE) has a solution in $\mathbf{A}$ for all T sufficiently small; in Chap. 5 we introduce an iterative method of solution of the IE; and in Chap. 6 we provide two explicit examples and its numerical solution. Appendix A provides a detailed step-by-step description of our method of solution including an explicit *Mathematica* program based on our explicit algorithmic procedure. Appendix B contains for demonstration purposes an explicit example of a result data file generated by our algorithm.

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Contents

Chapter 1
Introduction, PDE, and IE Formulations

Abstract In this chapter we first state the Navier–Stokes (N–S) problem as a system of nonlinear partial differential equations (PDE) along with initial conditions. We then convert this system of PDE to a system of integral equations (IE).

1.1 Introduction: The PDE Problem

The partial differential equation (PDE) problem which we shall consider is the following, as presented in [1], i.e., let $\mathbb{R} = (-\infty, \infty)$ denote the real line, let T denote a positive number, and let the velocity vector $\mathbf{u} = \mathbf{u}(\bar{r}, t) = \left(u^1,\ u^2,\ u^3\right)$ satisfy the *Navier–Stokes* (N–S) PDE problem

$$\frac{\partial u^j}{\partial t} - \varepsilon\, \Delta\, u^j = -\sum_{k=1}^{3} u^k \frac{\partial u^j}{\partial x_k} - \frac{\partial p}{\partial x_j}, \quad (\bar{r}, t) \in \mathbb{R}^3 \times [0, T]. \tag{1.1}$$

This problem is to be solved subject to the divergence-free condition[1]

$$\operatorname{div} \mathbf{u} = \sum_{j=1}^{3} \frac{\partial u^j}{\partial x_j} = 0, \quad (\bar{r}, t) \in \mathbb{R}^3 \times [0, T], \tag{1.2}$$

and subject to initial conditions

$$\mathbf{u}^0(\bar{r}) = \mathbf{u}(\bar{r}, 0) \quad \bar{r} \in \mathbb{R}^3. \tag{1.3}$$

Here, p denotes the pressure, $\mathbf{u}^0 = \left(u^{0,1}, u^{0,2}, u^{0,3}\right)$ is a given divergence-free velocity field on $\mathbb{R}^3$, ε is a positive coefficient (the viscosity), Δ denotes the Laplacian, $\Delta = \sum_{j=1}^{3} \frac{\partial^2}{(\partial x^j)^2}$, and p denotes the pressure.

[1]Fefferman actually had $[0, \infty)$ instead of $[0, T]$. We shall however consider only the case of $[0, T]$ in this monograph.

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1.2 The Classes $\mathbf{S}^n$ and $\mathbf{S}^n_T$

Let n denote a positive integer, and let $\mathbf{S}^n$ denote the family of all infinitely differentiable functions $g = g(\bar{r})$ defined on $\mathbb{R}^n$ such that

$$|\partial_{\bar{r}}^{\alpha} g(\bar{r})| \leq C_{\alpha,K}\,(1+|\bar{r}|)^{-K} \;\; \bar{r} \in \mathbb{R}^n, \;\; \text{for any constants } \alpha \;\text{ and }\; K, \tag{1.4}$$

for some constant $C_{\alpha,K}$ is a constant and similarly, let $\mathbf{S}^n_T$ denote the family of all functions $g = g(\bar{r}, t)$ defined on $\mathbb{R}^n \times [0, T]$, such that for any constants α, K and any integer $m \geq 0$,

$$|\partial_{\bar{r}}^{\alpha}\,\partial_t^m g(\bar{r})| \leq C_{\alpha;m,K}\,(1+|\bar{r}|)^{-K}, \;\; \bar{r} \in \mathbb{R}^n, \tag{1.5}$$

for some constant $C_{\alpha,m,K}$.

1.3 The Corresponding IE Problem

Let us denote by $\mathscr{G} = \mathscr{G}(\bar{r}, t)$, the well-known Green's function of the heat equation on $\mathbb{R}^3 \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$,

$$\mathscr{G}(\bar{r}, t) = \frac{1}{(4\,\pi\,\varepsilon\,t)^{3/2}}\,\exp\left(-\frac{r^2}{4\,\varepsilon\,t}\right); \quad r = |\bar{r}|. \tag{1.6}$$

This Green's function is the bounded solution to the following initial value problem on $\mathbb{R}^3 \times \mathbb{R}_+$:

$$\begin{aligned} \mathscr{G}_t(\bar{r}, t) - \varepsilon\,\Delta\,\mathscr{G}(\bar{r}, t) &= \delta(t)\,\delta^{(3)}(\bar{r}), \\ \mathscr{G}(\bar{r}, 0^+) &= 0, \end{aligned} \tag{1.7}$$

where $\delta(t)$ and $\delta^{(3)}$ denote the one- and three-dimensional delta functions, where $\mathbb{R} = (-\infty, \infty)$ and where $\mathbb{R}^+ = (0, \infty)$.

