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New Directions in Paraconsistent Logic

5th WCP, Kolkata, India, February 2014

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Foreword

This book constitutes the proceedings of the 5th World Congress of Paraconsistent Logic, a kind of logic whose systematic development effectively began in the middle of the twentieth century.

Loosely speaking, a paraconsistent system of logic can be defined as a system that may be the underlying logic of inconsistent but nontrivial theories, i.e., inconsistent theories, in which there are sentences of their languages that are not provable.

The systems of paraconsistent logic may be envisaged from two basic perspectives: (a) as rivals of classical logic, for instance, when inconsistent though apparently nontrivial set theories are built as rivals of classical set theories and (b) as systems complementary to classical logic, when, for example, a paraconsistent negation is present together with classical negation.

An important characteristic of paraconsistent logic is that it has found numerous applications in philosophy, quantum mechanics, artificial intelligence, traffic control, medicine, economics, finances, and computing. So, we are in the presence of an area of logic which opened up new research directions in philosophy, science, and technology.

This volume is valuable not only due to the technical works it includes, but also because it contributes to show the meaning of paraconsistent logic and its connection with other domains of knowledge. It also testifies to how paraconsistent logic is spreading around the world, the 5th edition of this congress having been organized in India.

Florianópolis
September 2015

Newton C.A. da Costa

Preface

This book is a collection of papers presented at the 5th World Congress on Paraconsistency, which was organized at the Indian Statistical Institute (ISI), Kolkata, India, February 13–17, 2014.

A paraconsistent logic is a logic where there is a nonexplosive negation, i.e., from a proposition and its paraconsistent negation it is not necessarily possible to deduce anything. The expression "paraconsistent logic" was coined in a discussion between Newton da Costa and the Peruvian philosopher Francisco Miró Quesada. This expression had a booming effect as recalled by da Costa:

> Several years ago, I needed a convenient and meaningful denomination for a logic that did not eliminate contradictions from the outset as being false, i.e., as absolutely unacceptable. Miró Quesada helped me. On the one hand, it should be recalled that, by that time, all logics unavoidably condemned contradictions. The new logic in which I worked faced too much resistance, it was badly divulged, and those that got to know it were in general sceptics. By that time I wrote to Miró Quesada, who saw the new logic with great enthusiasm, requesting a name for it. I remember as it was today that he answered with three proposals: it could be called metaconsistent, ultraconsistent or paraconsistent. After commenting on these possible denominations, he stated that, from his viewpoint, he preferred the latter. The term paraconsistent sounded splendid and I began to use it, suggesting that people interested on this logic did the same. Two or three months later, the miracle took place; the term spread through the world, all the centres directly or indirectly related to logic, from northern to southern hemisphere, began to employ it. I believe that few times in the history of science (definitely in the history of logic) something similar has happened, for not only the word run the whole world, but the very logic called by Miró Quesada "paraconsistent" received a formidable push. It became one of the most discussed theories of logic of our time. (da Costa, "La Filosofia de la Logica de Francisco Miró Quesada Cantuarias," in *Logica, Razon y Humanismo,* Lima, 1992, pp. 69–78.)

Previous world congresses on paraconsistency were organized in the following locations:

- 1st Word Congress on Paraconsistency: Ghent, Belgium (1997)
- 2nd Word Congress on Paraconsistency: Juquehy, Brazil (2000)
- 3rd Word Congress on Paraconsistency: Toulouse, France (2003)
- 4th Word Congress on Paraconsistency: Melbourne, Australia (2008)

In India, paraconsistent logic is still not very well known, but people do have interest in the subject, and a few researchers have taken it quite seriously. In Indian ancient methodology, there was "chatuskoti," which had four corners of which one was both "yes" and "no." This implies that contradiction was not altogether rejected. That is why it was decided to organize the 5th edition of the world congress on paraconsistency in this country.

And to make paraconsistent logic better known in India, we decided to organize tutorials during this event. Three tutorials were given, and they are included in the first part of this book. The other parts of the books contain papers presented during the event, and a few others are by people who were not able to come.

The event was nice and relaxing. The ISI is a charming place surrounded by nature and with a convenient guest house. The people from ISI were enthusiastic and animated to organize this event. The members of the local organizing ISI team included Sisir Roy, Rana Barua, Probal Dasgupta, Kuntal Ghosh, and Guruprasad Kar. Kuntal led the team in an efficient manner, supported by some local students, who helped to make this event a success. One evening a beautiful cruise was organized on the Ganga.

Jean-Yves Beziau
Mihir Chakraborty
Soma Dutta

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Tutorials

Chapter 1
Tutorial on Inconsistency-Adaptive Logics

Diderik Batens

Abstract This paper contains a concise introduction to a few central features of inconsistency-adaptive logics. The focus is on the aim of the program, on logics that may be useful with respect to applications, and on insights that are central for judging the importance of the research goals and the adequacy of results. Given the nature of adaptive logics, the paper may be read as a peculiar introduction to defeasible reasoning.

Keywords Paraconsistent logic · Inconsistency-adaptive logic

Mathematics Subject Classication (2000) 03-01, 03B53, 03B60, 03A05

1.1 Introduction

By a *logic* I shall mean a function that assigns a consequence set to any premise set. So where $\mathcal{L}$ is a language schema, with $\mathcal{F}$ as its set of formulas and $\mathcal{W}$ as its set of closed formulas, a logic is a function $\wp(\mathcal{W}) \to \wp(\mathcal{W})$. The standard predicative language schema, viz. that of **CL** (classical logic), will be called $\mathcal{L}_s$; $\mathcal{F}_s$ its set of formulas and $\mathcal{W}_s$ its set of closed formulas.

Adaptive logics are formal logics but are not deductive logics. They do not define the meaning of logical symbols and are certainly not in the competition for the title 'standard of deduction'—that is, for delineating deductively correct inferences from incorrect inferences and from non-deductive inferences. To the contrary, adaptive logics explicate reasoning processes that are typically not deductive, viz. defeasible reasoning processes.

I am indebted to Mathieu Beirlaen for careful comments on a previous draft.

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Sometimes deductive logics are opposed to inductive logics. The expression "inductive logic" may refer to constructions that proceed, for example in terms of probabilities, as in Carnap's work [31]. Where the expression refers to a logic in the sense of the previous paragraph, inductive logics are a specific form of defeasible reasoning, next to many others. Handling inconsistency as described in the present paper is just one of them.[1]

A logic is *formal* iff its consequence relation is defined in terms of logical form. Some people identify this with the Uniform Substitution rule,[2] but that is a mistake because Uniform Substitution defines just one way in which a logic may be formal. Let me quickly spell out a different one. A language or language schema $\mathcal{L}$ will comprise one or more sets of non-logical symbols, for example sentential letters, predicative letters, letters for individual constants, etc. Consider all total functions f that map every such set to itself. Extend f to formulas, $f(A)$ being the result of replacing every non-logical symbol ξ in A by $f(\xi)$. A logic $\mathbf{L}$ is clearly formal iff the following holds: $A_1, \ldots, A_n \vdash_{\mathbf{L}} B$ iff, for every such f, $f(A_1), \ldots, f(A_n) \vdash_{\mathbf{L}} f(B)$.

