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Reinhard Kahle
Thomas Strahm
Thomas Studer
Editors

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Advances in Proof Theory

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Preface

Advances in proof theory was the title of a symposium organized on the occasion of the 60th birthday of Gerhard Jäger. The meeting took place on December 13 and 14, 2013, at the University of Bern, Switzerland.

The aim of this symposium was to bring together some of the best specialists from the area of proof theory, constructivity, and computation and discuss recent trends and results in these areas. Some emphasis was put on ordinal analysis, reductive proof theory, explicit mathematics and type-theoretic formalisms, as well as abstract computations.

Gerhard Jäger has devoted his research to these topics and has substantially advanced and shaped our knowledge in these fields.

The program of the symposium was as follows:

Friday, December 13

Wolfram Pohlers: *From Subsystems of Classical Analysis to Subsystems of Set Theory: A personal account*

Wilfried Buchholz: *On the Ordnungszahlen in Gentzen's First Consistency Proof*

Andrea Cantini: *About Truth, Explicit Mathematics and Sets*

Peter Schroeder-Heister: *Proofs That, Proofs Why, and the Analysis of Paradoxes*

Roy Dyckhoff: *Intuitionistic Decision Procedures since Gentzen*

Grigori Mints: *Two Examples of Cut Elimination for Non-Classical Logics*

Rajeev Goré: *From Display Calculi to Decision Procedures via Deep Inference for Full Intuitionistic Linear Logic*

Pierluigi Minari: *Transitivity Elimination: Where and Why*

Saturday, December 14

Per Martin-Löf: *Sample Space-Event Time*

Anton Setzer: *Pattern and Copattern Matching*

Helmut Schwichtenberg: *Computational Content of Proofs Involving Coinduction*

Michael Rathjen: *When Kripke-Platek Set Theory Meets Powerset*

Stan Wainer: *A Miniaturized Predicativity*

Peter Schuster: *Folding Up*

Solomon Feferman: *The Operational Perspective*

This volume comprises contributions of most of the speakers and represents the wide spectrum of Gerhard Jäger's interests. We deeply miss Grisha Mints who planned to contribute to this Festschrift.

We acknowledge gratefully the financial support of Altonaer Stiftung für philosophische Grundlagenforschung, Burgergemeinde Bern, Swiss Academy of Sciences, Swiss National Science Foundation, and Swiss Society for Logic and Philosophy of Science. We further thank the other members of the program committee, namely Roman Kuznets, George Metcalfe, and Giovanni Sommaruga.

For the production of this volume, we thank the editors of the *Progress in Computer Science and Applied Logic (PCS)* Series, the staff members of Birkhäuser/Springer Basel, and the reviewers of the papers of this volume.

We dedicate this Festschrift to Gerhard Jäger and thank him for his great intellectual inspiration and friendship.

Lisbon	Reinhard Kahle
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December 2015	

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A Survey on Ordinal Notations Around the Bachmann-Howard Ordinal

Wilfried Buchholz

Dedicated to Gerhard Jäger on the occasion of his 60th birthday.

Abstract Various ordinal functions which in the past have been used to describe ordinals not much larger than the Bachmann-Howard ordinal are set into relation.

1 Introduction

In recent years a renewed interest in ordinal notations around the Bachmann-Howard ordinal $\phi_{\varepsilon_{\Omega+1}}(0)$ has evolved, amongst others caused by Gerhard Jäger's metapredicativity program. Therefore it seems worthwile to review some important results of this area and to present detailed and streamlined proofs for them. The results in question are mainly comparisons of various functions which in the past have been used for describing ordinals not much larger than the Bachmann-Howard ordinal. We start with a treatment of the Bachmann hierarchy $\left(\phi_\alpha\right)_{\alpha\leq\Gamma_{\Omega+1}}$ from [3]. This hierarchy consists of normal functions $\phi_\alpha : \Omega \to \Omega$ ($\alpha \leq \Gamma_{\Omega+1}$) which are defined by transfinite recursion on α referring to previously defined fundamental sequences $(\alpha[\xi])_{\xi<\tau_\alpha}$ (with $\tau_\alpha \leq \Omega$). The most important new concept in Bachmann's approach is the systematic use of ordinals $\alpha > \Omega$ as indices for functions from Ω into Ω. Bachmann describes his approach as a generalization of a method introduced by Veblen in [22]; according to him the initial segment $(\phi_\alpha)_{\alpha<\Omega^\Omega}$ is just a modified presentation of a system of normal functions defined by Veblen. But actually this connection is not so easy to see. At the end of Sect. 2 we will establish the connection between $(\phi_\alpha)_{\alpha<\Omega^\Omega}$ and Schütte's Klammersymbols [19] for which the relation to [22] is clear

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cf. [19, footnote 4]. In Sect. 3 we give an alternative characterization of the Bachmann hierarchy which instead of fundamental sequences $(\alpha[\xi])_{\xi<\tau_\alpha}$ uses finite sets $K\alpha \subseteq \Omega$ of *coefficients* ("Koeffizienten"). For $\alpha < \varepsilon_{\Omega+1}$, $K\alpha$ is almost identical to the set $C(\alpha)$ of *constituents* (i.e., ordinals $< \Omega$ which occur in the complete base Ω Cantor normal form of α) in [15], where it was shown how to construct a recursive system of ordinal notations on the basis of Bachmann's functions.

In the 1960s, the Bachmann method for generating hierarchies of normal functions on Ω was extended by Pfeiffer [17] and, much further, by Isles [16]. These extensions were highly complex; especially the Isles approach was so complicated that it was practically unusable for proof-theoretic applications. Therefore Feferman, in unpublished work around 1970, proposed an entirely different and much simpler method for generating hierarchies of normal functions θ_α ($\alpha \in On$) (see e.g. [14]). Aczel (in [1]) showed how the θ_α ($\alpha < \Gamma_{\Omega+1}$) correspond to Bachmann's ϕ_α. (Independently, Weyhrauch [23] established the same results for $\alpha < \varepsilon_{\Omega+1}$.) In addition, Aczel generalized Feferman's definition and conjectured that the generalized hierarchy (θ_α) matches up with the Isles functions. This conjecture was proved by Bridge in [4, 5]. In Sect. 4 of the present paper we show how Feferman's functions θ_α ($\alpha < \Gamma_{\Omega+1}$) can also be defined by use of the $K\alpha$'s. Together with the content of Sect. 3 this leads to an easy comparison of the hierarchies $(\phi_\alpha)_{\alpha<\Gamma_{\Omega+1}}$ and $(\theta_\alpha)_{\alpha<\Gamma_{\Omega+1}}$ which becomes particularly simple if one switches to the fixed-point-free versions: $\overline{\phi}_\alpha(\beta) = \overline{\theta}_\alpha(\beta)$ for all $\alpha < \Gamma_{\Omega+1}$, $\beta < \Omega$ (Theorem 4.7).

In Sects. 5, 6 we deal with the unary functions $\vartheta : \varepsilon_{\Omega+1} \to \Omega$ and $\psi : \varepsilon_{\Omega+1} \to \Omega$ which play an important rôle in [18]. We show that $\overline{\theta}_{1+\alpha}(\beta) = \vartheta(\Omega\alpha + \beta)$ (for $\alpha < \varepsilon_{\Omega+1}$, $\beta < \Omega$) and refine a result from [18] on the relationship between ϑ and ψ. In Sect. 7, largely following [23], we show how the Bachmann hierarchy below $\varepsilon_{\Omega+1}$ can be defined by means of functionals of finite higher types.

A nice survey on the history of the subject can be found in [13].

Preliminaries. The letters $\alpha, \beta, \gamma, \delta, \xi, \eta, \zeta$ always denote ordinals. *On* denotes the class of all ordinals and *Lim* the class of all limit ordinals. We are working in ZFC. So, every ordinal α is identical to the set $\{\xi \in On : \xi < \alpha\}$, and we have $\beta < \alpha \Leftrightarrow \beta \in \alpha$ and $\beta \leq \alpha \Leftrightarrow \beta \subseteq \alpha$. For $X \subseteq On$ we define: $X <(\leq)\, \alpha :\Leftrightarrow \forall x \in X\, (x <(\leq)\, \alpha)$ and $\alpha \leq X :\Leftrightarrow \exists x \in X (\alpha \leq x)$, i.e., $X < \alpha \Leftrightarrow X \subseteq \alpha$ and $\alpha \leq X \Leftrightarrow \neg(X < \alpha)$. By $\mathbb{H}$ we denote the class $\{\gamma \in On : \forall \alpha, \beta < \gamma(\alpha + \beta < \gamma)\} = \{\omega^\alpha : \alpha \in On\}$ of all *additive principal numbers* (*Hauptzahlen*), and by $\mathbb{E}$ the class $\{\alpha \in On : \omega^\alpha = \alpha\} = \{\varepsilon_\alpha : \alpha \in On\}$ of all *epsilon-numbers*. A *normal function* is a strictly increasing continuous function $F : On \to On$. The normal functions $\varphi_\alpha : On \to On$ ($\alpha \in On$) are defined by: $\varphi_0(\beta) := \omega^\beta$, and $\varphi_\alpha :=$ ordering (or enumerating) function of $\{\beta : \forall \xi < \alpha(\varphi_\xi(\beta) = \beta)\}$, if $\alpha > 0$. The family $(\varphi_\alpha)_{\alpha \in On}$ is called *the Veblen hierarchy over* $\lambda\xi.\omega^\xi$. An ordinal α is called *strongly critical* iff $\varphi_\alpha(0) = \alpha$. The class of all strongly critical ordinals is denoted by SC, and its enumerating function by $\lambda\alpha.\Gamma_\alpha$. It is well-known that $\lambda\alpha.\Gamma_\alpha$ is again a normal function, and that $\Gamma_\Omega = \Omega$, where Ω is the least regular ordinal $>\omega$.

2 Fundamental Sequences and the Bachmann Hierarchy

The following stems from Bachmann's seminal paper [3], but in some minor details we deviate from that paper. We start by assigning to each limit number $\alpha \leq \Gamma_{\Omega+1}$ a fundamental sequence $(\alpha[\xi])_{\xi<\tau_\alpha}$ with $\tau_\alpha \leq \Omega$. The definition of $\alpha[\xi]$ is based on the normal form representation of α in terms of $0, +, \cdot, F$, where $(F_\alpha)_{\alpha \in On}$ is the Veblen hierarchy over $\lambda x.\Omega^x$, i.e., $F_0(\beta) := \Omega^\beta$, and $F_\alpha :=$ ordering function of $\{\beta : \forall \xi < \alpha(F_\xi(\beta) = \beta)\}$, if $\alpha > 0$. The relationship between F_α and φ_α for $\alpha > 0$ is given by

$$F_\alpha(\beta) = \varphi_\alpha(\widetilde{\alpha} + \beta) \text{ with } \widetilde{\alpha} := \begin{cases} \Omega+1 & \text{if } 0 < \alpha < \Omega, \\ 1 & \text{if } \alpha = \Omega, \\ 0 & \text{if } \Omega < \alpha. \end{cases}$$

From this it follows that $\Gamma_{\Omega+1}$ is the least fixed point of $\lambda\alpha.F_\alpha(0)$.
For completeness note, that $F_0(\beta) = \varphi_0(\Omega\beta)$.