It is straightforward to use this Green's function $\mathscr{G}$ to transform the system (1.1) of three PDE into the following system of three integral equations (IE), for $j = 1, 2, 3$,

$$\begin{aligned} u^j(\bar{r}, t) = &\int_{\mathbb{R}^3} \mathscr{G}(\bar{r} - \bar{r}', t)\,u^{0,j}(\bar{r})\,d\bar{r}' \\ &- \int_0^t \int_{\mathbb{R}^3} \mathscr{G}(\bar{r} - \bar{r}', t - t')\left(\sum_{k=1}^3 u^k\,\frac{\partial u^j}{\partial x_k'} + \frac{\partial p}{\partial x_j'}\right) d\bar{r}'\,dt'. \end{aligned} \tag{1.8}$$

We omit the trivial proof of the following lemma.

Lemma 1.3.1. *If $g \in \mathbf{S}_T^3$, and if γ is defined by either of the integrals*

$$\gamma(\bar{r},t) = \int_{\mathbb{R}^3} \mathscr{G}(\bar{r}-\bar{r}',t)\, g(\bar{r}',t)\, d\bar{r}',$$
$$\gamma(\bar{r},t) = \int_0^t \int_{\mathbb{R}^3} \mathscr{G}(\bar{r}-\bar{r}',t-t')\, g(\bar{r}',t')\, d\bar{r}'\, dt', \tag{1.9}$$

in which $t \in (0,T)$, then $\gamma \in \mathbf{S}_T^3$. Moreover, we have for $j = 1,\ 2,\ 3$, that

$$\begin{aligned}\gamma_{x_j}(\bar{r},t) &= \int_{\mathbb{R}^3} \mathscr{G}_{x_j}(\bar{r}-\bar{r}',t)\, g(\bar{r}',t)\, d\bar{r}' \\ &= -\int_{\mathbb{R}^3} \mathscr{G}_{x_j'}(\bar{r}-\bar{r}',t)\, g(\bar{r}',t)\, d\bar{r}' \\ &= \int_{\mathbb{R}^3} \mathscr{G}(\bar{r}-\bar{r}',t)\, g_{x_j'}(\bar{r}',t)\, d\bar{r}'.\end{aligned} \tag{1.10}$$

Furthermore, the functions γ_{x_j} of (1.10) also belong to $\mathbf{S}_T^3$.

Lemma 1.3.1 enables differing IE expressions for the same system (1.8).

Next, in Theorem 1.3.1 which follows, we present an IE system which is equivalent to the one in (1.8), and with which we shall work within the remainder of this monograph.

Theorem 1.3.1. *If each component of $\mathbf{u}$ belongs to $\mathbf{S}_T^3$, and if $\mathbf{u}$ is divergence-free, then the system of differential equations (1.3) is equivalent to the integral equation formulation*

$$\mathbf{u} = \mathbf{v} + \mathbf{N}\,\mathbf{u}, \tag{1.11}$$

where the terms on the right-hand side are defined as follows:

$$\mathbf{v}(\bar{r},t) = \int_{\mathbb{R}^3} \mathscr{G}(\bar{r}-\bar{r}',t)\, \mathbf{u}^0(\bar{r})\, d\bar{r}'$$
$$\mathbf{N}\,\mathbf{u}(\bar{r},t) = \int_0^t \int_{\mathbb{R}^3} \left\{ \left(\left(\nabla' \mathscr{G}\right) \cdot \mathbf{u}(\bar{r}',t') \right) \mathbf{u}(\bar{r}',t') + \right. \tag{1.12}$$
$$\left. (\nabla' \mathscr{G})\ p(\bar{r}',t') \right\} d\bar{r}'\, dt'.$$

In (1.12) ∇' indicates the gradient taken with respect to $\bar{r}'$, and in (1.12), $\mathscr{G} = \mathscr{G}(\bar{r} - \bar{r}', t - t')$.