Logics may obviously be presented in very different ways. Formal logics are usually presented as sets of rules, possibly combined with the somewhat special rules that are called axioms (and axiom schemata). Apart from many types of 'axiomatizations', logics are standardly characterized by a semantics, which has a rather different function. Deductive logics are typically Tarski logics. This means that they are reflexive ($\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$),[3] transitive (if $\Delta \subseteq Cn_{\mathbf{L}}(\Gamma)$, then $Cn_{\mathbf{L}}(\Delta) \subseteq Cn_{\mathbf{L}}(\Gamma)$), and monotonic ($Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$ for all Γ'). Another interesting property, which is required if a logic has to have static proofs,[4] is compactness (if $A \in Cn_{\mathbf{L}}(\Gamma)$ then there is a finite $\Gamma' \subseteq \Gamma$ such that $A \in Cn_{\mathbf{L}}(\Gamma')$).

This paper follows several conventions that I better spell out from the start. Classical logic, **CL**, will be taken as the standard of deduction. This is a purely pragmatic decision, not a principled one. Next, all metalinguistic statements are meant in their classical sense. More specifically, the metalinguistic negation will always be classical. So where I say that A is not a **L**-consequence of Γ, I rule out that A is a **L**-consequence of Γ. Similarly, I shall use "false" in its classical sense; no statement can be true as well as false in this sense. An inconsistent situation will be one in which both A and $\neg A$ are true, not one in which A is both true and false. There is a

[1] See, for example, [17] for many other real-life examples of reasoning forms for which there is no positive test. The import of a positive test is discussed further in the text.

[2] Uniform Substitution is rule of propositional logic. Predicative classical logic is traditionally axiomatized in terms of a finite set of rules and axiom schemata, rather than axioms. So no substitution rule is then required. Substitution rules in predicate logic have been studied [56] and the outcome is very instructive.

[3] The **L**-consequence set of Γ is defined as $Cn_{\mathbf{L}}(\Gamma) =_{df} \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$.

[4] Just think about usual proofs. Every formula in the proof is a consequence of the premise set and every proof may be extended into a longer proof by applications of the rules.

rather deep divide between paraconsistent logicians on these matters. There are those who claim that 'the true logic' is paraconsistent and that it should always be used, in particular in its own metalanguage. Some of these even take it that classical negation is not coherent, lacks sense and the like. Other paraconsistent logicians, with whom I side, have no objections against the classical negation or against its occurrence in the same language as a paraconsistent negation. This is related to the fact that they are pluralists, either in general or with respect to contexts. They might argue, for example, that consistent domains, like most paraconsistent logics themselves, are more adequately described by **CL** than by a paraconsistent logic.

A warning of a different kind is that the materials discussed in the subsequent pages have been studied at the predicative level. That I shall offer mainly propositional toy examples has a pedagogical rationale.

The last general survey paper that I wrote on adaptive logics was [20]. Meanwhile new results were and are being obtained, some of them are still unpublished. This may be as expected, but one aspect needs to be mentioned from the start. Quite a group of people have contributed to adaptive logics and have published in the field, many more than I shall mention below. While I was always eager to retain the unity of the domain, not everyone attached the same value to unification. Such a situation was obviously very useful to prevent that interesting things are left out of the picture—in principle the aim is to integrate directly or under a translation all potentially realistic first-order defeasible reasoning forms. As we shall see, this integrating frame is the standard format. Little changes were introduced over the years in an attempt to make it as embracing as possible. While most were improvements or clarifications, there was one development that I now consider as misguided. In the end it resulted in the systematic introduction of a set of new symbols to any language. These new symbols had their **CL**-meaning, whence they were called classical. They were added even if they duplicated existing symbols. In the second half of Sect. 1.11, I shall discuss the idea of adding classical symbols and the reasons for not adding them any more today.

The present paper is by no means a summary of all available results on adaptive logics. It merely provides an introduction to the central highlights. Moreover, this paper is explicitly intended as an introduction to inconsistency-adaptive logics, viz. adaptive logics that handle inconsistency. They concern compatibility, inductive generalization, abduction, prioritized reasoning, the dynamics of discussions, belief revision, abstract argumentation theory, deontic logic and so on. Most adaptive logics in standard format are not inconsistency-adaptive and have no connection to paraconsistency. Nevertheless, the present paper can also be read as an introduction to adaptive logics in general, with special attention to handling inconsistency and with illustrations from that domain. The reference section is not a bibliography of inconsistency-adaptive logics.

1.2 The Original Problem

Consider a theory T that was intended as consistent and was given **CL** as its underlying logic: $T = \langle \Gamma, \mathbf{CL} \rangle$, in which Γ is the set of non-logical axioms of T and $Cn_{\mathbf{CL}}(\Gamma)$ is the set of theorems of T, often simply called T. Suppose, however, that T turns out to be inconsistent. There are several well-documented examples of such situation, both in mathematics (Newton's infinitesimal calculus, Cantor's set theory, Frege's set theory, ...) and in the empirical sciences [30, 43, 44, 47, 51–53, 62]. Actually, it is not difficult to find more examples, especially in creative episodes, for example in scientists' notes.

What scientists do in such situations, is look for a consistent replacement for T. As history teaches, however, they do not look for a consistent replacement from scratch. To the contrary, they reason from T, trying to locate the problems in it. This reasoning obviously cannot proceed in terms of **CL** because **CL** validates Ex Falso Quodlibet: $A, \neg A \vdash_{\mathbf{CL}} B$. So the theory T, viz. its set of theorems $Cn_{\mathbf{CL}}(\Gamma)$ is trivial; it contains each and every sentence of the language. If **CL** is the criterion, all one can do is give up the theory and restart from scratch; but scientists do not do so. The upshot is that one should reason about T in terms of a paraconsistent logic, a logic that allows for non-trivial inconsistent theories. Note that any such logic has a semantics that contains inconsistent models—models that verify inconsistent sets of sentences.

It is useful to make a little excursion at this point because many people underestimate the difficulties arising in inconsistent situations. Time and again, people argue that one should figure out where the inconsistency resides and next modify the theory in such a way that the inconsistency disappears. They apparently think that it is easy to separate the consistent parts of a theory from the inconsistencies. Next, if they are very uninformed, they will think that one may choose one half of the inconsistency (or inconsistencies) and add that to the consistent part. If they are a bit better informed, they will realize that a conceptual shift may very well be required, that the new consistent theory should only contain the important statements from the consistent parts, or even a good approximation of them, and should only contain an approximation of one of the 'halves' of the inconsistencies. What is wrong with this reasoning, even with the sophisticated version, is that it is in general impossible to identify the consistent parts of a predicative theory. There is no general positive test for consistency. Being a consistent set of predicative statements is *not* semi-decidable. The set of consistent subsets of a set of predicative statements is *not* semi-recursive. So there is no systematic method, no Turing machine, that is able to identify an arbitrary consistent set as consistent, independent of the number of steps that one allows the Turing machine (or the person who applies the method) to take. So the reasoning from an inconsistent theory can only be explicated in terms of a paraconsistent logic.