Abbreviations

1. $\Lambda := \Gamma_{\Omega+1} = \min\{\alpha : F_\alpha(0) = \alpha\}$.
2. $\alpha|\gamma \;:\Leftrightarrow\; \exists\xi(\gamma = \alpha\cdot\xi)$.
3. $\alpha =_{\text{NF}} \gamma + \Omega^\beta\eta \;:\Leftrightarrow\; \alpha = \gamma + \Omega^\beta\eta \;\&\; 0 < \eta < \Omega \;\&\; \Omega^{\beta+1}|\gamma$.
4. $\gamma =_{\text{NF}} F_\alpha(\beta) \;:\Leftrightarrow\; \alpha, \beta < \gamma = F_\alpha(\beta)$.

Proposition

(a) For each $0 < \delta < \Lambda$ *there are unique* γ, β, η *such that* $\delta =_{\text{NF}} \gamma + \Omega^\beta\eta$.
(b) For each $\delta \in \text{ran}(F_0) \cap \Lambda$ *there are unique* α, β *such that* $\delta =_{\text{NF}} F_\alpha(\beta)$.
(c) $\delta < \Lambda \;\Rightarrow\; (\delta =_{\text{NF}} F_\alpha(\beta) \Leftrightarrow \beta < \delta = F_\alpha(\beta))$.

Definition of a fundamental sequence $(\lambda[\xi])_{\xi<\tau_\lambda}$ for each limit number $\lambda \leq \Lambda$

1. $\lambda =_{\text{NF}} \gamma + \Omega^\beta\eta \notin \text{ran}(F_0)$:
1.1. $\eta \in \mathit{Lim}$: $\tau_\lambda := \eta$ and $\lambda[\xi] := \gamma + \Omega^\beta\cdot(1+\xi)$.
1.2. $\eta = \eta_0+1$: $\tau_\lambda := \tau_{\Omega^\beta}$ and $\lambda[\xi] := \gamma + \Omega^\beta\eta_0 + \Omega^\beta[\xi]$.
2. $\lambda =_{\text{NF}} F_\alpha(\beta)$:
2.1. $\beta \in \mathit{Lim}$: $\tau_\lambda := \tau_\beta$ and $\lambda[\xi] := F_\alpha(\beta[\xi])$.
2.2. $\beta \notin \mathit{Lim}$: Let $\lambda^- := \begin{cases} 0 & \text{if } \beta = 0, \\ F_\alpha(\beta_0)+1 & \text{if } \beta = \beta_0+1. \end{cases}$
2.2.0. $\alpha = 0$: Then $\beta = \beta_0+1$. $\tau_\lambda := \Omega$ and $\lambda[\xi] := \Omega^{\beta_0}\cdot(1+\xi)$.
2.2.1. $\alpha = \alpha_0+1$: $\tau_\lambda := \omega$ and $\lambda[n] := F_{\alpha_0}^{(n+1)}(\lambda^-)$.
2.2.2. $\alpha \in \mathit{Lim}$: $\tau_\lambda := \tau_\alpha$ and $\lambda[\xi] := F_{\alpha[\xi]}(\lambda^-)$.
3. $\tau_\Lambda := \omega$ and $\Lambda[0] := 1$, $\Lambda[n+1] := F_{\Lambda[n]}(0)$.

Definition
For each limit $\lambda \le \Lambda$ we set $\lambda[\tau_\lambda] := \lambda$.
Further $\tau_0 := 0$, $0[\xi] := 0$ and $\tau_{\alpha+1} := 1$, $(\alpha{+}1)[\xi] := \alpha$.

Lemma 2.1 $\lambda =_{\text{NF}} F_\alpha(\beta) < \Lambda \;\&\; \beta \in \mathit{Lim} \;\&\; 1 \le \xi < \tau_\beta \;\Rightarrow$ $\lambda[\xi] =_{\text{NF}} F_\alpha(\beta[\xi])$.

Proof Cf. Appendix.

Lemma 2.2 *Let* $\lambda \in \mathit{Lim} \cap (\Lambda{+}1)$.

(a) $\xi < \eta \le \tau_\lambda \;\Rightarrow\; \lambda[\xi] < \lambda[\eta]$.
(b) $\lambda = \sup_{\xi<\tau_\lambda} \lambda[\xi]$.
(c) $\eta \in \mathit{Lim} \cap (\tau_\lambda + 1) \;\Rightarrow\; \lambda[\eta] \in \mathit{Lim} \;\&\; \tau_{\lambda[\eta]} = \eta \;\&\; \forall\xi < \eta(\lambda[\eta][\xi] = \lambda[\xi])$.
(d) $\xi < \tau_\lambda \;\&\; \lambda[\xi] < \delta \le \lambda[\xi{+}1] \;\Longrightarrow\; \lambda[\xi] \le \delta[1]$.

The proof of (a), (b), (c) is left to the reader. The proof of (d) will be given in the Appendix.

We now introduce a binary relation $\underline{\ll}$ which corresponds to Bachmann's $\to$ (cf. [3] p. 123, 130) and is essential for proving the basic properties of the Bachmann hierarchy. The advantage of $\ll$ over $\to$ is that its definition does not refer to the functions ϕ_α but only to the fundamental sequences $(\alpha[\xi])_{\xi<\tau_\alpha}$.

Definition of $\ll^1$, $\ll$ and $\underline{\ll}$

1. $\beta \ll^1 \alpha \;:\Leftrightarrow\; \alpha \le \Lambda \;\&\; \beta \in \{\alpha[\xi] : \xi < \tau^\circ_\alpha\}$, where $\tau^\circ_\alpha := \begin{cases} \omega & \text{if } \tau_\alpha = \Omega, \\ \tau_\alpha & \text{otherwise.} \end{cases}$
2. $\ll$ ($\underline{\ll}$) is the transitive (transitive and reflexive) closure of $\ll^1$.

Lemma 2.3 *Let* $\alpha \le \Lambda$.

(a) $\alpha \in \mathit{Lim} \;\&\; \xi{+}1 < \tau_\alpha \;\Rightarrow\; \alpha[\xi]{+}1 \;\underline{\ll}\; \alpha[\xi{+}1]$.
(b) $\alpha \in \mathit{Lim} \;\&\; \xi < \eta < (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; \alpha[\xi] \ll \alpha[\eta]$.
(c) $\beta \ll \alpha \;\Rightarrow\; \beta{+}1 \;\underline{\ll}\; \alpha$.
(d) $n < \omega \;\&\; n \le \alpha \;\Rightarrow\; n \;\underline{\ll}\; \alpha$.

Proof
(a) By induction on δ we prove: $\alpha[\xi] < \delta \le \alpha[\xi{+}1] \Rightarrow \alpha[\xi] + 1 \;\underline{\ll}\; \delta$.

1. $\delta = \delta_0{+}1$ with $\alpha[\xi] \le \delta_0$: Then either $\alpha[\xi]{+}1 = \delta$ or $\alpha[\xi]{+}1 \overset{\text{IH}}{\underline{\ll}} \delta_0 \ll^1 \delta$.
2. $\delta \in \mathit{Lim}$:

By Lemma 2.2a, d, $\alpha[\xi] < \delta[2] < \alpha[\xi{+}1]$. Hence $\alpha[\xi]{+}1 \overset{\text{IH}}{\underline{\ll}} \delta[2] \ll^1 \delta$.

(b) Induction on η:

1. $\eta = \eta_0{+}1 < \tau_\alpha$: $\alpha[\xi] \overset{\text{IH}}{\underline{\ll}} \alpha[\eta_0] \ll^1 \alpha[\eta_0] + 1 \overset{\text{(a)}}{\underline{\ll}} \alpha[\eta]$.
2. $\eta \in \mathit{Lim}$: Then $\tau_{\alpha[\eta]} = \eta$ and $\alpha[\xi] = \alpha[\eta][\xi] \ll^1 \alpha[\eta]$.

(c) We may assume $\beta \ll^1 \alpha$, i.e. $\beta = \alpha[\xi]$ with $\xi < \tau^\circ_\alpha$.

Then either $\tau^\circ_\alpha = 1$ & $\beta{+}1 = \alpha$ or $\tau^\circ_\alpha \in \mathit{Lim}$ & $\alpha[\xi] + 1 \overset{\text{(a)}}{\underline{\ll}} \alpha[\xi{+}1] \ll^1 \alpha$.

(d) Induction on n:

1. Using Lemma 2.2a we get $0 \underline{\ll} \alpha$ by transfinite induction on α.
2. $n{+}1 \le \alpha \;\Rightarrow\; n < \alpha \;\&\; n \overset{\text{IH}}{\underline{\ll}} \alpha \;\Rightarrow\; n \ll \alpha \overset{\text{(c)}}{\Rightarrow} n{+}1 \underline{\ll} \alpha$.

Definition
An Ω-*normal function* is a strictly increasing continuous function $f : \Omega \to \Omega$.
A set $M \subseteq \Omega$ is Ω-*club* (*closed and unbounded in* Ω) iff

$$\forall X \subseteq M (X \neq \emptyset \;\&\; \sup(X) < \Omega \Rightarrow \sup(X) \in M) \text{ and } \forall \alpha < \Omega \exists \beta \in M (\alpha < \beta).$$

It is well-known that $M \subseteq \Omega$ is Ω-club if, and only if, M is the range of some Ω-normal function. Hence the ordering function of any Ω-club set is Ω-normal.

The collection of Ω-club sets has the following closure properties:

1. If f is Ω-normal then $\{\beta \in \Omega : f(\beta) = \beta\}$ is Ω-club.
2. If $(M_\xi)_{\xi<\alpha}$ is a sequence of Ω-club sets with $0 < \alpha < \Omega$ then $\bigcap_{\xi<\alpha} M_\xi$ is Ω-club.
3. If $(M_\xi)_{\xi<\Omega}$ is a sequence of Ω-club sets then also $\{\alpha \in \Omega : \alpha \in \bigcap_{\xi<\alpha} M_\xi\}$ is Ω-club.

Drawing upon 1.–3. and upon the above assignment of fundamental sequences we now define Bachmann's hierarchy of Ω-normal functions ϕ_α ($\alpha \le \Lambda$).

Definition $\phi_\alpha : \Omega \to \Omega$ is the ordering function of the Ω-club set R_α, where R_α is defined by recursion on α as follows:

$$R_0 := \mathbb{H} \cap \Omega,$$
$$R_{\alpha+1} := \{\beta \in \Omega : \phi_\alpha(\beta) = \beta\},$$
$$R_\alpha := \begin{cases} \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} & \text{if } \tau_\alpha \in \Omega \cap \mathit{Lim}, \\ \{\beta \in \Omega \cap \mathit{Lim} : \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}\} & \text{if } \tau_\alpha = \Omega. \end{cases}$$

Notes

1. In Lemma 2.5d we will show that $R_\alpha = \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$ if $\tau_\alpha = \Omega$.
2. As mentioned above, our definition of the Bachmann hierarchy (and of F_α) diverges in some minor points from [3]. As a consequence of this, Bachmann's ordinals $H(1) = \varphi_{F_{\Omega}(1)+1}(1)$ and $\varphi_{F_{\omega_2+1}(1)}(1)$ are $\phi_{F_\Omega(0)}(0)$ and $\phi_\Lambda(0)$, respectively, in the present paper. For more details cf. [2, Note on p. 35].