Proof. Using (1.6), (1.7) as well as Lemma 1.3.1, and assuming that $\mathbf{u}$ is divergence-free, i.e., that $\sum_{j=1}^{3} u^j_{x_j} = 0$, we get the vector IE (1.11).

■

1.4 The Pressure p

We here derive an integral expression for the pressure p. Our derived expression is obtained under the assumption that if $\mathbf{u}$ is divergence-free, then so is the vector on the right-hand side of the IE (1.11).

Theorem 1.4.1. *Let each component of $\mathbf{u}^0 \in \mathbf{S}^3$ and let each vector $\mathbf{u} \in \mathbf{S}^3_T$ on the right-hand side of (1.12) be divergence-free. Let the pressure p be selected such that the vector on the right-hand side of (1.12) is also divergence-free. Then p is given by*

$$p(\bar{r}, t) = \int_{\mathbb{R}^3} \mathscr{G}_0(\bar{r} - \bar{r}')\, g(\bar{r}', t)\, d\,\bar{r}' \tag{1.13}$$

where

$$\begin{aligned} \mathscr{G}_0(\bar{r}) &= \frac{1}{4\,\pi\, r}, \\ g(\bar{r}, t) &= \sum_{j=1}^{3} \sum_{k=1}^{3} u^j_{x^k}(\bar{r}, t)\, u^k_{x^j}(\bar{r}, t). \end{aligned} \tag{1.14}$$

Proof. By differentiating each term on the right-hand side of (1.8) with respect to x_j, summing over j and in that way, "forcing" (1.2), we get

$$\sum_{j=1}^{3} u^j_{x_j} = 0 = \sum_{j=1}^{3} \left(A^j - B^j - C^j\right), \tag{1.15}$$

where

$$\begin{aligned} A^j &= \int_{\mathbb{R}^3} \mathscr{G}_{x_j}(\bar{r} - \bar{r}', t)\, u^{0,j}(\bar{r}')\, d\bar{r}' \\ B^j &= \int_0^t \int_{\mathbb{R}^3} \mathscr{G}_{x_j}(\bar{r} - \bar{r}', t - t') \sum_{k=1}^{3} u^k(\bar{r}', t')\, u^j_{x_k}(\bar{r}', t')\, d\bar{r}'\, dt' \\ C^j &= \int_0^t \int_{\mathbb{R}^3} \mathscr{G}_{x_j}(\bar{r} - \bar{r}', t - t')\, p_{x'_j} d\bar{r}'\, dt'. \end{aligned} \tag{1.16}$$

For the term A^j in (1.16) we clearly have $\sum_{j=1}^3 A^j = 0$, since we assumed that $\mathbf{u}^0$ is divergence-free. Next, by our assumption that the right-hand side of (1.12) is also divergence-free, we must also have $\sum_{j=1}^3 (B^j + C^j) = 0$. But by application of Lemma 1.3.1 to this sum, we arrive at the equation

$$0 = \int_0^t \int_{\mathbb{R}^3} \mathscr{G}(\bar{r} - \bar{r}', t - t') \left\{ g(\bar{r}', t') + \Delta_{\bar{r}'} p(\bar{r}', t') \right\} d\bar{r}' \, dt', \tag{1.17}$$

in which g is given in (1.14) above.

In this case, because the right-hand side of (1.12) will also be divergence-free we determine $p(\bar{r}, t)$ such that

$$\Delta_{\bar{r}} p(\bar{r}, t) = -g(\bar{r}, t), \quad (\bar{r}, t) \in \mathbb{R}^3 \times [0, T], \tag{1.18}$$

with g given in (1.14). There is only one solution to this PDE problem which satisfies (1.18) and which is bounded on $\mathbb{R}^3$ for all $t \in [0, T]$; this solution is given by the statement of Theorem 1.4.1 above.

■

1.5 Modifying the IE

The above derived expression (1.13) for the pressure p leads to a more complicated kernel in the N–S IE, which we will now simplify.

If we substitute p as given by Theorem 1.4.1 into the IE (1.12), then the resulting right-hand side of the IE involves an integral over $\mathbb{R}^6 \times (0, T)$. We can reduce this integral into a lower dimensional integral based on the result of the following lemma.