Moving from **CL** to a paraconsistent logic has some drastic consequences. Not only Ex Falso Quodlibet, but many other rules are invalidated. Which rules will be invalidated will depend on the chosen paraconsistent logic. If one chooses a compact Tarski logic in which negation is paraconsistent but in which all other logical symbols have the same meaning as in **CL**, then Disjunctive Syllogism and several other rules are definitely invalidated. Incidentally, the *weakest* compact Tarski logic in which negation is paraconsistent but not paracomplete[5] and in which all other logical symbols have their **CL**-meaning is **CLuN**, to which I return in Sect. 1.3.

Let us first have a look at Disjunctive Syllogism (or rather at one of its forms), for example $A \vee B, \neg A/B$. Reasoning about the classical semantics one shows: if $A \vee B$ and $\neg A$ are true, then B is true. Here is one version of the reasoning.

1	$A \vee B$ and $\neg A$ are true	supposition
2	$A \vee B$ is true	from 1
3	$\neg A$ is true	from 1
4	A is true or B is true	from 2
5	A is false	from 3
6	B is true	from 4 and 5

Reasoning about the paraconsistent semantic leads to a very different result because 5 is not derivable from 3. Indeed, both A and $\neg A$ may be true in a paraconsistent model. If that is the case, however, then both $A \vee B$ and $\neg A$ are true even if B is false. So there are models in which $A \vee B$ and $\neg A$ are true and B is false.

Remember that we were considering **CLuN** and paraconsistent extensions of it. We have seen that Disjunctive Syllogism is invalid in **CLuN**. Moreover, as Addition (in particular the variant $A/A \vee B$) is valid, extending **CLuN** with Disjunctive Syllogism would make Ex Falso Quodlibet derivable, whence we would be back at **CL**. Other **CL**-rules are also invalid in **CLuN**, but **CLuN** may be extended with them. Double Negation is among those rules, for example the axiom $\neg\neg A \supset A$ and also its converse. If A is false, $\neg A$ is bound to be true, but $\neg\neg A$ may still be true also. So some paraconsistent models verify $\neg\neg A$ and falsify A. Although $\neg\neg A \supset A$ is invalid in **CLuN**, extending **CLuN** with it results in a paraconsistent logic. This holds for many **CL**-theorems, for example $\neg(\neg A \wedge \neg B) \supset (A \vee B)$. However, extending **CLuN** with several such **CL**-theorems may again result in **CL**.

1.3 Paraconsistent Tarski Logics

The basic paraconsistent logic **CLuN** was already mentioned in the previous section. It is obtained in two steps. First, full positive logic $\mathbf{CL}^+$ is retained. Next, for the negation, Excluded Middle ($\vdash A \vee \neg A$, which is contextually equivalent to $\vdash (A \supset \neg A) \supset \neg A$) is retained, but Ex Falso Quodlibet is not.[6] To avoid confusion, let

[5] A logic **L** is paracomplete (with respect to a negation $\neg$) iff some A may false together with its negation $\neg A$; syntactically: iff there are Γ, A and B such that $\Gamma, A \vdash_{\mathbf{L}} B$ and $\Gamma, \neg A \vdash_{\mathbf{L}} B$, but $\Gamma \nvdash_{\mathbf{L}} B$.

[6] In the context of $\mathbf{CL}^+$, Excluded Middle together with Ex Falso Quodlibet define the classical negation.

me characterize **CLuN** semantically. It is obtained from the **CL**-semantics by first removing the clause for negation—the result of this removal is $\mathbf{CL}^+$—and next adding "If $v_M(A) = 0$, then $v_M(\neg A) = \ldots$"[7]

That **CLuN** contains $\mathbf{CL}^+$ warrants that, for example, $\neg p \vdash_{\mathbf{CLuN}} q \supset (\neg p \wedge q)$ because $A \vdash_{\mathbf{CL}^+} B \supset (A \wedge B)$. This is because $\mathbf{CL}^+$ theorem schemata hold for all formulas, formulas of the form $\neg A$ included. However, $\mathbf{CL}^+$ does not have any effect *within* such formulas, in other words within the scope of a negation symbol. As a result of this, Replacement of Equivalents is invalid: $\vdash_{\mathbf{CLuN}} p \equiv (p \wedge p)$ and $\vdash_{\mathbf{CLuN}} \neg p \equiv \neg p$ but $\nvdash_{\mathbf{CLuN}} \neg p \equiv \neg(p \wedge p)$. For the same reason, Replacement of Identicals is invalid: $a = b \vdash_{\mathbf{CLuN}} Pa \equiv Pb$ but $a = b \nvdash_{\mathbf{CLuN}} \neg Pa \equiv \neg Pb$. However, it is easy to extend **CLuN** with Replacement of Identicals.

In the previous section, I referred several times to **CLuN**-models. The reader may wonder what these models precisely look like. For all that was said until now, the **CLuN**-semantics is *indeterministic*. Excluded Middle is retained, $v_M(\neg A) = 1$ whenever $v_M(A) = 0$, but the converse obviously cannot hold because, if it did, Ex Falso Quodlibet would be valid. It is not difficult to restore determinism and the method is interesting because it can be applied rather generally. Two functions play an important role in connection with models. The assignment v is part of the model itself: $M = \langle D, v \rangle$.[8] The assignment fixes the 'meaning' of non-logical symbols. Next, the valuation v_M fixes the 'meaning' of logical symbols. A decent semantics presupposes a complexity ordering $<$ which is such that if $A < B$, then all non-logical symbols that occur in A also occur in B. If the semantics is deterministic, the valuation function defines the valuation value $v_M(A)$ in terms of the assignment function and in terms of valuation values $v_M(B_1), \ldots, v_M(B_n)$ such that $B_1 < A, \ldots, B_n < A$. So every valuation value $v_M(A)$ is a function of assignment values of formulas B such that $B < A$ and of non-logical symbols that occur in those B. Actually, a deterministic semantics is the standard. If two models are identical $M = \langle D, v \rangle = \langle D', v' \rangle = M'$, whence $D = D'$ and $v = v'$, then they better verify the same formulas. If they do not, then we should describe a semantics in terms of model variants rather than models. Nevertheless, indeterministic semantic systems have been around for more than 30 years, never caused any confusion and were the subject of several interesting systematic studies [3–6].

The official deterministic semantics for **CLuN** is obtained from the indeterministic one by replacing the clause "if $v_M(A) = 0$, then $v_M(\neg A) = 1$" by

$$v_M(\neg A) = 1 \text{ iff } v_M(A) = 0 \text{ or } v(\neg A) = 1\,.$$

Obviously, for this to work, v needs to assign a value to formulas of the form $\neg A$. Note that $v_M(\neg A)$ is still not a function of $v_M(A)$ in the deterministic **CLuN**-semantics. Determinism does not entail truth-functionality.