Lemma 2.4

(a) $\alpha_0 \underline{\ll} \alpha \;\Rightarrow\; R_\alpha \subseteq R_{\alpha_0}$.
(b) $\alpha_0 \ll \alpha \;\Rightarrow\; \phi_{\alpha_0}(0) < \phi_\alpha(0)$.
(c) $n < \alpha \cap \omega \;\&\; \beta \in R_\alpha \;\Rightarrow\; \omega{\cdot}n < \beta \in \mathit{Lim}$.

Proof

(a) It suffices to prove $R_\alpha \subseteq R_{\alpha_0}$ for $\alpha_0 \ll^1 \alpha$.

1. $\alpha = \alpha_0{+}1$: Then $R_\alpha = \{\beta \in \Omega : \phi_{\alpha_0}(\beta) = \beta\} \subseteq R_{\alpha_0}$.
2. $\tau_\alpha \in \Omega \cap \mathit{Lim}$: Then $\alpha_0 \in \{\alpha[\xi] : \xi < \tau_\alpha\}$ and thus $R_\alpha = \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} \subseteq R_{\alpha_0}$.
3. $\tau_\alpha = \Omega$: $\beta \in R_\alpha \Rightarrow \omega \le \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]} \Rightarrow \beta \in \bigcap_{\xi<\omega} R_{\alpha[\xi]} \subseteq R_{\alpha_0}$, since $\alpha_0 \in \{\alpha[\xi] : \xi < \omega\}$.

(b) 1. $\alpha = \alpha_0{+}1$: $\beta := \phi_\alpha(0) \in R_\alpha \;\Rightarrow\; \phi_{\alpha_0}(0) < \phi_{\alpha_0}(\beta) = \beta$.

2. $\alpha_0{+}1 < \alpha$: $\alpha_0 \ll \alpha \overset{2.3c}{\Rightarrow} \alpha_0{+}1 \underline{\ll} \alpha \overset{(a)}{\Rightarrow} R_\alpha \subseteq R_{\alpha_0+1} \;\Rightarrow\; \phi_{\alpha_0}(0) \overset{1.}{<} \phi_{\alpha_0+1}(0) \le \phi_\alpha(0)$.

(c) We have $1 \le \phi_0(0) < \phi_1(0) < \cdots$ and $\phi_{k+1}(0) \in \mathit{Lim}$. Hence $\omega{\cdot}n < \phi_{n+1}(0)$. Further: $n{<}\alpha \overset{2.3d}{\Rightarrow} n{+}1 \underline{\ll} \alpha \overset{(a)}{\Rightarrow} R_\alpha \subseteq R_{n+1} \subseteq \{\beta : \phi_{n+1}(0) \le \beta \in \mathit{Lim}\}$.

Lemma 2.5 *For each $\alpha \in \mathit{Lim} \cap (\Lambda{+}1)$ the following holds:*

(a) $\xi < \eta < (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; R_{\alpha[\eta]} \subseteq R_{\alpha[\xi]}$ & $\phi_{\alpha[\xi]}(0) < \phi_{\alpha[\eta]}(0)$.
(b) $\xi < (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; \xi \le \phi_{\alpha[\xi]}(0)$.
(c) $\lambda \in \mathit{Lim} \cap (\tau_\alpha{+}1) \cap \Omega \;\Rightarrow\; R_{\alpha[\lambda]} = \bigcap_{\xi<\lambda} R_{\alpha[\xi]}$.
(d) $\tau_\alpha = \Omega \;\Rightarrow\; R_\alpha = \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$.
(e) $n < \omega \;\Rightarrow\; \phi_{\alpha[n]}(0) < \phi_\alpha(0)$.

Proof

(a) follows from Lemmata 2.3b, 2.4a, b.
(b) follows from (a).
(c) By Lemma 2.2c we have $\tau_{\alpha[\lambda]} = \lambda$ and $\alpha[\lambda][\xi] = \alpha[\xi]$. Hence, by definition, $R_{\alpha[\lambda]} = \bigcap_{\xi<\lambda} R_{\alpha[\xi]}$.
(d) $R_\alpha = \{\beta \in \Omega \cap \mathit{Lim} : \beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}\} \overset{(c)}{=} \{\beta \in \Omega : \beta \in R_{\alpha[\beta]}\} \overset{(b)}{=} \{\beta \in \Omega : \phi_{\alpha[\beta]}(0) = \beta\}$.
(e) follows from Lemma 2.4b.

Schütte's Klammersymbols

In [19], building on [22], Schütte introduced a system of ordinal notations based on so-called 'Klammersymbols'. A Klammersymbol is a matrix $\begin{pmatrix}\xi_0 & \dots & \xi_n\\ \alpha_0 & \dots & \alpha_n\end{pmatrix}$ with $0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_n < \Omega$ and $\xi_0, \dots, \xi_n < \Omega$. Two Klammersymbols are defined to be equal if they are identical after deleting all columns of the form $\binom{0}{\alpha_i}$. This means that one can identify the Klammersymbol $\begin{pmatrix}\xi_0 & \dots & \xi_n\\ \alpha_0 & \dots & \alpha_n\end{pmatrix}$ with the ordinal $\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0$. Under this identification the $<$-relation between ordinals induces a well-ordering $\prec$ on the Klammersymbols. To each Ω-normal function f and each Klammersymbol A an ordinal $fA < \Omega$ is assigned by $\prec$-recursion: $f\binom{\xi}{0} := f(\xi)$, and for $\xi_1 > 0$, the function $\lambda x.f\begin{pmatrix}x & \xi_1 & \dots & \xi_n\\ 0 & \alpha_1 & \dots & \alpha_n\end{pmatrix}$ is the ordering function of the set $\{\beta \in \Omega : \forall\xi < \xi_1 \forall\alpha_0 < \alpha_1 [f \begin{pmatrix} \beta & \xi & \xi_2 & \dots & \xi_n \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix} = \beta]\}$. In this subsection we will locate the values $\phi_0 A$ within the Bachmann hierarchy, i.e., we will prove $\phi_0 \begin{pmatrix} \beta & \xi_0 & \dots & \xi_n \\ 0 & 1{+}\alpha_0 & \dots & 1{+}\alpha_n \end{pmatrix} = \phi_{\Omega^{\alpha_n}\xi_n + \dots + \Omega^{\alpha_0}\xi_0}(\beta)$.

Lemma 2.6 *Assume* $\alpha =_{\mathrm{NF}} \gamma + \Omega^{\delta_1}\xi_1$ *with* $\delta_1 < \Omega$.

(a) $\xi < \xi_1 \;\Rightarrow\; \gamma + \Omega^{\delta_1}\xi + 1 \underline{\ll} \gamma + \Omega^{\delta_1}(\xi{+}1) \underline{\ll} \alpha$.
(b) $\xi < \xi_1 \;\&\; \delta_0 < \delta_1 \;\Rightarrow\; \gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_0+1} \;\underline{\ll}\; \alpha$.
(c) $\beta \in R_\alpha \;\Leftrightarrow\; \forall\xi < \xi_1 [\, \phi_{\gamma+\Omega^{\delta_1}\xi}(\beta) = \beta \;\&\; \forall\delta_0 < \delta_1 (\phi_{\gamma+\Omega^{\delta_1}\xi+\Omega^{\delta_0}\beta}(0) = \beta)\,]$.

Proof
(a) Let $\hat{\alpha} := \gamma + \Omega^{\delta_1+1}$, $\eta := -1 + (\xi + 1)$, and $\eta_1 := -1 + \xi_1$. Then $\hat{\alpha}[\eta] = \gamma + \Omega^{\delta_1}(\xi{+}1)$, $\hat{\alpha}[\eta_1] = \gamma + \Omega^{\delta_1}\xi_1 = \alpha$, and $\eta \le \eta_1 < \tau_{\hat{\alpha}}$. Hence $\gamma + \Omega^{\delta_1}(\xi{+}1) \underline{\ll} \alpha$ by Lemma 2.3b. For the first inequality one needs the following auxiliary lemma (to be proved by induction on δ_1): $\Omega^{\delta_1}|\gamma_1 \;\Rightarrow\; \gamma_1 + 1 \underline{\ll} \gamma_1 + \Omega^{\delta_1}$.

(b) $\gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_0+1} \overset{(*)}{\underline{\ll}} \gamma + \Omega^{\delta_1}\xi + \Omega^{\delta_1} = \gamma + \Omega^{\delta_1}(\xi{+}1) \overset{(a)}{\underline{\ll}} \gamma + \Omega^{\delta_1}\xi_1 = \alpha$.

$(*)$ Let $\gamma_1 := \gamma + \Omega^{\delta_1}\xi$. We have $\delta_1 = \delta + n$ with ($\delta_0 < \delta \in \mathit{Lim}$ or $\delta = \delta_0 + 1$).

Further, $\gamma_1 + \Omega^{\delta_0+1} \underline{\ll} \gamma_1 + \Omega^{\delta} \ll \gamma_1 + \Omega^{\delta+1} \ll \cdots \ll \gamma_1 + \Omega^{\delta+n}$.

(c) We have to show:
$\beta \in R_\alpha \;\Leftrightarrow\; \forall\xi < \xi_1 [\, \beta \in R_{\gamma+\Omega^{\delta_1}\xi+1} \;\&\; \forall\delta_0 < \delta_1 (\beta \in R_{\gamma+\Omega^{\delta_1}\xi+\Omega^{\delta_0+1}})\,]$.
"$\Rightarrow$": Cf. Lemma 2.4a and (a), (b).
"$\Leftarrow$": We distinguish the following cases:
1. $\xi_1 \in \mathit{Lim}$: $\beta \in \bigcap_{\xi<\xi_1} R_{\gamma+\Omega^{\delta_1}(1+\xi)} = R_\alpha$.
2. $\xi_1 = \xi_0{+}1$:
2.1. $\delta_1 = 0$: Then $\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+1} = R_\alpha$.
2.2. $\delta_1 = \delta_0{+}1$: $\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+\Omega^{\delta_0+1}} = R_\alpha$.
2.3. $\delta_1 \in \mathit{Lim}$: Since $\delta_1 < \Omega$, we then have $\tau_\alpha = \delta_1$ and $\alpha[\xi] = \gamma + \Omega^{\delta_1}\xi_0 + \Omega^{1+\xi}$.
From $\forall\xi < \delta_1 (\beta \in R_{\gamma+\Omega^{\delta_1}\xi_0+\Omega^{\xi+1}})$ we get $\beta \in \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi+1]} \overset{2.5a}{\subseteq} \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} = R_\alpha$.