Lemma 1.5.1. *Let $\mathscr{K}$ be defined by*

$$\mathscr{K}(\bar{r}, t) = \int_{\mathbb{R}^3} \mathscr{G}(\bar{r} - \bar{r}', t) \, \mathscr{G}_0(\bar{r}') \, d\bar{r}', \tag{1.19}$$

where $\mathscr{G}$ and $\mathscr{G}_0$ are given in (1.6) and (1.14) respectively. Then $\mathscr{K}$ is also given by

$$\mathscr{K}(\bar{r}, t) = \frac{1}{4\,\pi^{3/2}\,(\varepsilon t)^{1/2}} \int_0^1 \exp\left(-\frac{r^2}{4\,\varepsilon\, t}\, y^2 \right) dy. \tag{1.20}$$

Proof. We set $\bar{\Lambda} = (\lambda_1, \lambda_2, \lambda_3)$, and $\Lambda = |\bar{\Lambda}|$. It is well known that the Fourier transforms of $\mathscr{G}(\bar{r}, t)$ and $\mathscr{G}_0(\bar{r})$ taken with respect to $\bar{r}$ are $\exp(-\Lambda^2\, \varepsilon\, t)$ and Λ^{-2}

respectively. Hence, it follows from (1.20) that $\mathscr{K}$ is just the inverse transform of the product of these transforms, i.e.,

$$\mathscr{K}(\bar{r}, t) = \frac{1}{8\,\pi^3} \int_{\mathbb{R}^3} e^{-i\bar{\Lambda}\cdot\bar{r}}\, \Lambda^{-2}\, \exp(-\Lambda^2\,\varepsilon\, t)\, d\bar{\Lambda}. \tag{1.21}$$

Transforming this integral from Cartesian to spherical coordinates, so that the element of volume $d\bar{\Lambda}$ becomes $\Lambda^2\, d\Omega(\widehat{\bar{\Lambda}})$ where $\widehat{(\cdot)}$ denotes a unit vector and where $d\Omega(\widehat{\bar{\Lambda}})$ denotes the solid angle, and using the easily verifiable result [3],

$$\int_{|\hat{\bar{\Lambda}}|=1} e^{-i\bar{\Lambda}\cdot\bar{r}}\, d\Omega(\bar{\Lambda}) = 4\,\pi\, \frac{\sin(\Lambda\, r)}{\Lambda\, r}, \tag{1.22}$$

that is obtainable by expanding $(\Lambda\, r)^{-1}\, \sin(\Lambda\, r)$ in powers of $\Lambda\, r$ and then performing term wise integration, we arrive at (1.20).

■

1.6 The Theoretical Iteration Scheme

In this section we introduce two iterative methods for solving (1.11)–(1.12), the first, a Neumann-type iterative method, which we shall use to prove existence and uniqueness of the solution, and the second, a Gauss–Seidel type method, which we shall use in Chap. 5 below to get a numerical solution.

1.6.1 Von Neumann Iteration

Let us write the IE (1.11)–(1.12) in the IE form

$$\mathbf{u} = \mathbf{v} + \mathbf{N}\,\mathbf{u}; \quad \mathbf{N}\,\mathbf{u} = \mathbf{P}\,\mathbf{u} + \mathbf{Q}\,\mathbf{u}, \tag{1.23}$$

where $\mathbf{v}$ is given in (1.12), and where

$$\begin{aligned} \mathbf{P}\,\mathbf{u}(\bar{r}, t) &\\ = \int_0^t \int_{\mathbb{R}^3} \left(\left(\nabla' \mathscr{G}(\bar{r} - \bar{r}', t - t')\right) \cdot \mathbf{u}(\bar{r}', t') \right) \mathbf{u}(\bar{r}', t')\, d\bar{r}'\, dt' &\\ \mathbf{Q}\,\mathbf{u}(\bar{r}, t) = \int_0^t \int_{\mathbb{R}^3} \left(\left(\nabla' \mathscr{K}(\bar{r} - \bar{r}', t - t')\right) g(\bar{r}', t') \right) d\bar{r}'\, dt'. \end{aligned} \tag{1.24}$$

In (1.24), g, $\mathscr{G}$, and $\mathscr{K}$ are given in (1.14), (1.6), and (1.19)–(1.20), respectively.

One way to solve (1.23) for sufficiently small T is by use of the classical successive approximation scheme

$$U_{n+1} = \mathbf{v} + \mathbf{N}(U_n), \quad n = 0, 1, \ldots, \tag{1.25}$$

with $U_0 = \mathbf{v}$. In Chap. 4 we shall prove that this procedure converges on $\mathbb{R}^3 \times [0, T]$ for suitably restricted ranges of T and $\mathbf{v}$.