[7] So $p \wedge \neg q \vDash_{\mathbf{CL}^+} \neg q$, $\forall x \neg Px \vDash_{\mathbf{CL}^+} \neg Pa$, and $a = b, Px \vDash_{\mathbf{CL}^+} Pb$, but $a = b, \neg Px \nvDash_{\mathbf{CL}^+} \neg Pb$.

[8] Names and notation may obviously be different and the model may be more complex.

A useful observation is the following. Precisely because, in the two-valued semantics of paraconsistent logics, $v_M(\neg A)$ is not a function of $v_M(A)$, the truth-value of $\neg A$ depends on information not contained in the truth-value of A. Information of this type must naturally be conveyed by the assignment v. Indeed, a model itself, viz. $M = \langle D, v\rangle$, represents a possible situation (or possible state of the world, etc.), whereas the valuation describes the conventions by which we define logical symbols in order to build complex statements—formulas at the schematic level—that enable us to describe the situation. So all information should obviously come from the model itself—the situation, the world, or however you prefer to call it. Moreover, in order to handle not only negation gluts, viz. inconsistencies, but gluts and gaps with respect to any logical symbol, one better lets the assignment map every formula of the language to the set of truth values $\{0, 1\}$.[9]

Incidentally, the view on models presented in the previous paragraph throws some doubt on claims to the effect that classical negation is not a sensible logical operator, among other things because it would be tonk-like. Unless a different approach to logic and models is elaborated, such claims seem not to refer to the situation or world, but to the way in which we handle language. If that is so, one wonders why a modification to our logical operators (for example banning classical negation) is more legitimate than modifying the way in which we handle language.[10]

As already suggested in the previous section, several **CL**-theorems (as well as the corresponding rules) are lost in **CLuN**. Moreover, some of these are such that if **CLuN** is extended with them, even separately, then Ex Falso Quodlibet is derivable, whence we are back to **CL**, or Ex Falso Quodlibet Falsum ($A, \neg A \vdash \neg B$) is derivable, whence we are back to something almost as explosive as **CL**. Disjunctive Syllogism is such a rule. Other examples of such rules are (full) Contraposition, Modus Tollens, Reductio ad Absurdum and Replacement of Equivalents. Let me illustrate the matter for Modus Tollens. In view of $A \vdash_{\mathbf{CLuN}} B \supset A$ and reflexivity, $B \supset A, \neg A \in Cn_{\mathbf{CLuN}}(\{A, \neg A\})$. So extending **CLuN** with Modus Tollens results in $A, \neg A \vdash_{\mathbf{CLuN}} \neg B$ in view of transitivity.

As was also suggested in the preceding section, some **CL**-theorems and **CL**-rules are invalid in **CLuN**, but adding them (separately) to **CLuN** results in a richer paraconsistent logic. Among the striking examples are $\neg\neg A/A$; de Morgan properties; $A, \neg A \vdash B$ for non-atomic A; Replacement of Identicals; and so on. Note that some combinations of such **CL**-theorems and **CL**-rules still result in the validity of Ex Falso Quodlibet or of Ex Falso Quodlibet Falsum.

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[9]Take conjunction as an example. The clause allowing for gluts: $v_M(A \wedge B) = 1$ iff ($v_M(A) = 1$ and $v_M(B) = 1$) or $v(A \wedge B) = 1$; the one allowing for gaps: $v_M(A \wedge B) = 1$ iff ($v_M(A) = 1$ and $v_M(B) = 1$) and $v(A \wedge B) = 1$; the one allowing for both: $v_M(A \wedge B) = v(A \wedge B)$.

[10]I heard the claim that restricting the formation rules of natural language so as to classify "this sentence is false" as non-grammatical is illegitimate because the sentence is 'perfect English'. I also heard the claim that invalidating Disjunctive Syllogism is illegitimate because this reasoning form is 'perfectly sound'.

It still seems useful to mention a result from an almost 35-year-old publication [8]. There is an infinity of logics between the propositional fragments of **CLuN** and **CL**. These logics form a mesh. Some of them are maximally paraconsistent in that every extension of them is either propositional **CL** or the trivial logic **Tr**, characterized by $\Gamma \vdash_{\mathbf{Tr}} A$, in other words $Cn_{\mathbf{Tr}}(\Gamma) = \mathcal{W}$. Many propositional paraconsistent logics have a place in this mesh—exceptions are extensions of **CLuN** that validate non-**CL**-theorems like $\neg(A \supset \neg A)$.[11] Other paraconsistent logics are fragments of logics in this mesh, for example Priest's **LP**, which has no detachable implication. Other paraconsistent propositional logics are obviously not within the mesh, for example relevant logics, modal paraconsistent logics, logics that display other gluts or gaps and so on.

An example of a maximal paraconsistent logic is the propositional fragment of a logic which is called **CLuNs** in Ghent because Schütte [59] was the first to describe that propositional fragment. **CLuNs**, fragments of it and slight variants of it were heavily studied and are known under many names [1, 2, 8, 25, 33, 35–40, 57, 61]. **CLuNs** is obtained by extending **CLuN** with axiom schemas to 'drive negations inwards' as well as with an axiom schema that restores Replacement of Identicals: $\neg\neg A \equiv A$, $\neg(A \supset B) \equiv (A \wedge \neg B)$, $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$, $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$, $\neg(A \equiv B) \equiv ((A \vee B) \wedge (\neg A \vee \neg B))$, $\neg(\forall\alpha)A \equiv (\exists\alpha)\neg A$, $\neg(\exists\alpha)A \equiv (\forall\alpha)\neg A$, and $\alpha = \beta \supset (A \supset B)$, in which B is obtained by replacing in A an occurrence of α by β. **CLuNs** has a nice two-valued semantics and several other semantic systems, among which a three-valued one, are adequate for it. I refer the reader elsewhere [25] for this. Priest's **LP** is obtained from **CLuNs** by removing the axioms and semantic clauses for implication and equivalence and defining the symbols in a non-detachable way: $A \supset B =_{df} \neg A \vee B$ and $A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$.

Several paraconsistent logics having been described, we may now return to the original problem and phrase things in a more precise way.

1.4 The Original Problem Revisited

We considered a $T = \langle \Gamma, \mathbf{CL} \rangle$ that turned out inconsistent. T itself is obviously too strong, viz. trivial, to offer a sensible view on 'what T was intended to be'. But we know a way to avoid triviality: replace **CL** by a paraconsistent logic. So let us pick **CLuN** or any other paraconsistent Tarski logic. For nearly all sensible Γ, $T' = \langle \Gamma, \mathbf{CLuN} \rangle$ offers a non-trivial interpretation of 'what T was intended to be'. A little reflection reveals, however, that this T' is too weak.

A toy example will be helpful. Specify the Γ in T to be $\Gamma_1 = \{p, q, \neg p \vee r, \neg q \vee s, \neg q\}$. Note that $\Gamma \nvdash_{\mathbf{CLuN}} s$ and $\Gamma \nvdash_{\mathbf{CLuN}} r$. However, there seems to be a clear difference between p and q. Intuitively speaking, Γ_1 obviously requires that q behaves

[11] This formula is **CL**-equivalent to A but not **CLuN**-equivalent to it.

inconsistently but does not require that p behaves inconsistently. However, and this is interesting, **CLuN** leads to exactly the same insight. Indeed, $\Gamma_1 \vdash_{\mathbf{CLuN}} q \wedge \neg q$ whereas $\Gamma_1 \vdash_{\mathbf{CLuN}} p$ but $\Gamma_1 \nvdash_{\mathbf{CLuN}} \neg p$. Let us see whether something interesting can be done with the help of this apparently interesting distinction.