Definition Due to the fact that every ordinal can be uniquely represented in the form $\Omega\alpha + \beta$ with $\beta < \Omega$ it is possible to code the binary function $(\alpha, \beta) \mapsto \phi_\alpha(\beta)$ $(\alpha \le \Lambda, \beta < \Omega)$ into a unary one by $\phi\langle\Omega\alpha + \beta\rangle := \phi_\alpha(\beta)$ $(\alpha \le \Lambda, \beta < \Omega)$.

Using $\phi\langle\cdot\rangle$, the values of the Klammersymbols can be presented in a particularly nice way (cf. Theorem 2.8a below).

Lemma 2.7 *Assume* $\tilde{\alpha} =_{\mathrm{NF}} \gamma_1 + \Omega^{\alpha_1}\xi_1$ *with* $0 < \alpha_1 < \Omega$.

(a) $\lambda x.\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi_1 + x\rangle$ *enumerates*
$Q := \{\beta \in \Omega : \forall\xi < \xi_1 \forall\alpha_0 < \alpha_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$.
(b) *If* $\alpha_1 = \alpha_0 + 1$ *then* $Q = \{\beta \in \Omega : \forall\xi < \xi_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$.

Proof There are δ_1 and γ such that $\alpha_1 = 1 + \delta_1$ and $\gamma_1 = \Omega\gamma$. Let $\alpha := \gamma + \Omega^{\delta_1}\xi_1$. From (the proof of) Lemma 2.6c we get

$$\begin{aligned} R_\alpha &= \{\beta \in \Omega : \forall\xi < \xi_1[\,\phi\langle\Omega\gamma + \Omega^{1+\delta_1}\xi + \beta\rangle = \beta\ \& \\ &\qquad\qquad \forall\delta_0 < \delta_1(\phi\langle\Omega\gamma + \Omega^{1+\delta_1}\xi + \Omega^{1+\delta_0}\beta\rangle = \beta)\,]\} \\ &= \{\beta \in \Omega : \forall\xi < \xi_1\forall\alpha_0 < \alpha_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}, \text{ and} \\ R_\alpha &= \{\beta \in \Omega : \forall\xi < \xi_1[\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}, \text{ if } \alpha_1 = \alpha_0+1. \end{aligned}$$

On the other side, $\lambda x.\phi\langle\gamma_1 + \Omega^{\alpha_1}\xi_1 + x\rangle = \lambda x.\phi\langle\Omega\alpha + x\rangle$ enumerates R_α.

Theorem 2.8 *For* $\alpha_0 < \cdots < \alpha_n < \Omega$ *and* $\xi_0, \ldots, \xi_n < \Omega$*:*

(a) $\phi_0\begin{pmatrix}\xi_0 & \ldots & \xi_n \\ \alpha_0 & \ldots & \alpha_n\end{pmatrix} = \phi\langle\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0\rangle$.

(b) $\phi_0\begin{pmatrix}\beta & \xi_0 & \ldots & \xi_n \\ 0 & 1{+}\alpha_0 & \ldots & 1{+}\alpha_n\end{pmatrix} = \phi_{\Omega^{\alpha_n}\xi_n+\cdots+\Omega^{\alpha_0}\xi_0}(\beta)$.

Proof
(a) W.l.o.g. $\alpha_0 = 0$.

1. $n = 0$: $\phi\langle\Omega^0\xi_0\rangle = \phi\langle\Omega{\cdot}0 + \xi_0\rangle = \phi_0(\xi_0) = \phi_0\begin{pmatrix}\xi_0 \\ 0\end{pmatrix}$.
2. $n > 0$: W.l.o.g. $\xi_1 > 0$.

By Lemma 2.7a, $\lambda x.\phi\langle\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_1}\xi_1 + x\rangle$ is the ordering function of $\{\beta \in \Omega : \forall\xi < \xi_1\forall\alpha_0 < \alpha_1[\phi\langle\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_1}\xi + \Omega^{\alpha_0}\beta\rangle = \beta]\}$.

Combining this with the above given definition of $\phi_0 A$ (for Klammersymbols A) the assertion is established by induction on $\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0$.

(b) $\phi_0\begin{pmatrix}\beta & \xi_0 & \ldots & \xi_n \\ 0 & 1{+}\alpha_0 & \ldots & 1{+}\alpha_n\end{pmatrix} \overset{\text{(a)}}{=} \phi\langle\Omega^{1+\alpha_n}\xi_n + \cdots + \Omega^{1+\alpha_0}\xi_0 + \Omega^0\beta\rangle =$
$$= \phi\langle\Omega{\cdot}(\Omega^{\alpha_n}\xi_n + \cdots + \Omega^{\alpha_0}\xi_0) + \beta\rangle.$$

Lemma 2.9 *For* $\xi_0, \ldots, \xi_n < \Omega$ *let* $\varphi^{n+1}(\xi_n, \ldots, \xi_0) := \phi\langle\Omega^n\xi_n + \cdots + \Omega^0\xi_0\rangle$. *Then the following holds:*

(i) $\varphi^{n+1}(0, \ldots, 0, \beta) = \phi_0(\beta)$.

(ii) *If* $0 < k \le n$ *and* $\xi_k > 0$, *then* $\lambda x.\varphi^{n+1}(\xi_n, \ldots, \xi_k, 0, \ldots, 0, x)$ *enumerates* $\{\beta \in \Omega : \forall\xi < \xi_k(\varphi^{n+1}(\xi_n, \ldots, \xi_{k+1}, \xi, \beta, 0, \ldots, 0) = \beta)\}$.

Proof of (ii):
By definition, $\varphi^{n+1}(\xi_n, \ldots, \xi_k, \vec{0}, x) = \phi\langle\gamma + \Omega^k\xi_k + \Omega^0 x\rangle$ with
$\gamma := \Omega^n\xi_n + \cdots + \Omega^{k+1}\xi_{k+1}$.

Therefore by Lemma 2.7a, b, $\lambda x.\varphi^{n+1}(\xi_n, \ldots, \xi_k, \vec{0}, x)$ enumerates $\{\beta \in \Omega : \forall \xi < \xi_k[\phi\langle\gamma + \Omega^k\xi + \Omega^{k-1}\beta\rangle = \beta]\}$.

Note
φ^{n+1} $(n \geq 1)$ is known as the $n+1$-ary Veblen function.
Usually it is *defined* by (i), (ii).

3 Characterization of ϕ_α via $K\alpha$

In [15] the Bachmann hierarchy (ϕ_α) restricted to $\alpha < \varepsilon_{\Omega+1}$ is studied, and thereby, as a technical tool, the sets $C(\alpha)$ and $ND(\alpha)$ (of *constituents* and *nondistinguished constituents* of α) are defined. From Lemmata 4.1, 4.2 and Theorems 3.1, 3.3 of this paper one can derive the following interesting result which provides an alternative definition of the Bachmann hierarchy not referring to fundamental sequences:

$$R_\alpha = \{\gamma \in R_0 : C(\alpha) \leq \gamma \ \& \ ND(\alpha) < \gamma \ \& \ \forall \xi < \alpha(C(\xi) < \gamma \Rightarrow \phi_\xi(\gamma) = \gamma)\} \ (\alpha < \varepsilon_{\Omega+1}). \tag{G}$$

In the following we will directly prove an analogue of (G), namely Theorem 3.4, and then exemplarily derive Gerber's Theorems 5.1, 4.3 (our Lemmas 3.7, 3.8) from that.

Definition of $K\alpha$ for $\alpha \leq \Lambda$

1. $K\alpha := \begin{cases} \emptyset & \text{if } \alpha \in \{0, \Omega\}, \\ \{\alpha\} & \text{if } \alpha \in Lim \cap \Omega, \\ K\alpha_0 & \text{if } \alpha = \alpha_0{+}1 < \Omega. \end{cases}$
2. $\Omega < \alpha =_{\text{NF}} \gamma + \Omega^\beta\eta \notin \text{ran}(F_0)$: $K\alpha := K\gamma \cup K\beta \cup K\eta$.
3. $\Omega < \alpha =_{\text{NF}} F_\xi(\eta) < \Lambda$: $K\alpha := K'\xi \cup K\eta$ with $K'\xi := \begin{cases} \emptyset & \text{if } \xi = 0, \\ \{\omega\} \cup K\xi & \text{if } \xi > 0. \end{cases}$
4. $K\Lambda := \{\omega\}$.

Remark $K(\alpha_0{+}1) = K\alpha_0$.

Lemma 3.1 $\lambda \in Lim \ \& \ 1 \leq \xi \leq \tau_\lambda \ \Rightarrow \ K\lambda[\xi] = K\lambda[1] \cup K\xi$.

Proof

1. $\lambda =_{\text{NF}} \gamma + \Omega^\beta\eta \notin \text{ran}(F_0)$:

1.1. $\eta \in Lim$: $\tau_\lambda = \eta$ and $\lambda[\xi] = \gamma + \Omega^\beta(1{+}\xi)$.
$\xi \leq \eta \ \Rightarrow \ K\lambda[\xi] = K\gamma \cup K\beta \cup K\xi$.

1.2. $\eta = \eta_0{+}1$: $\tau_\lambda = \tau_{\Omega^\beta}$ and $\lambda[\xi] = \gamma + \Omega^\beta\eta_0 + \Omega^\beta[\xi]$.
$K\lambda[\xi] = K\gamma \cup K(\Omega^\beta\eta_0) \cup K(\Omega^\beta[\xi]) \overset{\text{IH}}{=} K\gamma \cup K(\Omega^\beta\eta_0) \cup K(\Omega^\beta[1]) \cup K\xi$.

2. $\lambda =_{\text{NF}} F_\alpha(\beta)$:

2.1. $\beta \in Lim$: Then by Lemma 2.1, $\lambda[\xi] =_{\text{NF}} F_\alpha(\beta[\xi])$ and thus $K\lambda[\xi] = K'\alpha \cup K(\beta[\xi]) \overset{\text{IH}}{=} K'\alpha \cup K\beta[1] \cup K\xi = K\lambda[1] \cup K\xi$.

2.2. $\beta \notin Lim$: Then $K\lambda^- = \begin{cases} K'\alpha \cup K\beta \text{ if } \beta = \beta_0{+}1 \ \& \ \beta_0 < F_\alpha(\beta_0), \\ K\beta \quad\quad\quad \text{otherwise.} \end{cases}$
Hence $K\lambda = K'\alpha \cup K\beta = K'\alpha \cup K\lambda^-$.
2.2.0. $\alpha = 0$: Then $\lambda = \Omega^{\beta_0+1}$, $\tau_\lambda = \Omega$ and $\lambda[\xi] = \Omega^{\beta_0}(1{+}\xi)$.
Hence $K\lambda[\xi] = K\beta_0 \cup K\xi$.
2.2.1. $\alpha = \alpha_0{+}1$: Then $\tau_\lambda = \omega$ and, for $\xi < \omega$, $K\lambda[\xi] = K(F_{\alpha_0}^{(\xi+1)}(\lambda^-)) = K'\alpha \cup K\lambda^-$ and $K\xi = \emptyset$.
Further $K\lambda[\omega] = K\lambda = K'\alpha \cup K\lambda^- = K'\alpha \cup K\lambda^- \cup K\omega$.
2.2.2. $\alpha \in Lim$: For $\xi < \tau_\lambda = \tau_\alpha$ we have $K\lambda[\xi] = KF_{\alpha[\xi]}(\lambda^-) = K\alpha[\xi] \cup \{\omega\} \cup K\lambda^- \overset{\text{IH}}{=} K\alpha[1] \cup \{\omega\} \cup K\lambda^- \cup K\xi$.
Further $K\lambda = K\alpha \cup \{\omega\} \cup K\lambda^- \overset{\text{IH}}{=} K\alpha[1] \cup \{\omega\} \cup K\lambda^- \cup K\tau_\alpha$.
3. $\lambda = \Lambda$: For $1 \le \xi \le \omega$ we have $K\Lambda[\xi] = \{\omega\}$, whence $K\Lambda[\xi] = K\Lambda[1] \cup K\xi$.