1.6.2 Gauss–Seidel Iteration

Instead of using the scheme (1.25) to solve (1.12) we shall use the following *Gauss–Seidel*–like scheme,

$$\begin{aligned} u^1_{n+1} &= v^1 + \mathbf{N}^1(u^1_n, u^2_n, u^3_n), \\ u^2_{n+1} &= v^2 + \mathbf{N}^2(u^1_{n+1}, u^2_n, u^3_n), \\ u^3_{n+1} &= v^3 + \mathbf{N}^3(u^1_{n+1}, u^2_{n+1}, u^3_n), \end{aligned} \tag{1.26}$$

starting with $\mathbf{u}_0 = \mathbf{v}$, where

$$\begin{aligned} v^j(\bar{r}, t) &= \int_{\mathbb{R}^3} \mathscr{G}(\bar{r} - \bar{r}', t)\, u^{0,j}(\bar{r}')\, d\bar{r}' \\ \mathbf{N}^j\mathbf{u} &= P^j\mathbf{u} + Q^j\mathbf{u}, \end{aligned} \tag{1.27}$$

and where

$$\begin{aligned} &\mathbf{P}^j\mathbf{u}(\bar{r}, t) \\ &\quad = \int_0^t \int_{\mathbb{R}^3} \left(\nabla' \mathscr{G}(\bar{r} - \bar{r}', t - t') \cdot \mathbf{u}(\bar{r}', t')\right) u^j(\bar{r}', t')\, d\bar{r}'\, dt', \\ &\mathbf{Q}^j(\mathbf{u})(\bar{r}, t) = \int_0^t \int_{\mathbb{R}^3} \mathscr{K}_{x_j}(\bar{r} - \bar{r}', t - t')\, g(\bar{r}', t')\, d\bar{r}'\, dt', \end{aligned} \tag{1.28}$$

where g is given in (1.14). In general, the scheme (1.26) converges more rapidly than the scheme (1.25), and indeed, our programming examples comparing this Gauss–Seidel scheme with the one in [2] bears this out. It thus enables a convergent iterative solution for larger values of T than that for (1.25).

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Chapter 2
Spaces of Analytic Functions

Abstract We present here spaces of analytic functions $\mathbf{A}^n_{\alpha,d} \subset \mathbf{S}^n$ as well as spaces, $\mathbf{A}^n_{\alpha,d,T} \subset \mathbf{S}^n_T$, $n = 1, 2, 3$. In this chapter, we shall study the properties of these spaces, we shall prove in Chap. 3 that if the components of the initial condition vector $\mathbf{u}^0$ belong to $\mathbf{A}^3_{\alpha,d}$ then each component of $\mathbf{N}\,\mathbf{u}$ of (1.23) belongs to $\mathbf{A}^3_{\alpha,d,T}$, and we shall furthermore prove in Chap. 4 that the solution to (1.23) belongs to $\mathbf{A}^3_{\alpha,d,T}$, for all T sufficiently small. These spaces are in fact special cases of the spaces $\mathbf{S}^n$ and $\mathbf{S}^n_T$ introduced in Sect. 1.2. They provide several conveniences, such as enabling sharper error bounds and yielding exponential convergence of our approximate solution which we obtain in Chap. 5.

2.1 The Spaces $\mathbf{A}^n_{\alpha,d}$ and $\mathbf{A}^n_{\alpha,d,T}$

Let n denote a positive integer, and let us now define vector spaces of functions $\mathbf{A}^n_{\alpha,d}$ and $\mathbf{A}^n_{\alpha,d,T}$.

Definition 2.1.1. Set $\bar{r}^* = \bar{r} + i\,\bar{\rho}$, with $\bar{r} \in \mathbb{R}^n$, $\bar{\rho} \in \mathbb{R}^n$, set $r = |\bar{r}|$, and $\rho = |\bar{\rho}|$.

(a) Corresponding to some positive numbers α and d, let $\mathbf{A}_{\alpha,d}$ denote the family of all functions f with the following properties:

(i) *Analyticity property.* There exists a positive number $d' > d$, such that each f is analytic in the domain

$$\mathscr{D}^n_{d'} = \{\bar{r}^* = \bar{r} + i\,\bar{\rho} \in \mathbb{C}^n : \; \rho < d'\}; \tag{2.1}$$

and

(ii) *Asymptotic property.* There exist positive numbers $C = C(f, d')$ and $\alpha' > \alpha$, such that for all $\bar{r}^* \in \mathscr{D}^n_{d'}$,

$$|f(\bar{r}^*)| < C\,\exp(-\alpha'\,r). \tag{2.2}$$

Notice that if f and g belong either to $\mathbf{A}_{\alpha,d}$ or to $\mathbf{A}_{\alpha,d,T}$, then so does h, where for any constants a, b, and c, $h = af + b\,g$, or $h = cf$.