As p and $\neg p \vee r$ are T-theorems, r was intended as a T-theorem. Similarly, as q and $\neg q \vee s$ are T-theorems, s was *intended* as a T-theorem. However, s better be not a T-theorem. Indeed, intuitively and by **CLuN**, q and $\neg q \vee A$ are T-theorems for every A. So if, relying q, we obtain the conclusion s from $\neg q \vee s$, then, by exactly the same move we obtain the conclusion A from $\neg q \vee A$. The justification for deriving s justifies deriving every formula A because $\neg q \vee A$ is just as much a **CLuN** consequence of Γ_1 as is $\neg q \vee s$. In other words, this kind of reasoning leads to triviality. The matter is very different in the case of r. Indeed, r can be a T-theorem. Relying on p one obtains the conclusion r from$\neg p \vee r$ and there is no other formula of the form $\neg p \vee A$ to which the same move might sensibly be applied.[12] A different way to phrase the matter is by saying that applications of Disjunctive Syllogism of which q is the minor result in triviality, but that applications of Disjunctive Syllogism of which p is the minor do not result in triviality. The reason for the difference is clear: Γ_1 requires q to behave inconsistently, but does not require p to behave inconsistently.

One might take that the preceding paragraphs led to the following insight: what was intended as a T-theorem and can be retained as a T-theorem, should be retained as a T-theorem. Alas, this will not do. Consider another toy example for the non-logical axioms: $\Gamma_2 = \{\neg p, \neg q, p \vee r, q \vee s, \neg t, u \vee t, p \vee q\}$. Clearly, r was intended as a theorem and indeed it can be retained. However, then q, which was also intended as a theorem, should by the same reasoning also be retained. Moreover, if q is retained, then so is $q \vee A$ for every formula A. So, although s was also intended as a theorem, it cannot be retained because, relying on $\neg q$ we cannot only obtain s from $q \vee s$, but we can obtain every formula A from $q \vee A$.

That may seem all right at first sight, but it is not. If you take a closer look at Γ_2, you will see that p and q are strictly on a par. The reasoning in the preceding paragraph relied on the consistent behaviour of p to derive s and q and hence to find out that q behaves inconsistently. However, one may just as well start off by relying on the consistent behaviour of q to obtain s as well as p and hence to find out that p behaves inconsistently. So the insight mentioned at the outset of the previous paragraph should be corrected. Here is the correct version: what was intended as a T-theorem and can be retained as a T-theorem *in view of a systematic and formal account,* should be retained as a T-theorem. A little reflection on the part of the reader will readily reveal that neither r nor s can be retained as consequences of Γ_2, but that u can be so retained.

What is the upshot? We want to replace T by a consistent theory. Obviously, there is no point in pursuing a consistent replacement for a trivial theory—every

[12]As q is **CLuN**-derivable from the premises, so is $\neg p \vee q$. However, relying on p to repeat the move described in the text delivers a formula that was already derivable, viz. q. The same story may be retold for every **CLuN**-consequence of Γ_1 and each time the move will be harmless because nothing new will come out of it.

consistent theory is equally qualified. Moreover, T', in which **CL** is replaced by **CLuN** will be non-trivial for most Γ, but is clearly too weak. However, for most Γ one may strengthen T' by adding certain instances of applications of **CL**-rules that are **CLuN**-invalid. These instances of applications may be added to T' in view of the fact that a systematic distinction can be made between formulas that behave consistently with respect to Γ and others that do not. In this way one obtains T "in its full richness, except for the pernicious consequences of its inconsistency"; one obtains an 'interpretation' of T that is as consistent as possible, and also as much as possible in agreement with the intention behind T.

Of course the matter should still be made precise. This will be done in the next section, but a central clue is the following:

$$\neg A, A \vee B \nvdash_{\mathbf{CLuN}} B \text{ but } \neg A, A \vee B \vdash_{\mathbf{CLuN}} B \vee (A \wedge \neg A)\,.$$

In view of this, one may consider formulas of the form $A \wedge \neg A$ as false, unless and until proven otherwise—unless it turns out that the premises do not permit to consider them as false on systematic grounds. In the first toy example Γ_1 requires that $q \wedge \neg q$ is true, but not that $p \wedge \neg p$ is true: $\Gamma_1 \vdash_{\mathbf{CLuN}} q \wedge \neg q$ whereas $\Gamma_1 \nvdash_{\mathbf{CLuN}} p \wedge \neg p$. Relying on the presumed falsehood of $p \wedge \neg p$, we may take r to be true. The second toy example shows that the matter is slightly more complicated: $\Gamma_2 \vdash_{\mathbf{CLuN}} (p \wedge \neg p) \vee (p \wedge \neg p)$ whereas neither $\Gamma_2 \vdash_{\mathbf{CLuN}} p \wedge \neg p$ nor $\Gamma_2 \vdash_{\mathbf{CLuN}} p \wedge \neg p$. We shall deal with this in the next section.

In order to avoid circularity, it is essential to distinguish between **CLuN**-consequences of a premise set and defeasible consequences derived in view of **CLuN**-consequences. Which formulas behave consistently with respect to a given premise set, will typically be decided in terms of the **CLuN**-consequences of Γ.

1.5 Dynamic Proofs

Dynamic proofs are a typical feature of adaptive logics. The logics were 'discovered' in terms of the proofs. In the first paper written on the topic [10], not the first published, only a rather clumsy semantics was available. The semantics for what became later known as the Minimal Abnormality strategy was described in an article [9] that was written 6 years later but published earlier. A decent semantics for the Reliability strategy appears only in [12]. Dynamic proofs are also typical for adaptive logics because nearly no other approaches to defeasible reasoning present proofs and certainly not proofs that resemble Hilbert proofs. A theoretic account of static proofs as well as dynamic proofs, which turn out to be a generalization of the former, is published [21]; a more extensive account is available on the web [24, Sect. 4.7].

Let us, very naively, have a look at some examples of dynamic proofs. More precise definitions follow in Sect. 1.7, but obtaining a clear and intuitive insight may be more important for the reader. Let us start with a dynamic proof from Γ_1. First have a look at stage 7 of the proof—a stage is a sequence of lines; think about stage 0 as the empty sequence and let the addition of a line to stage n result in stage $n + 1$.