Lemma 3.2

(a) $\alpha \in Lim \ \& \ \alpha[\xi] \le \delta \le \alpha[\xi{+}1] \ \Rightarrow \ K\alpha[\xi] \subseteq K\delta$.
(b) $\delta < \alpha \ \& \ K\delta < \xi \in Lim \cap \tau_\alpha \ \Rightarrow \ \delta < \alpha[\xi]$.

Proof

(a) Induction on δ:

1. $\delta = \alpha[\xi]$: trivial.
2. $\delta = \delta_0 + 1$ with $\alpha[\xi] \le \delta_0$: Then $K\alpha[\xi] \overset{\text{IH}}{\subseteq} K\delta_0 = K\delta$.
3. $\alpha[\xi] < \delta \in Lim$: Then, by Lemma 2.2d, $\alpha[\xi] \le \delta[1]$. Hence $K\alpha[\xi] \overset{\text{IH}}{\subseteq} K\delta[1] \overset{3.1}{\subseteq} K\delta$.

(b) Assume $\alpha[0] \le \delta$. Then by Lemma 2.2a, b, c there exists $\zeta < \tau_\alpha$ such that $\alpha[\zeta] \le \delta < \alpha[\zeta{+}1]$. By (a) and Lemma 3.1 we get $K\zeta \subseteq K\alpha[\zeta] \subseteq K\delta < \xi \in Lim$. Hence $\delta < \alpha[\zeta{+}1] < \alpha[\xi]$.

Definition
$\mathsf{k}(\alpha) := \max(K\alpha \cup \{0\})$. $\mathsf{k}^+(\alpha) := \max\{\mathsf{k}(\alpha[1]){+}1, \mathsf{k}(\alpha)\}$.

Lemma 3.3

(a) $\mathsf{k}(\alpha) \le \mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1$;
(b) $\mathsf{k}^+(\alpha{+}1) = \mathsf{k}(\alpha) + 1$;
(c) $\mathsf{k}^+(\alpha) \le \phi_\alpha(0)$.

Proof

(a) By Lemma 3.1, $\mathsf{k}(\alpha) = \max\{\mathsf{k}(\alpha[1]), \mathsf{k}(\tau_\alpha)\}$ and thus
$\mathsf{k}^+(\alpha) = \max\{\mathsf{k}(\alpha[1]) + 1, \mathsf{k}(\tau_\alpha)\}$ $\quad(*)$.
(b) $\mathsf{k}^+(\alpha{+}1) = \max\{\mathsf{k}(\alpha){+}1, \mathsf{k}(\alpha{+}1)\} = \mathsf{k}(\alpha){+}1$.
(c) Induction on α:

1. $\mathsf{k}^+(0) = 1 \le \phi_0(0)$.

2. $\alpha > 0$: By IH and Lemma 2.5e, $\mathsf{k}(\alpha[1]) \le \phi_{\alpha[1]}(0) < \phi_\alpha(0)$. By Lemma 2.5b, $\mathsf{k}(\tau_\alpha) \le \phi_\alpha(0)$. Hence $\mathsf{k}^+(\alpha) \overset{(*)}{=} \max\{\mathsf{k}(\alpha[1])+1, \mathsf{k}(\tau_\alpha)\} \le \phi_\alpha(0)$.

Theorem 3.4 $R_\alpha = \{\beta \in R_0 : \mathsf{k}^+(\alpha) \le \beta \;\&\; \forall \xi < \alpha(K\xi < \beta \Rightarrow \phi_\xi(\beta) = \beta)\}$.

Proof "$\subseteq$": Assume $\beta \in R_\alpha$. By Lemmata 2.4a, 3.3a, c we get $\mathsf{k}^+(\alpha) \le \beta \in R_0$. The second part is proved by induction on α. So let $\delta < \alpha \;\&\; K\delta < \beta \in R_\alpha$.

1. $\alpha = \delta + 1$: $\beta \in R_{\delta+1}$ implies $\phi_\delta(\beta) = \beta$.
2. $\alpha = \alpha_0 + 1 \;\&\; \delta < \alpha_0$: From $\delta < \alpha_0 \;\&\; K\delta < \beta \in R_\alpha \subseteq R_{\alpha_0}$ we obtain $\phi_\delta(\beta) = \beta$ by IH.
3. $\alpha \in Lim \;\&\; \tau_\alpha < \Omega$: Then $\beta \in \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]}$ and $\delta < \alpha$. From this we get $\exists \xi < \tau_\alpha(\beta \in R_{\alpha[\xi]} \;\&\; \delta < \alpha[\xi])$ and then $\phi_\delta(\beta) = \beta$ by IH.
4. $\tau_\alpha = \Omega$: By Lemmata 2.4c, 2.5c we get $\beta \in Lim \cap R_{\alpha[\beta]}$. From $\delta < \alpha$ and $K\delta < \beta \in Lim \cap \tau_\alpha$ we get $\delta < \alpha[\beta]$ by Lemma 3.2b. Now we have $\beta \in R_{\alpha[\beta]}$ and $\delta < \alpha[\beta] < \alpha \;\&\; K\delta < \beta$ which by IH yields $\phi_\delta(\beta) = \beta$.

"$\supseteq$": Assume (1) $\mathsf{k}^+(\alpha) \le \beta \in R_0$, and (2) $\forall \delta < \alpha(K\delta < \beta \Rightarrow \beta \in R_{\delta+1})$. From $\mathsf{k}^+(\alpha) \le \beta$ we get (3) $K\alpha[1] < \beta$.

1. $\alpha = 0$: trivial.
2. $\alpha = \alpha_0+1$: From $\alpha_0 < \alpha \;\&\; K\alpha_0 = K\alpha[1] < \beta$ by (2) we obtain $\beta \in R_{\alpha_0+1} = R_\alpha$.
3. $\alpha \in Lim \;\&\; \tau_\alpha < \Omega$: By Lemma 3.1 and (1) we have $\tau_\alpha \le \mathsf{k}(\alpha) \le \beta$. From $0 < \xi < \tau_\alpha \le \beta$ by Lemma 3.1 and (3) we conclude $\alpha[\xi] < \alpha \;\&\; K\alpha[\xi] \subseteq K\alpha[1] \cup K\xi < \beta$, and then by (2), $\beta \in R_{\alpha[\xi]+1}$. Hence $\beta \in \bigcap_{\xi<\tau_\alpha} R_{\alpha[\xi]} = R_\alpha$.
4. $\tau_\alpha = \Omega$: From $0 < \alpha \;\&\; K0 = \emptyset < \beta$ by (2) we get $\beta \in R_1$, thence $\beta \in Lim$. Similarly as above we obtain $\beta \in \bigcap_{\xi<\beta} R_{\alpha[\xi]}$. Hence $\beta \in R_\alpha$.

The Fixed-point-free Functions $\overline{\phi}_\alpha$

Definition

$\overline{\phi}_\alpha(\beta) := \phi_\alpha(\beta + \tilde{\iota}\alpha\beta)$ where

$\tilde{\iota}\alpha\beta := \begin{cases} 1 & \text{if } \beta = \beta_0 + n \text{ with } \phi_\alpha(\beta_0) \in K\alpha \cup \{\beta_0\}, \\ 0 & \text{otherwise.} \end{cases}$

$\overline{R}_\alpha := \mathrm{ran}(\overline{\phi}_\alpha)$.

Notation. From now on we mostly write $\phi\alpha\beta$, $\overline{\phi}\alpha\beta$ for $\phi_\alpha(\beta)$, $\overline{\phi}_\alpha(\beta)$.

Theorem 3.5

(a) $\overline{\phi}_\alpha$ *is order preserving.*
(b) $\overline{R}_\alpha = \{\phi\alpha\beta : K\alpha \cup \{\beta\} < \phi\alpha\beta\} = \{\gamma \in R_\alpha \setminus R_{\alpha+1} : K\alpha < \gamma\}$.
(c) $\overline{\phi}\alpha\beta = \min\{\gamma \in R_\alpha : \forall \eta < \beta(\overline{\phi}\alpha\eta < \gamma) \;\&\; K\alpha \cup \{\beta\} < \gamma\}$.

Proof

(a) If $\beta_1 < \beta_2$ then $\beta_1 + \tilde{\iota}\alpha\beta_1 < \beta_2$ or $\beta_1 + \tilde{\iota}\alpha\beta_1 = \beta_2$.
In the latter case $\tilde{\iota}\alpha\beta_2 = \tilde{\iota}\alpha\beta_1 = 1$.

(b) The first equation follows immediately from the definition, since $\mathsf{k}(\alpha) \le \phi\alpha 0$ and $\eta{+}1 < \phi\alpha(\eta{+}1)$ for all $\eta < \Omega$. The second equation follows from the first, since $\phi\alpha\beta \in R_{\alpha+1} \Leftrightarrow \beta = \phi\alpha\beta$.

(c) Let $X := \{\gamma \in R_\alpha : \forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma)\ \&\ K\alpha \cup \{\beta\} < \gamma\}$. By (a) and (b) we have $\overline{\phi}\alpha\beta \in X$. It remains to prove $\forall\gamma \in X(\overline{\phi}\alpha\beta \le \gamma)$. So let $\gamma \in X$, i.e. $\gamma = \phi\alpha\delta$ with $\forall\eta < \beta(\phi\alpha(\eta + \tilde{\iota}\alpha\eta) < \phi\alpha\delta)\ \&\ K\alpha \cup \{\beta\} < \phi\alpha\delta$ $(*)$.

To prove: $\overline{\phi}\alpha\beta \le \phi\alpha\delta$, i.e. $\beta + \tilde{\iota}\alpha\beta \le \delta$.

From $\forall\eta < \beta(\phi\alpha(\eta + \tilde{\iota}\alpha\eta) < \phi\alpha\delta)$ we get $\beta \le \delta$. Therefore if $\beta < \delta$ or $\tilde{\iota}\alpha\beta = 0$, we are done.

Assume now $\beta = \delta\ \&\ \tilde{\iota}\alpha\beta = 1$. Then $\delta = \beta = \beta_0 + n$ with $\phi\alpha\beta_0 \in K\alpha \cup \{\beta_0\}$.