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(b) We also define the space $\mathbf{A}^n_{\alpha,d,T}$ of functions $f = f(\bar{r}^*, t^*)$ such that $f(\cdot, t) \in \mathbf{A}^n_{\alpha,d}$ for each fixed $t \in [0, T]$, and such that $f(\bar{r}, \cdot)$ is an analytic and uniformly bounded function of $t = t^*$ in the "eye-shaped" region

$$\mathscr{D}_{d',T} = \{t^* \in \mathbb{C} : |\arg(t^*/(T - t^*))| < d'\}. \tag{2.3}$$

(c) The spaces $\mathbf{A}^n_{\alpha,d}$ and $\mathbf{A}^n_{\alpha,d,T}$, are normed for $p \in [1, \infty)$ by $\|\cdot\|_p$, i.e.,

$$\begin{aligned} \|f\|_p &= \left(\int_{\mathbb{R}^n} |f(\bar{r})|^p \, d\bar{r}\right)^{1/p}, \text{ if } p \in [1, \infty) \text{ and} \\ \|f\|_\infty &= \sup_{\bar{r}\in\mathbb{R}^n} |f(\bar{r})|. \end{aligned} \tag{2.4}$$

and similarly,

$$\begin{aligned} \|f\|_{p,T} &= \left(\int_{\mathbb{R}^n} \int_0^T |f(\bar{r}, t)|^p \, d\bar{r} \, dt\right)^{1/p}, \text{ if } p \in [1, \infty) \text{ and} \\ \|f\|_{\infty,T} &= \sup_{\bar{r}\in\mathbb{R}^n, t\in(0,T)} |f(\bar{r}, t)|. \end{aligned} \tag{2.5}$$

The following theorem describes an important and beautiful property of the class of functions $\mathbf{A}^n_{\alpha,d}$.

Theorem 2.1.1. *In the notation of Definition 2.1.1, let $f \in \mathbf{A}^n_{\alpha,d}$. Let $\hat{f}$ denote the Fourier transform of $\hat{f}$, i.e.,*

$$\hat{f}(\bar{\Lambda}) = \int_{\mathbb{R}^n} f(\bar{r}') \exp(i \bar{\Lambda} \cdot \bar{r}') \, d\bar{r}'. \tag{2.6}$$

Then $\hat{f} \in \mathbf{A}^n_{d,\alpha}$.

Proof. The one-dimensional version of this result is found in Theorem 26 of Sect. 1.27 of [3]. The proof of the three-dimensional case is similar, and we omit it.

■

Upon recalling the inverse Fourier transform formula for $\mathbb{R}^3$ [see, e.g., (1.21)] we also have by Parseval's theorem that

$$\|\hat{f}\|_2 = (2\pi)^{3/2} \|f\|_2, \tag{2.7}$$

and similarly for $\|\hat{f}\|_{2,T}$.

Definition 2.1.2. Let n denote a positive integer, and set $\mathbf{b} = (b_1, \ldots, b_n)$, with b_j nonnegative integers and with $|\mathbf{b}| = \sum_{j=1}^n b_j$, and define

$$D^{\mathbf{b}} f = \frac{\partial^{|\mathbf{b}|} f}{\partial(x^1)^{b_1}, \ \ldots, \ \partial(x^n)^{b_n}}. \tag{2.8}$$

Theorem 2.1.2. *Let d, d', α, and α' defined as in Definition 2.1.1 be given.*

(i) If $f \in \mathbf{A}^n_{\alpha,d}$, then $D^{\mathbf{b}} f \in \mathbf{A}^n_{\alpha,d}$;
(ii) If f is analytic in the domain $\mathscr{D}^n_{d'}$ and if for all $\bar{r}^* = \bar{r} + i\,\bar{\rho}$ in $\mathscr{D}^n_{d'}$ and constants $C > 0$, $m \geq 0$ and $\alpha'' \in (\alpha, \alpha')$, we have $f(\bar{r}^*) \leq C\, r^m \exp(-\alpha'' r)$, then $f \in \mathbf{A}^n_{\alpha,d}$.