1	p	Prem	$\emptyset$
2	q	Prem	$\emptyset$
3	$\neg p \vee r$	Prem	$\emptyset$
4	$\neg q \vee s$	Prem	$\emptyset$
5	$\neg q$	Prem	$\emptyset$
6	r	1, 3; RC	$\{p \wedge \neg p\}$
7	s	2, 4; RC	$\{q \wedge \neg q\}$

So the premises were introduced and next two conditional steps were taken. Line 6 informs us that r is derivable on the condition that $p \wedge \neg p$ is false and line 7 that s is derivable on the condition that $q \wedge \neg q$ is false. Incidentally, a line with a non-empty condition corresponds nicely and directly with a line from a static proof—in the present case a Hilbert-style **CLuN**-proof. The condition, Δ, of a line is always a finite set of contradictions. Where a line of the dynamic proof contains a line at which A is derived on the condition Δ, the corresponding static **CLuN**-proof contains a line at which $A \vee \bigvee(\Delta)$ is derived—as expected, $\bigvee(\Delta)$ is the disjunction of the members of Δ. So in a sense stage 7 of this dynamic proof is nothing but a static proof in disguise. Note that the rule applied at lines 6 and 7 is called RC (conditional rule) because, as explained, a formula $A \vee \bigvee(\Delta)$ is **CLuN**-derivable from previous members of the proof, but Δ is pushed into the condition.

The way in which dynamics is introduced appears from the continuation of the proof. I do not repeat 1–5, which merely introduce the premises.

6	r	1, 3; RC	$\{p \wedge \neg p\}$	
7	s	2, 4; RC	$\{q \wedge \neg q\}$	✓
8	$q \wedge \neg q$	2, 5; RU	$\emptyset$	

At stage 8 of the proof, $q \wedge \neg q$ is unconditionally derived, viz. at line 8. So the supposition of line 7, viz. that $\{q \wedge \neg q\}$ is false, cannot be upheld. As a result, line 7 is marked, which means that its formula is considered as not derived from the premise set Γ_1.[13] Incidentally, the rule applied at line 8 is called RU (unconditional rule) because (the formula of) 8 is a **CLuN**-consequence of (the formulas of) 2 and 5.

So the dynamics is controlled by marks. Which lines are marked or unmarked is decided by a marking *definition*, which is typical for a strategy. More information on this follows in Sect. 1.7. For now, it is important that the reader understands why line 7 is marked and other lines are unmarked. As far as this specific proof stage is concerned, nothing interesting happens when the proof is continued. No mark will be removed or added to any of these 8 lines.[14] Incidentally, the only line that might

[13] Do not read the "not derived" as "not derivable". Indeed, a formula may be derivable in several ways from the same premise set.

[14] A more accurate wording requires that one adds: in a proof from Γ_1 that extends the present stage 8. Indeed, the logic we are considering is non-monotonic. So extending the premise set may result in line 6 being marked.

become marked is line 6. The formulas derived on lines with an empty condition are **CLuN**-consequences of the premises. These are the stable consequences of the premise set. The marks pertain to the supplementary, defeasible consequences of the premise set.

How can I be so sure that the marks of lines 1–8 will not be changed in an extension of the proof from Γ_1? The example is propositional and propositional **CLuN** is decidable in the same sense as propositional **CL**. It is easy enough to prove that $q \wedge \neg q$ is the only contradiction that is **CLuN**-derivable from Γ_1.[15] Beware. As is the case for **CL**, only some fragments of **CLuN** are decidable. So arguing that a predicative proof is stable with respect to certain lines will often be much more complicated than in the present case.

Before we proceed, allow me to summarize that the two components governing dynamic proofs are rules (of inference) and the marking definition. The rules are applied at will by the people who devise the proof—if they are smart, they will follow a certain heuristics. As we shall see, the marking definition operates independently of any human intervention. In view of the stage of the proof, the marking definition determines which lines are marked.

When we consider more examples, a little complication will catch our attention. Here is a dynamic proof from $\Gamma_2 = \{\neg p, \neg q, p \vee r, q \vee s, \neg t, u \vee t, p \vee q\}$.

1	$\neg p$	PREM	$\emptyset$	
2	$\neg q$	PREM	$\emptyset$	
3	$p \vee r$	PREM	$\emptyset$	
4	$q \vee s$	PREM	$\emptyset$	
5	$\neg t$	PREM	$\emptyset$	
6	$u \vee t$	PREM	$\emptyset$	
7	$p \vee q$	PREM	$\emptyset$	
8	r	1, 3; RC	$\{p \wedge \neg p\}$	✓
9	s	2, 4; RC	$\{q \wedge \neg q\}$	✓
10	u	5, 6; RC	$\{t \wedge \neg t\}$	
11	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 2, 7; RC	$\emptyset$	

At stage 10 of the proof—when the proof consists of lines 1–10 only—no line is marked. At stage 11, however, lines 8 and 9 are both marked. Why is that? Line 11 gives us the information that either p or q behaves inconsistently on Γ_2, but does not inform us which of both behaves inconsistently. So a natural reaction is to consider both $p \wedge \neg p$ and $q \wedge \neg q$ as unreliable. This is the reaction that agrees with the Reliability strategy—we shall come across other strategies later. According to the Reliability strategy a line is marked if one of the members of its condition is unreliable. At this point in the paper, consider the unreliable formulas as the

[15] The reader might think that, as p is also a **CLuN**-consequence of Γ_1, $(p \wedge q) \wedge \neg(p \wedge q)$ is also a **CLuN**-consequence of Γ_1. This however is mistaken. $\neg q \nvdash_{\mathbf{CLuN}} \neg(p \wedge q)$.

disjuncts of the *minimal* disjunctions of contradictions. If the "minimal" was not there, Addition would cause every contradiction to be unreliable as soon as one contradiction is unreliable.

In both example proofs, some lines were unmarked at a stage and marked at a later stage. The converse move is also possible, as is illustrated by a proof from $\Gamma_3 = \{(p \wedge q) \wedge t, \neg p \vee r, \neg q \vee s, \neg p \vee \neg q, t \supset \neg p\}$.

1	$(p \wedge q) \wedge t$	PREM	$\emptyset$	
2	$\neg p \vee r$	PREM	$\emptyset$	
3	$\neg q \vee s$	PREM	$\emptyset$	
4	$\neg p \vee \neg q$	PREM	$\emptyset$	
5	$t \supset \neg p$	PREM	$\emptyset$	
6	r	1, 2; RC	$\{p \wedge \neg p\}$	$\surd$
7	s	1, 3; RC	$\{q \wedge \neg q\}$	$\surd$
8	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 4; RU	$\emptyset$	

Both lines 6 and 7 are marked at stage 8 because $(p \wedge \neg p) \vee (q \wedge \neg q)$ is a minimal disjunction of contradictions that is derived at the stage. However, look what happens if stage 9 looks as follows—I do not repeat 1–5.

6	r	1, 2; RC	$\{p \wedge \neg p\}$	$\surd$
7	s	1, 3; RC	$\{q \wedge \neg q\}$	
8	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 4; RU	$\emptyset$	
9	$p \wedge \neg p$	1, 5; RU	$\emptyset$	

At stage 9 of this proof, $(p \wedge \neg p) \vee (q \wedge \neg q)$ is not a minimal disjunction of abnormalities because (the 'one disjunct disjunction') $p \wedge \neg p$ was derived. We knew already that either $p \wedge \neg p$ or $q \wedge \neg q$ was unreliable and now obtain the more specific information that it is actually $p \wedge \neg p$ that is unreliable. So $q \wedge \neg q$ is off the hook, whence line 7 is unmarked. Stage 9 of this proof is stable: no mark will be removed or added to lines 1–9 if the stage is extended. Actually, nothing interesting happens in any such extension.