1. $0 < n$: Then $\eta := \beta_0 + (n{-}1) < \beta = \eta + 1$ and therefore $\beta = \eta + \tilde{\iota}\alpha\eta \overset{(*)}{<} \delta = \beta$. Contradiction.
2. $n = 0$: Then $\phi\alpha\beta \in K\alpha \cup \{\beta\} \overset{(*)}{<} \phi\alpha\delta = \phi\alpha\beta$. Contradiction.

Corollary 3.6

(a) $\xi < \alpha\ \&\ K\xi \cup \{\eta\} < \overline{\phi}\alpha\beta \ \Rightarrow\ \overline{\phi}\xi\eta < \overline{\phi}\alpha\beta$.
(b) $K\alpha \cup \{\beta\} < \overline{\phi}\alpha\beta$.

Proof

(a) $\xi < \alpha\ \&\ K\xi \cup \{\eta\} < \overline{\phi}\alpha\beta \in R_\alpha \ \Rightarrow\ \overline{\phi}\xi\eta \le \phi\xi(\eta{+}1) < \phi\xi\overline{\phi}\alpha\beta \overset{3.4}{=} \overline{\phi}\alpha\beta$.
(b) follows immediately from Theorem 3.5c.

Lemma 3.7 *Let* $\gamma_i = \overline{\phi}\alpha_i\beta_i$ *(*$i = 1, 2$*).*

(a) $\gamma_1 < \gamma_2$ *if, and only if, one of the following holds:*

(i) $\alpha_1 < \alpha_2\ \&\ K\alpha_1 \cup \{\beta_1\} < \gamma_2$*;*
(ii) $\alpha_1 = \alpha_2\ \&\ \beta_1 < \beta_2$*;*
(iii) $\alpha_2 < \alpha_1\ \&\ \gamma_1 \le K\alpha_2 \cup \{\beta_2\}$*.*

(b) $\gamma_1 = \gamma_2 \ \Rightarrow\ \alpha_1 = \alpha_2\ \&\ \beta_1 = \beta_2$*.*

Proof
(a) Let $Q(\alpha_1, \beta_1, \alpha_2, \beta_2) :\equiv (i) \vee (ii) \vee (iii)$.
To prove: $\gamma_1 < \gamma_2 \ \Leftrightarrow\ Q(\alpha_1, \beta_1, \alpha_2, \beta_2)$.
From Theorem 3.5a and Corollary 3.6 we get the implications
(1) $Q(\alpha_1, \beta_1, \alpha_2, \beta_2) \ \Rightarrow\ \gamma_1 < \gamma_2$ and (2) $Q(\alpha_2, \beta_2, \alpha_1, \beta_1) \ \Rightarrow\ \gamma_2 < \gamma_1$.
Obviously,
(3) $\neg Q(\alpha_1, \beta_1, \alpha_2, \beta_2) \ \Rightarrow\ Q(\alpha_2, \beta_2, \alpha_1, \beta_2) \vee (\alpha_1 = \alpha_2\ \&\ \beta_1 = \beta_2)$.
From (2) and (3) we get: $\neg Q(\alpha_1, \beta_1, \alpha_2, \beta_2) \ \Rightarrow\ \neg(\gamma_1 < \gamma_2)$.
(b) Proof by contradiction. Assume $\gamma_1 = \gamma_2\ \&\ \alpha_1 < \alpha_2$. Then by Corollary 2.6b we have $\alpha_1 < \alpha_2\ \&\ K\alpha_1 \cup \{\beta_1\} < \gamma_1 = \gamma_2$. Hence $\gamma_1 < \gamma_2$ by Corollary 2.6a.

Lemma 3.8 *For each* $\gamma \in R_0 \cap \phi_\Lambda(0)$ *there exists* $\alpha < \Lambda$ *such that* $\gamma \in \overline{R}_\alpha$*.*

Proof
Assume $\omega < \gamma$. Then $K\Lambda < \gamma \notin R_\Lambda$. Let α_1 be the least ordinal such that $K\alpha_1 < \gamma \notin R_{\alpha_1}$. Then by Theorem 3.4 there exists $\alpha < \alpha_1$ such that $K\alpha < \gamma \notin R_{\alpha+1}$. By minimality of α_1 we get $\gamma \in R_\alpha$. Hence $\gamma \in \overline{R}_\alpha$ by Theorem 3.5b.

The following will prove useful in Sect. 5.

Theorem 3.9 *Let* $\overline{\phi}\langle\Omega\alpha + \beta\rangle := \overline{\phi}\alpha\beta$ *(*$\alpha \leq \Lambda$, $\beta < \Omega$*). Then for all* $\alpha < \Lambda + \Omega$, $\overline{\phi}\langle\alpha\rangle = \min\{\gamma \in R_0 : \forall\xi < \alpha(K\xi < \gamma \Rightarrow \overline{\phi}\langle\xi\rangle < \gamma)\ \&\ K\alpha < \gamma\}$.

Proof

$$\overline{\phi}\langle\Omega\alpha + \beta\rangle = \overline{\phi}\alpha\beta \overset{3.5c}{=}$$
$$\min\{\gamma \in R_\alpha : \forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma)\ \&\ K\alpha \cup \{\beta\} < \gamma\} \overset{3.4}{=}$$
$$\min\{\gamma \in R_0 : \forall\xi < \alpha\forall\eta(K\xi \cup \{\eta\} < \gamma \Rightarrow \overline{\phi}\xi\eta < \gamma)\ \&$$
$$\forall\eta < \beta(\overline{\phi}\alpha\eta < \gamma)\ \&\ K\alpha \cup \{\beta\} < \gamma\} \overset{(*)}{=}$$
$$\min\{\gamma \in R_0 : \forall\xi < \alpha\forall\eta(K\xi \cup K\eta < \gamma \Rightarrow \overline{\phi}\langle\Omega\xi + \eta\rangle < \gamma)\ \&$$
$$\forall\eta < \beta(K\alpha \cup K\eta < \gamma \Rightarrow \overline{\phi}\langle\Omega\alpha + \eta\rangle < \gamma)\ \&\ K\alpha \cup K\beta < \gamma\} =$$
$$\min\{\gamma \in R_0 : \forall\zeta < \Omega\alpha{+}\beta(K\zeta < \gamma \Rightarrow \overline{\phi}\langle\zeta\rangle < \gamma)\ \&\ K(\Omega\alpha + \beta) < \gamma\}.$$

$(*)$ For $\alpha = \beta = 0$ the equation is trivial. Otherwise it follows from the fact that for $1 < \gamma \in R_0$ we have $\forall\eta < \Omega(K\eta < \gamma \Leftrightarrow \eta < \gamma)$.

4 Comparison of $\phi_\alpha, \overline{\phi}_\alpha$ with $\theta_\alpha, \overline{\theta}_\alpha$

In this section we will compare the Bachmann functions ϕ_α with Feferman's functions θ_α. We will prove that $\phi\alpha\beta = \theta\alpha(\widehat{\alpha} + \beta)$ for all $\alpha \leq \Lambda, \beta < \Omega$, where $\widehat{\alpha} := \min\{\eta : \mathsf{k}^+(\alpha) \leq \theta\alpha\eta\}$. This result is already stated in [1], Theorem 3[1] and, for $\alpha < \varepsilon_{\Omega+1}$, proved in [23].

Before we can turn to the proper subject of this section we have to do some elementary ordinal arithmetic.

Definition $E_\Omega(\alpha) = \begin{cases} \emptyset & \text{if } \alpha \in \{0, \Omega\}, \\ \{\alpha\} & \text{if } \alpha \in \mathbb{E} \setminus \{\Omega\}, \\ \bigcup_{i \leq n} E_\Omega(\alpha_i) & \text{if } \alpha = \omega^{\alpha_0} \# \cdots \# \omega^{\alpha_n} \notin \mathbb{E}. \end{cases}$

Definition A set $C \subseteq On$ is *nice* iff
$0 \in C\ \&\ \forall n \forall\alpha_0, \ldots, \alpha_n(\omega^{\alpha_0}\#\cdots\#\omega^{\alpha_n} \in C \Leftrightarrow \{\alpha_0, \ldots, \alpha_n\} \subseteq C)$.

Lemma 4.1

(a) $E_\Omega(\Omega + \alpha) = E_\Omega(\Omega{\cdot}\alpha) = E_\Omega(\Omega^\alpha) = E_\Omega(\alpha)$.
(b) $\alpha =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \;\Rightarrow\; E_\Omega(\alpha) = E_\Omega(\gamma) \cup E_\Omega(\beta) \cup E_\Omega(\eta)$.
(c) If C is nice and $\Omega \in C$ *then* $\forall\alpha(\alpha \in C \Leftrightarrow E_\Omega(\alpha) \subseteq C)$.

[1] Actually Aczel's Theorem 3 looks somewhat different, but it implies the above formulated result. A proof of Theorem 3 can be extracted from the proof of Theorem 3.5 in [5].

(d) $\alpha < \varepsilon_{\Omega+1}$ & $\delta \in \mathbb{E} \;\Rightarrow\; (E_\Omega(\alpha) < \delta \Leftrightarrow K\alpha < \delta)$.

Proof
(a) Let $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$ with $\alpha_1 \geq \cdots \geq \alpha_n$.
1. $E_\Omega(\Omega + \alpha) = \begin{cases} E_\Omega(\alpha) & \text{if } \Omega < \alpha_0, \\ E_\Omega(\Omega) \cup E_\Omega(\alpha) & \text{if } \Omega \geq \alpha_0. \end{cases}$
2. $E_\Omega(\Omega{\cdot}\alpha) = E_\Omega(\omega^{\Omega+\alpha_0} + \cdots + \omega^{\Omega+\alpha_n}) = \bigcup_{i \leq n} E_\Omega(\Omega + \alpha_i) \overset{1.}{=} \bigcup_{i \leq n} E_\Omega(\alpha_i) = E_\Omega(\alpha)$.
3. $E_\Omega(\Omega^\alpha) = E_\Omega(\omega^{\Omega\cdot\alpha}) = E_\Omega(\Omega \cdot \alpha) \overset{2.}{=} E_\Omega(\alpha)$.

(b) Let $\eta = \omega^{\eta_0} + \cdots + \omega^{\eta_m}$ with $\eta_0 \geq \cdots \geq \eta_m$.
Then $\Omega^\beta \eta = \omega^{\Omega\cdot\beta} \cdot (\omega^{\eta_0} + \cdots + \omega^{\eta_m}) = \omega^{\Omega\beta+\eta_0} + \cdots + \omega^{\Omega\beta+\eta_m}$.
Hence $E_\Omega(\Omega^\beta \eta) = \bigcup_{i \leq m} E_\Omega(\Omega\beta + \eta_0) = \bigcup_{i \leq m}(E_\Omega(\beta) \cup E_\Omega(\eta_i)) = E_\Omega(\beta) \cup \bigcup_{i \leq m} E_\Omega(\eta_i) = E_\Omega(\beta) \cup E_\Omega(\eta)$.