Proof. Part (i): In the notation of Definition 2.1.1, taking $d'' \in (d, d')$ and $\alpha'' \in (\alpha, \alpha')$, we have for any $\bar{r}^* = (x^1, x^2, x^3) \in \mathscr{D}^n_{d''}$ and any $\varepsilon \in \min(0, d' - d'', (\alpha' - \alpha''))$, that

$$\frac{\partial f}{\partial x^j} = \frac{k\,!}{2\,\pi\, i} \int_{|\bar{r}^* - \bar{\rho}^*| = \varepsilon} \frac{f(\bar{\rho}^*)\, d\xi^j}{(\xi^j - x^j)^k}. \tag{2.9}$$

Hence by our assumption that for all $\bar{r}^* \in \mathscr{D}^n_{\alpha',d'}$ we have $|f(\bar{r}^*)| \leq C \exp(-\alpha'\, r)$, it follows that $|f(\bar{\rho}^*| \leq C\, \exp(-\alpha'\, \rho + \varepsilon) < C\, \exp(-\alpha''\, \rho)$, and so

$$\left|\frac{\partial f}{\partial x^j}\right| \leq \frac{C}{2\,\pi\,\varepsilon^k}\, \exp(-\alpha''\, r), \tag{2.10}$$

This proves Theorem 2.1.2 for the case of $f = f(\bar{r}^*)$ when taking one derivative with respect to x^j. The proof for the case of $D^{\mathbf{b}} f$ is similar, just by repeating the one-dimensional argument. The proof for the case of $f = f(\bar{r}^*, t)$ with $t \in [0, T]$ is also similar, and we omit it.

Part (ii). Let us select $\varepsilon > 0$, such that $\alpha'' - \varepsilon > \alpha$, and let us then select $R > 0$, such that if $r \geq R$, then $r^m \leq \exp(\varepsilon r)$. Then we have $C\, r^m\, e^{\alpha''\, r} < C\, e^{\beta\, r}$, for $r \geq R$, where $\beta = \alpha'' - \varepsilon > \alpha$. We now select $C' > 0$ such that $C'\, e^{\varepsilon\, R} = C\, R^m$. Then $C'\, e^{\beta\, r} \leq C\, r^m\, e^{\alpha''\, r}$ for $r \in (0, \infty)$.

∎

2.2 Denseness of $\mathbf{A}^n_{\alpha,d}$ in $\mathbf{S}^n$

As we already mentioned, our preference is to work with the spaces $\mathbf{A}^n_{\alpha,d}$, not only for computing the solution of the N–S equations efficiently and accurately, but also for obtaining a simpler proof of existence of the solution to the equations. We first show in this section that the Sinc spaces are dense in the spaces $\mathbf{S}^n$. We then use this result in Sect. 2.2.2 to show that the spaces $\mathbf{A}^n_{\alpha,d}$ defined above are dense in the spaces $\mathbf{S}^n$.

Now, let $h > 0$, let $k \in \mathbb{Z}$, let $\xi \in \mathbb{C}$, let $\mathbf{k} \in \mathbb{Z}^n$, and let $\bar{r} \in \mathbb{R}^n$. Let us define the Sinc function $S(k,h)(\xi)$ in one dimension and a product $\mathscr{S}(\mathbf{k},h)(\bar{r}^*)$, in n dimensions, where now $\bar{r}^* \in \mathbb{C}^n$.

$$\begin{aligned} S(k,h)(x) &= \frac{\sin\left(\frac{\pi}{h}(x - k\,h)\right)}{\frac{\pi}{h}(x - k\,h)}, \\ \mathscr{S}(\mathbf{k},h)(\bar{r}^*) &= \prod_{j=1}^{n} S(k_j,h)(\xi^j), \quad \xi^j \in \mathbb{C}. \end{aligned} \tag{2.11}$$

2.2.1 Denseness of Sinc Approximation in $\mathbf{S}^n$

We shall prove the denseness of Sinc approximation only with respect to the "sup" norm; we omit the proofs of $\mathbf{L}^2$—approximation at this time, since such proofs follow almost verbatim from the $\mathbf{L}^\infty$ ones of this section. We thus prove only the following theorem in the remainder of this section.