It is time to make the marking more precise. Dynamic proofs need to explicate the dynamic reasoning. So, at the level of the proofs, the dynamics needs to be controlled. The central features for this control are the conditions and the marking definition. The way in which conditions are introduced should be clear by now—precise generic rules follow in Sect. 1.7. However, how does one precisely figure out which lines are marked?

Only some adaptive logics are inconsistency-adaptive. So allow me to use a slightly more general terminology. The formulas that occur in conditions of lines—in the previous examples these were contradictions—are called abnormalities and Ω is the usual name for the set of abnormalities.

A classical disjunction of abnormalities will be called a *Dab-formula*—it goes without saying that a disjunction of formulas is always a disjunction of finitely many formulas. I shall often write $Dab(\Delta)$ to refer to the classical disjunction of the members of a finite $\Delta \subset \Omega$. A *Dab*-formula that is derived in a proof stage by RU at a line with condition $\emptyset$ will be called a *inferred Dab-formula* of the proof stage. Note that a *Dab*-formula introduced by Prem is not an inferred *Dab*-formula in the sense of this definition. $Dab(\Delta)$ is a *minimal inferred Dab-formula* of a proof stage if it is an inferred *Dab*-formula of the proof stage and there is no $\Theta \subset \Delta$ such that $Dab(\Theta)$ is an inferred *Dab*-formula of the proof stage. Where $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ are the minimal inferred *Dab*-formulas of stage s, the set of *unreliable formulas of stage* s is $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$. Where Θ is the condition of line i, line i is *marked* iff $\Theta \cap U_s(\Gamma) \neq \emptyset$. This is the marking definition for the Reliability strategy—every strategy has its own marking definition.

Marks come and go. As they determine which formulas are considered as derived, derivability seems to be unstable; it changes from stage to stage. Let this unstable derivability be called *derivability at a stage*. Apart from it, we want a stable form of derivability, which is called *final derivability* and is noted as $\Gamma \vdash_{\mathbf{CLuN}^r} A$. There are several ways to define final derivability. At this point in my story, the following seems most handy. If A is derived at an unmarked line i of a stage of a proof from Γ and the stage is *stable* with respect to i—line i is not marked in any extension of the stage—then A is finally derived from Γ.

Just as we wanted the stable entity called final derivability, we also want to have some further entities that refer to what is **CLuN**-derivable from the premise set Γ rather than referring to a stage of a proof from Γ.

Definition 1.1 $Dab(\Delta)$ is a *minimal Dab-consequence of* Γ iff $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$ and, for all $\Delta' \subset \Delta$, $\Gamma \nvdash_{\mathbf{CLuN}} Dab(\Delta')$.

Definition 1.2 Where $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ are the minimal *Dab*-consequences of Γ, $U(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$.

The set $U(\Gamma)$ is defined in view of the Reliability strategy. A very different set will be introduced later in view of Minimal Abnormality.

The reader may expect a section on semantics at this point, but I shall only deal with the semantics as defined by the standard format.

1.6 The Standard Format SF

There is a large diversity of adaptive logics. Every new adaptive logic requires that one delineates its syntax (proof theory), its semantics (models), and, what is the hard bit, its metatheory (study of properties of the system). This suggested the search for a common structure for a large set of adaptive logics, if possible for all of them. The idea was that the structure would take care of most of the work beforehand, that the proof theory and semantics would be defined in terms of the common structure and that the metatheoretic properties would be provable from the structure. The common structure would be a function of certain parameters and specifying these would result in a specific adaptive logic with all required features available. This common structure is called the *standard format*.

An adaptive logic **AL** in Standard Format is defined as a triple comprising[16]:

- a *lower limit logic* **LLL**: a logic that has static proofs and contains classical disjunction,
- a *set of abnormalities* Ω, a set of formulas that share a (possibly restricted) logical form or a union of such sets,
- a *strategy* (Reliability, Minimal Abnormality, ...).

That the lower limit logic contains a classical disjunction means that one of the logical symbols is implicitly or explicitly defined in such a way that it has the meaning of the **CL**-disjunction. Explaining the notion of static proofs goes beyond the scope of the present paper, but the reader may for all useful purposes replace the requirement by: a formal and compact Tarski logic.

"Abnormality" is a technical term, different adaptive logics require that different formulas are seen as abnormalities. Only the abnormalities of corrective adaptive logics—those with **LLL** weaker than **CL**—are **CL**-falsehoods. In nearly all inconsistency-adaptive logics, existentially closed contradictions are abnormalities. Also other formulas may belong to the Ω, for example Universally closed contradictions or formulas of the form $A \wedge \neg(A \vee B)$. Some examples of restricted and unrestricted logical forms will be presented below.

Adaptive strategies will be discussed at some length later in this section.

If the lower limit logic **LLL** is extended with a set of rules or axioms that trivialize abnormalities (and no other formulas), then one obtains a logic called the *upper limit logic* **ULL**. Examples follow but it should be clear by now that, for all $A \in \Omega$ and for all $B \in \mathcal{W}$, A/B should be a derivable rule in **ULL**. As Ω is characterized by a logical form, it is in possible to obtain **ULL** by extending **LLL** with a set of rules.

I shall suppose that a characteristic semantics of **LLL** is available. This will enable me to define the semantics of **AL** in terms of the standard format. The **LLL**-models that verify no member of Ω form a semantics for **ULL**.[17] A premise set that has **ULL**-models is often called a *normal premise set*; it does not require that any abnormality is true.

It is instructive to have a closer look at the difference between **ULL** and **AL**. **ULL** extends **LLL** by validating some further rules of inference. **AL** extends **LLL** by validating certain *applications* of **ULL**-rules. The point is easily illustrated in connection to Disjunctive Syllogism. **CL** validates this rule, while in the (not yet precise) toy examples of proofs from Sect. 1.5, some but not all applications of Disjunctive Syllogism were sanctioned as correct. As those examples clarify, it depends on the premises—or should one say on the content of the premises—which applications

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[16]Names like **LLL**, **AL**, **AL**^{r}, and **ULL** are used as generic names to define the standard format and to study its features. The names refer to arbitrary logics that stand in a certain relation to each other.

[17]Similarly for those models together with the trivial model—the model that verifies all formulas.

turn out valid. In other words, adaptive logics display a form of *content guidance*.[18] A different way of phrasing the matter is that $Cn_{\mathbf{AL}}(\Gamma)$ comes to $Cn_{\mathbf{LLL}}(\Gamma)$ extended with what is derivable if *as many* abnormalities are false *as* the premises permit. This phrase is obviously ambiguous, but strategies disambiguate it, as we shall see.