(c) 1. $\alpha \in \{0, \Omega\}$: $E_\Omega(\alpha) = \emptyset \subseteq C$ and $\alpha \in C$.
2. $\alpha \in \mathbb{E}$: $E_\Omega(\alpha) = \{\alpha\}$.
3. $\alpha = \omega^{\alpha_0} \# \cdots \# \omega^{\alpha_n} \notin \mathbb{E}$: $E_\Omega(\alpha) = E_\Omega(\alpha_0) \cup \cdots \cup E_\Omega(\alpha_n)$ and therefore:
$E_\Omega(\alpha) \subseteq C \;\Leftrightarrow\; \forall i \leq n(E_\Omega(\alpha_i) \subseteq C) \overset{\text{IH}}{\Leftrightarrow} \forall i \leq n(\alpha_i \in C) \overset{C\,\text{nice}}{\Leftrightarrow} \alpha \in C$.

(d) 1. $\alpha \in \{0, \Omega\}$: $E_\Omega(\alpha) = \emptyset = K\alpha$.
2. $\alpha < \Omega$: $E_\Omega(\alpha) < \delta \Leftrightarrow \alpha < \delta \Leftrightarrow K\alpha < \delta$.
3. $\Omega < \alpha =_{\text{NF}} \gamma + \Omega^\beta \eta$: $E_\Omega(\alpha) < \delta \overset{\text{(b)}}{\Leftrightarrow} E_\Omega(\gamma) \cup E_\Omega(\beta) \cup E_\Omega(\eta) < \delta \overset{\text{IH}}{\Leftrightarrow} K\gamma \cup K\beta \cup K\eta < \delta \Leftrightarrow K\alpha < \delta$.

Basic Properties of the Functions θ_α
The functions $\theta_\alpha : On \to On$ and sets $C(\alpha, \beta) \subseteq On$ are defined simultaneously by recursion on α (cf. [5], p. 174, [7], p. 6, [20], p. 225). Instead of giving this definition we present a list of basic properties which are sufficient for proving Theorems 4.6, 4.7 below.—*Notation:* $\theta\alpha\beta := \theta_\alpha(\beta)$.

(θ1) $\theta_\alpha : On \to On$ is a normal function and $\text{In}_\alpha := \text{ran}(\theta_\alpha)$.
(θ2) (i) $\text{In}_0 = \mathbb{H}$,
(ii) $\text{In}_{\alpha+1} = \{\beta \in \text{In}_\alpha : \alpha \in C(\alpha, \beta) \Rightarrow \beta = \theta\alpha\beta\}$,
(iii) $\text{In}_\alpha = \bigcap_{\xi<\alpha} \text{In}_\xi$ if $\alpha \in Lim$.
(θ3) $\theta\alpha\Omega = \Omega$.
(θ4) $\text{In}_\alpha \cap \Omega = \{\beta \in \Omega : C(\alpha, \beta) \cap \Omega \subseteq \beta\}$.
(θ5) $\{0\} \cup \beta \subseteq C(\alpha, \beta)$, and if $\alpha > 0$ then $C(\alpha, \beta)$ is nice and $\Omega \in C(\alpha, \beta)$.
(θ6) $\xi < \alpha \leq \Lambda$ & $\Omega < \eta < \theta\xi\eta \;\Rightarrow\; (\xi, \eta \in C(\alpha, \beta) \Leftrightarrow \theta\xi\eta \in C(\alpha, \beta))$.

Remark (θ4)–(θ6) are only needed for the proof of Lemma 4.3c (via Lemmas 4.2 and 4.3a, b). Having established Lemma 4.3c we will make use only of (θ1)–(θ3) with (θ2*ii*) replaced by Lemma 4.3c.

Lemma 4.2

(a) $\alpha < \theta\alpha(\Omega{+}1)$ & $\Omega \le \beta \;\Rightarrow\; (\beta \in \mathrm{In}_{\alpha+1} \Leftrightarrow \beta = \theta\alpha\beta)$.
(b) $0 < \alpha \le \Lambda \;\Rightarrow\; F_\alpha(\beta) = \theta\alpha(\Omega + 1 + \beta)$.

Proof
(a) "$\Leftarrow$": immediate consequence of $(\theta 2ii)$ (and $(\theta 1)$).
"$\Rightarrow$": Assume $\beta \in \mathrm{In}_\alpha$ and $(\alpha \in C(\alpha, \beta) \Rightarrow \beta = \theta\alpha\beta)$. For $\beta = \Omega$ the claim follows directly from $(\theta 3)$. Otherwise:

$$\theta\alpha\Omega \overset{(\theta 3)}{=} \Omega < \beta \in \mathrm{In}_\alpha \Rightarrow \theta\alpha(\Omega{+}1) \le \beta \Rightarrow \alpha < \beta \overset{(\theta 5)}{\Rightarrow} \alpha \in C(\alpha, \beta) \Rightarrow \beta = \theta\alpha\beta.$$

(b) Let $J := \{\beta : \Omega < \beta\}$. We prove $\mathrm{ran}(F_\alpha) = \mathrm{In}_\alpha \cap J$ which is equivalent to the claim $\forall\beta(F_\alpha(\beta) = \theta\alpha(\Omega + 1 + \beta))$.

The proof proceeds by induction on α.

1. $\alpha = 1$: $\mathrm{ran}(F_1) = \{\beta : \beta = \Omega^\beta\} = \{\beta : \Omega < \beta = \omega^\beta\} \overset{(\theta 2)}{=} \mathrm{In}_1 \cap J$.
2. $\alpha = \alpha_0{+}1$ with $1 \le \alpha_0$: $\mathrm{ran}(F_\alpha) = \{\beta : \beta = F_{\alpha_0}(\beta)\} \overset{\mathrm{IH}}{=}$
   $\{\beta : \beta = \theta\alpha_0(\Omega{+}1{+}\beta)\} = \{\beta : \Omega < \beta = \theta\alpha_0\beta\} \overset{(*)}{=} \mathrm{In}_\alpha \cap J$.
   $(*)$ $\alpha_0{<}\Lambda \Rightarrow \alpha_0{<}F_{\alpha_0}(0) \overset{\mathrm{IH}}{=} \theta_{\alpha_0}(\Omega{+}1) \overset{(a)}{\Rightarrow} \forall\beta > \Omega(\beta = \theta\alpha_0\beta \Leftrightarrow \beta \in \mathrm{In}_\alpha)$.
3. $\alpha \in \mathit{Lim}$: $\mathrm{ran}(F_\alpha) = \bigcap_{\xi<\alpha} \mathrm{ran}(F_\xi) \overset{\mathrm{IH}}{=} \bigcap_{\xi<\alpha} \mathrm{In}_\xi \cap J \overset{(\theta 2\mathrm{iii})}{=} \mathrm{In}_\alpha \cap J$.

Lemma 4.3 *For* $\alpha < \Lambda$ *we have:*

(a) $\xi < \alpha$ & $\eta < F_\xi(\eta) < \Lambda \;\Rightarrow\; (\xi, \eta \in C(\alpha, \beta) \;\Leftrightarrow\; F_\xi(\eta) \in C(\alpha, \beta))$.
(b) $\forall\delta \le \alpha(\delta \in C(\alpha, \beta) \;\Leftrightarrow\; K\delta \subseteq C(\alpha, \beta))$.
(c) $\mathrm{In}_{\alpha+1} = \{\beta \in \mathrm{In}_\alpha : K\alpha < \beta \Rightarrow \beta = \theta\alpha\beta\}$.

Proof
(a) For $\xi = 0$ the claim follows from Lemma 4.1a, c and $(\theta 5)$.
Assume now $\xi > 0$ and let $\gamma := F_\xi(\eta)$.
Then $\xi, \eta_1 < \gamma = \theta\xi\eta_1$ with $\eta_1 := \Omega{+}1{+}\eta$.
By $(\theta 5)$ and Lemma 4.1a, c we have $(\eta \in C(\alpha, \beta) \Leftrightarrow \eta_1 \in C(\alpha, \beta))$.
Hence: $\xi, \eta \in C(\alpha, \beta) \Leftrightarrow \xi, \eta_1 \in C(\alpha, \beta) \overset{(\theta 6)}{\Leftrightarrow} \gamma \in C(\alpha, \beta)$.
(b) Induction on δ: Assume $\delta \le \alpha$, and let $C := C(\alpha, \beta)$.

1. $\delta \in \{0, \Omega\}$: $\delta \in C$ & $K\delta = \emptyset$.
2. $\delta = \delta_0{+}1$: $\delta \in C \Leftrightarrow \delta_0 \in C$, and $K\delta = K\delta_0$.
3. $\delta \in \mathit{Lim} \cap \Omega$: $K\delta = \{\delta\}$.
4. $\delta =_{\mathrm{NF}} \gamma + \Omega^\beta\eta \notin \mathrm{ran}(F_0)$: $\delta \in C \overset{4.1c}{\Leftrightarrow} E_\Omega(\delta) \subseteq C \overset{4.1b}{\Leftrightarrow} E_\Omega(\gamma) \cup E_\Omega(\beta) \cup E_\Omega(\eta) \subseteq C \overset{4.1c}{\Leftrightarrow} \gamma, \beta, \eta \in C \overset{\mathrm{IH}}{\Leftrightarrow} K\gamma \cup K\beta \cup K\eta \subseteq C \Leftrightarrow K\delta \subseteq C$.
5. $\delta =_{\mathrm{NF}} F\xi\eta$: $\delta \in C \overset{(a)}{\Leftrightarrow} \xi, \eta \in C \overset{\mathrm{IH}}{\Leftrightarrow} K\xi \cup K\eta \subseteq C \overset{(*)}{\Leftrightarrow} K\delta \subseteq C$.
$(*)$ $\omega = \theta 01 \in C$.
(c) follows from $(\theta 2ii)$, $(\theta 4)$, (b) and the fact that $K\alpha \subseteq \Omega$.

Theorem 4.4 $\alpha \le \Lambda \;\Rightarrow\; \mathrm{In}_\alpha = \{\beta \in \mathbb{H} : \forall \xi < \alpha (K\xi < \beta \;\Rightarrow\; \theta\xi\beta = \beta)\}$.

Proof by induction on α

1. $\alpha = 0$: By $(\theta 2i)$ we have $\mathrm{In}_0 = \mathbb{H}$.
2. $\alpha = \alpha_0{+}1$: $\mathrm{In}_\alpha \overset{4.3c}{=} \{\beta \in \mathrm{In}_{\alpha_0} : K\alpha_0 < \beta \Rightarrow \beta = \theta\alpha_0\beta\} \overset{\text{IH}}{=}$ $\{\beta \in \mathbb{H} : \forall \xi < \alpha_0 (K\xi < \beta \Rightarrow \beta = \theta\xi\beta)\ \&\ (K\alpha_0 < \beta \Rightarrow \beta = \theta\alpha_0\beta)\}$.
3. $\alpha \in \mathit{Lim}$: Then, by $(\theta 2iii)$, $\mathrm{In}_\alpha = \bigcap_{\xi<\alpha} \mathrm{In}_\xi$ and the assertion follows immediately from the IH.