Theorem 2.2.1. *Let $\bar{r} = (x^1, \ldots, x^n) \in \mathbb{R}^n$, let $g \in \mathbf{S}^n$, let $h > 0$, let N be a positive integer, and set*

$$g_{N,h}(\bar{r}) = \sum_{\mathbf{k} \in \mathbb{Z}_N^n} g(\mathbf{k}\,h)\,\mathscr{S}(\mathbf{k},h)(\bar{r}), \tag{2.12}$$

where

$$\mathbb{Z}_N^n = \{\mathbf{k} = (k^1, \ldots, k^n) \in \mathbb{Z}^n : -N \le k^j \le N,\ j = 1, \ldots, n\}. \tag{2.13}$$

Given any positive number ε, we can select $h > 0$ and N such that

$$\|g - g_{N,h}\|^\infty = \sup_{\bar{r} \in \mathbb{R}^n} |g(\bar{r}) - g_{N,h}(\bar{r})| < \varepsilon. \tag{2.14}$$

We split the proof of this theorem into the proofs of some lemmas.

We omit the proof of the following lemma since the (a)-Part of it is well known ([2], Theorem 1.2.1), and since the (b)-Part follows directly from the (a)-Part.

Lemma 2.2.1.

(a) If $(x,y) \in \mathbb{C} \times (-\pi/h, \pi/h)$, then

$$e^{i\,x\,y} = \sum_{j \in \mathbb{Z}} S(j,h)(x)\,e^{i\,j\,h\,y}. \tag{2.15}$$

(b) Let Q^n be defined by

$$Q^n = \left\{\bar{\rho} = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n : |\xi^j| < \pi/h,\ j = 1, \ldots, n\right\}. \tag{2.16}$$

Then we have the identity

$$\exp(i\,\bar{r} \cdot \bar{\rho}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathscr{S}(\mathbf{k}, h)(\bar{r})\, \exp(i\,j\,h\,\bar{\rho} \cdot \mathbf{k}) \tag{2.17}$$

for all $(\bar{r}, \bar{\rho}) \in \mathbb{C}^n \times Q^n$.

Remark 2.2.1. The function on the right-hand side of (2.15) can be extended as a function of y to the real line $\mathbb{R}$, where for arbitrary integer m it is a periodic copy of e^{ixy} on $(-\pi/h, \pi/h)$ to the interval $((2\,m-1)\,\pi/h, (2\,m+1)\,\pi/h)$. The infinite series on the right-hand side of (2.15) is discontinuous at each of the points $(2\,m+1)\,\pi/h$ where it takes on the value $\cos(\pi\,x)$. In particular, both functions, that on the left-hand side of (2.15) and that on the right-hand side, are bounded by 1 on $\mathbb{R} \times \mathbb{R}$. Similarly, both sides of (2.17) are identically equal on $\mathbb{C} \times Q^n$, and also the right-hand side of (2.17) has periodic extension to all of R^n, similar to that of (2.15) for the $n = 1$ case, and so both sides of (2.17) are bounded by 1 on $\mathbb{R}^n \times \mathbb{R}^n$.

Let $g_{N,h}$ be defined as in (2.12) and set

$$g_h(\bar{r}) = \lim_{N\to\infty} g_{N,h}(\bar{r}). \tag{2.18}$$

Lemma 2.2.2. *Let $g \in \mathbf{S}^n$, and let g_h be defined as in (2.18). Given $\varepsilon > 0$ there exists $h > 0$ such that*

$$\|g - g_h\|^\infty = \sup_{\bar{r}\in\mathbb{R}^n} |g(\bar{r}) - g_h(\bar{r})| < \frac{\varepsilon}{2}. \tag{2.19}$$

Proof. Let $\hat{g}$ denote the n-dimensional Fourier transform of g, i.e., with $\bar{r}$ and $\bar{\rho}$ in $\mathbb{R}^n$,

$$\hat{g}(\bar{r}) = \int_{\mathbb{R}^n} \exp(i\,\bar{r} \cdot \bar{\rho})\, g(\bar{\rho})\, d\bar{\rho}. \tag{2.20}$$

It then follows immediately, upon using (2.17) and applying the inverse of the Fourier transform formula of (2.20), that

$$\begin{aligned} & g(\bar{r}) - g_h(\bar{r}) \\ & = \frac{1}{(2\,\pi)^n} \int_{\mathbb{R}^n} \left(\exp(-i\,\bar{r} \cdot \bar{\rho}) - \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathscr{S}(\mathbf{k}, h)(\bar{r})\, \exp(-i\,h\,\mathbf{k} \cdot \bar{\rho})\right) \cdot \\ & \quad \cdot \hat{g}(\bar{\rho})\, d\bar{\rho}. \end{aligned} \tag{2.21}$$