An important supposition on the language $\mathcal{L}$ of **AL** is that it contains a classical disjunction. It may of course contain several disjunctions, but one of them should be classical. In the sequel of this paper, the symbol $\hat{\vee}$ will always refer to this disjunction.[19] Similarly, $\sim$ will always refer to a classical negation. This is *not* supposed to occur in every considered language schema.

As we already have seen in Sect. 1.5, we need $\hat{\vee}$ for *Dab*-formulas—but see Sect. 1.11 for an alternative. In Sect. 1.5, I also introduced inferred *Dab*-formulas and minimal inferred *Dab*-formulas of a proof stage as well as the notation $Dab(\Delta)$.

Let us consider some examples of adaptive logics. Expressions $\exists A$ will denote the existential closure of A, viz. A preceded by an existential quantifier over every variable free in A.

The adaptive logic $\mathbf{CLuN}^m$ is defined by the following triple:

- lower limit logic: **CLuN**,
- set of abnormalities $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$
- strategy: Minimal Abnormality.

The upper limit logic is **CL**, obtained by extending **CLuN** with, for example, the axiom schema $(A \wedge \neg A) \supset B$.[20] It is not difficult to prove that the **CLuN**-models that verify no abnormality form a semantics of **CL**.

The logic $\mathbf{CLuNs}^m$ is defined by:

- lower limit logic: **CLuNs**,
- set of abnormalities $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s^a\}$
- strategy: Minimal Abnormality,

in which $\mathcal{F}_s^a$ is the set of atomic (open and closed) formulas of $\mathcal{L}_s$—atomic formulas are those in which no logical symbols occur except possibly for identity $=$. The upper limit logic is **CL**, obtained by extending **CLuNs** with, for example, the axiom schema $(A \wedge \neg A) \supset B$.[21] Semantically: the **CLuNs**-models that verify no abnormality form a **CL**-semantics.

[18]The notion played a rather central role in discussions on scientific heuristics. A very clear and argued position was for example proposed by Dudley Shapere [60].

[19]This obviously does not mean that $\hat{\vee}$ is a symbol of the language. It is a conventional name to refer to a symbol of the language that has the meaning of classical disjunction. It may even refer ambiguously: if there are several classical disjunctions, $\hat{\vee}$ need not always refer to the same one.

[20]Axioms are suppose to be closed formulas. So $A \in \mathcal{W}_s$. The idea is that **CLuN**-valid rules are fully retained in the extension. One of these rules is: from $\vdash A(a) \supset B$ to derive $\vdash \exists x A(x) \supset B$ provided a does not occur in B.

[21]The axiom schema may be restricted to $A \in \mathcal{W}_s^a$, but there is no need to do so.

Some further examples are easy variants. **CLuN**^{r} is like **CLuN**^{m}, except that Minimal Abnormality is replaced by Reliability. **LP**^{m} is like **CLuNs**^{m} except that **CLuNs** is replaced by Priest's **LP**—see Sect. 1.3 for the relation between **CLuNs** and **LP**.

In these examples **LLL** or the strategy are varied. What about the difference between the set of abnormalities of **CLuN**^{m} as opposed to **CLuNs**^{m}? In a sense this is just a variation. Yet, if the Ωs are exchanged, the resulting variant of **CLuN**^{m} is still an inconsistency-adaptive logic, but its **ULL** is weaker than **CL**—a feature that is difficult to justify with respect to applications. If the Ω are exchanged, the resulting variant of **CLuNs**^{m} is also still an inconsistency-adaptive logic, but it is a flip-flop logic—see Sect. 1.12, where also more variation will be considered.

If an adaptive logic is in standard format, this fact (not specific properties of the logic) provides it with:

- its proof theory,
- its semantics (models),
- most of its metatheory (*including* soundness and completeness).

So the standard format provides guidance in devising new adaptive logics. Moreover, once a new adaptive logic is phrased in standard format, most of the hard work is over.

1.7 SF: Proof Theory

As we already know, every adaptive logic requires a set of rules of inference and a marking definition. The rules of inference are determined by **LLL** and Ω; the marking definition is determined by Ω and by the strategy. We also know that the dynamics of the proofs is controlled by attaching conditions (finite subsets of Ω) to derived formulas, or, if you prefer, to lines at which formulas are derived. We also have seen what is special about annotated dynamic proofs: their lines consist of four rather than three elements: a number, a formula, a justification and a condition. The rules govern the addition of lines, the marking definition determines for every line i at every stage s of a proof whether i is unmarked or marked— this means that it is respectively IN or OUT—in view of (i) the condition of i and (ii) the minimal inferred *Dab*-formulas of stage s.

The rules of inference can be presented as three generic rules. Let Γ be the premise set and let

$$A \qquad \Delta$$

abbreviate that A occurs in the proof on the condition Δ.

Prem If $A \in \Gamma$:

$$\frac{\ldots \quad \ldots}{A \quad \emptyset}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:

$$\frac{\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \end{array}}{B \quad \Delta_1 \cup \ldots \cup \Delta_n}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \mathbin{\hat{\vee}} Dab(\Theta)$:

$$\frac{\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \end{array}}{B \quad \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta}$$

Only RC *introduces* new non-empty conditions (adds a non-empty set to the conditions of the local premises). Prem introduces empty conditions and RU merely carries conditions over and adds them up in a union.

Easy illustrations: RU may be applied in view of $p, p \supset q \vdash_{\mathbf{CLuN}} q$; RC may be applied in view of $p, \neg p \vee q \vdash_{\mathbf{CLuN}} q \mathbin{\hat{\vee}} (p \wedge \neg p)$. In view of the formulation of the antecedent of RU and RC, all rules are *finitary*—have a finite number of local premises. This formulation does not in any way affect the adaptive logic **AL** because **LLL** is a compact logic anyway. Incidentally, it is instructive to review the toy examples in terms of the precise formulation of the rules.

Marking definitions proceed in terms of the minimal inferred *Dab*-formulas at the proof stage. Where $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ are the minimal inferred *Dab*-formulas at stage s, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$.

Definition 1.3 Marking for Reliability: where Δ is the condition of line i, line i is marked at stage s iff $\Delta \cap U_s(\Gamma) \neq \emptyset$.

The idea behind the definition consists of two steps. First, the minimal inferred *Dab*-formulas of stage s of a proof from Γ provide, at stage s, the best available estimate of the minimal *Dab*-consequences of Γ. So their disjuncts, which are abnormalities, cannot be safely considered as false. Next, the formula of a line can only be considered as derived (by present insights) if the abnormalities in the condition of the line can be considered as false. If they cannot, the line is marked.

However sensible this may sound, Minimal Abnormality offers a more refined approach. A *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ is a set that contains one element out of each member of Σ. A *minimal choice set* of Σ is a choice set of Σ of which no proper subset is a choice set of Σ. Where $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ are the minimal inferred *Dab*-formulas of stage s, $\Phi_s(\Gamma)$ is the set of the minimal choice sets of $\{\Delta_1, \ldots, \Delta_n\}$.

Definition 1.4 Marking for Minimal Abnormality: where A is the formula and Δ is the condition of line i, line i is marked at stage s iff (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, *or* (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.