Definition $\widehat{\alpha} := \min\{\eta : \mathsf{k}^+(\alpha) \le \theta\alpha\eta\}$.

Lemma 4.5 $\alpha \le \Lambda\ \&\ K\alpha < \theta\alpha\beta \;\Rightarrow\; (\theta\alpha(\widehat{\alpha} + \beta) = \beta \Leftrightarrow \theta\alpha\beta = \beta)$.

Proof
"$\Rightarrow$": This follows from $\beta \le \theta\alpha\beta \le \theta\alpha(\widehat{\alpha} + \beta)$.
"$\Leftarrow$": If $K\alpha < \beta = \theta\alpha\beta$ then $\widehat{\alpha} \le \mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1 < \beta \in \mathbb{H}$ and thus $\widehat{\alpha} + \beta = \beta$.

Theorem 4.6 *If* $\alpha \le \Lambda$, *then* $R_\alpha = \{\gamma \in \Omega : \mathsf{k}^+(\alpha) \le \gamma \in \mathrm{In}_\alpha\}$, *and thus* $\forall \beta < \Omega\, (\phi\alpha\beta = \theta\alpha(\widehat{\alpha} + \beta))$.

Proof by induction on α:
For $\beta < \Omega$ we have:
$\beta \in R_\alpha \overset{3.4}{\Leftrightarrow} \mathsf{k}^+(\alpha) \le \beta \in \mathbb{H}\ \&\ \forall \xi < \alpha (K\xi < \beta \Rightarrow \phi\xi\beta = \beta) \overset{IH+4.5}{\Leftrightarrow} \mathsf{k}^+(\alpha)$ $\le \beta \in \mathbb{H}\ \&\ \forall \xi < \alpha (K\xi < \beta \Rightarrow \theta\xi\beta = \beta) \overset{4.4}{\Leftrightarrow} \mathsf{k}^+(\alpha) \le \beta \in \mathrm{In}_\alpha$.

The Functions $\overline{\theta}_\alpha$
In [7] the fixed-point-free functions $\overline{\theta}_\alpha$ are introduced, which are more suitable for proof-theoretic applications than the θ_α's. By definition, $\overline{\theta}_\alpha$ is the $<$-isomorphism from $\{\eta \in \mathit{On} : S\mu(\alpha) \le \eta\}$ onto $\overline{\mathrm{In}}_\alpha$ where $\overline{\mathrm{In}}_\alpha := \mathrm{In}_\alpha \setminus \mathrm{In}_{\alpha+1}$, $\mu(\alpha) := \min\{\eta : \theta\alpha\eta \in \overline{\mathrm{In}}_\alpha\}$, $S\mu(\alpha) := \min\{\Omega_\xi : \mu(\alpha) < \Omega_{\xi+1}\}$ where $\Omega_0 := 0$.

As we will show in a moment, $S\mu(\alpha) = 0$ for all $\alpha < \Lambda$, and therefore, if $\alpha < \Lambda$ then $\overline{\theta}_\alpha$ is the ordering function of $\overline{\mathrm{In}}_\alpha$. On the other side, by Theorem 3.5, $\overline{\phi}_\alpha$ is the ordering function of $\overline{R}_\alpha = \{\gamma \in R_\alpha \setminus R_{\alpha+1} : K\alpha < \gamma\}$. Using Theorem 4.6 one easily sees that $\overline{R}_\alpha = \overline{\mathrm{In}}_\alpha \cap \Omega$. So we arrive at the following theorem.

Theorem 4.7 $\overline{\phi}\alpha\beta = \overline{\theta}\alpha\beta$ *for all* $\alpha < \Lambda$, $\beta < \Omega$.

Proof
I. From $\alpha < \Lambda$ by Lemma 4.3c and $(\theta 3)$ we obtain $\forall \beta \in \Omega\, (\mathsf{k}(\alpha) \le \beta \Rightarrow \theta\alpha(\beta{+}1) \in \overline{\mathrm{In}}_\alpha \cap \Omega)$. Hence $S\mu(\alpha) = 0$, and $\overline{\mathrm{In}}_\alpha \cap \Omega$ is unbounded in Ω. This implies that $\overline{\theta}_\alpha{\restriction}\Omega$ is the ordering function of $\overline{\mathrm{In}}_\alpha \cap \Omega$.
II. As mentioned above, $\overline{\phi}_\alpha$ is the ordering function of $\overline{R}_\alpha$. So it remains to prove that $\overline{R}_\alpha = \overline{\mathrm{In}}_\alpha \cap \Omega$. First note that

(1) $\mathsf{k}^+(\alpha) \le \mathsf{k}(\alpha){+}1 = \mathsf{k}^+(\alpha{+}1)$ and (2) $\forall \gamma \in \overline{\mathrm{In}}_\alpha (\mathsf{k}(\alpha) < \gamma)$ (by Lemma 4.3c).

Then for $\gamma < \Omega$ we get: $\gamma \in \overline{R}_\alpha \Leftrightarrow \mathsf{k}(\alpha) < \gamma \in R_\alpha\ \&\ \gamma \notin R_{\alpha+1} \overset{4.6,(1)}{\Leftrightarrow} \mathsf{k}(\alpha)$ $< \gamma \in \mathrm{In}_\alpha\ \&\ (\mathsf{k}(\alpha) < \gamma \Rightarrow \gamma \notin \mathrm{In}_{\alpha+1}) \overset{(2)}{\Leftrightarrow} \gamma \in \overline{\mathrm{In}}_\alpha$.

5 The Unary Functions $\vartheta^{\mathbb{X}}$ and $\psi^{\mathbb{X}}$

As we have seen above, $\overline{\theta}_\alpha$ is the ordering function of $\overline{\mathrm{In}}_\alpha = \mathrm{In}_\alpha \setminus \mathrm{In}_{\alpha+1}$ (if $\alpha < \Lambda$). From this together with ($\theta 2ii$) and ($\theta 4$) one easily derives the following equation
(1) $\overline{\theta}\alpha 0 = \min\{\beta : C(\alpha, \beta) \cap \Omega \subseteq \beta \;\&\; \alpha \in C(\alpha, \beta)\}$
which motivates the definition of $\vartheta\alpha$ in [18]:
(2) $\vartheta\alpha := \min\{\beta : \tilde{C}(\alpha, \beta) \cap \Omega \subseteq \beta \;\&\; \alpha \in \tilde{C}(\alpha, \beta)\}$ $(\alpha < \varepsilon_{\Omega+1})$
where $\tilde{C}(\alpha, \beta)$ is the closure of $\{0, \Omega\} \cup \beta$ under $+$, $\lambda\xi.\omega^\xi$ and $\vartheta{\restriction}\alpha$.
On the other side, by Theorems 4.7, 3.9 we have:
(3) $\overline{\theta}\alpha 0 = \overline{\phi}\langle\Omega\alpha\rangle = \min\{\beta \in \mathbb{H} : \forall\xi < \Omega\alpha(K\xi < \beta \Rightarrow \overline{\phi}\langle\xi\rangle < \beta) \;\&\; K\alpha < \beta\}$.

In the light of (1)–(3) the following theorem suggests itself.

Theorem 5.1
$\alpha < \varepsilon_{\Omega+1} \;\Rightarrow\; \vartheta\alpha = \min\{\beta \in \mathbb{E} : \forall\xi < \alpha(K\xi < \beta \Rightarrow \vartheta\xi < \beta) \;\&\; K\alpha < \beta\}$.

Proof
I. From [18], Lemma 2.1 and 2.2(1)–(4) we obtain
$\vartheta\alpha \in \mathbb{E} \;\&\; \forall\xi < \alpha(E_\Omega(\xi) < \vartheta\alpha \Rightarrow \vartheta\xi < \vartheta\alpha) \;\&\; E_\Omega(\alpha) < \vartheta\alpha$.
II. Assume $\beta \in \mathbb{E} \;\&\; \forall\xi < \alpha(E_\Omega(\xi) < \beta \Rightarrow \vartheta\xi < \beta) \;\&\; E_\Omega(\alpha) < \beta$.
We will prove that $\vartheta\alpha \le \beta$.

For this let $Q := \{\gamma : E_\Omega(\gamma) \subseteq \beta\}$. Since $\beta \in \mathbb{E}$, we have $Q \subseteq \beta$. Moreover, as one easily sees, $\{0, \Omega\} \subseteq Q$ and Q is closed under $+$, $\lambda\xi.\omega^\xi$ and $\vartheta{\restriction}\alpha$. Hence $\tilde{C}(\alpha, \beta) \subseteq Q$ and thus $\tilde{C}(\alpha, \beta) \cap \Omega \subseteq Q \cap \Omega \subseteq \beta$. It remains to show that $\alpha \in \tilde{C}(\alpha, \beta)$. But this follows immediately from $E_\Omega(\alpha) \subseteq \beta \subseteq \tilde{C}(\alpha, \beta)$ and [18, 1.2(4)].

From I. and II. we get
$\vartheta\alpha = \min\{\beta \in \mathbb{E} : \forall\xi < \alpha(E_\Omega(\xi) < \beta \Rightarrow \vartheta\xi < \beta) \;\&\; E_\Omega(\alpha) < \beta\}$,
which together with Lemma 4.1 d yields the claim.

Relativization
Comparing the recursion equations for $\vartheta\alpha$ and $\overline{\phi}\langle\alpha\rangle$ in Theorems 5.1, 3.9 one notices that these equations are almost identical. The only difference is that in the equation for $\vartheta\alpha$ there appears $\mathbb{E}$ where in the equation for $\overline{\phi}\langle\alpha\rangle$ we have R_0 (i.e. $\mathbb{H}$). In order to establish the exact relationship between ϑ and $\overline{\phi}$ we go back to the definition of the Bachmann hierarchy in Sect. 2 and replace the initial clause "$R_0 := \mathbb{H} \cap \Omega$" of this definition by "$R_0 := \mathbb{X} \cap \Omega$" where here and in the sequel $\mathbb{X}$ *always denotes a subclass of* $\{1\} \cup$ *Lim such that* $\mathbb{X} \cap \Omega$ is Ω-*club*. Then the whole of Sects. 2, 3 remains valid as it stands. To make the dependency on $\mathbb{X}$ visible we write $R^{\mathbb{X}}_\alpha$, $\overline{R}^{\mathbb{X}}_\alpha$, $\phi^{\mathbb{X}}_\alpha$, $\overline{\phi}^{\mathbb{X}}_\alpha$, $\phi^{\mathbb{X}}\langle\alpha\rangle$, $\overline{\phi}^{\mathbb{X}}\langle\alpha\rangle$ instead of R_α, $\overline{R}_\alpha$, ...

Remark
Theorems 5.1, 3.9 yield $\vartheta\alpha = \overline{\phi}^{\mathbb{E}}\langle\alpha\rangle$ and $\vartheta(\Omega\alpha + \beta) = \overline{\phi}^{\mathbb{E}}_\alpha(\beta)$ $(\alpha < \varepsilon_{\Omega+1}, \beta < \Omega)$
The previous explanations motivate the following definition.

Definition
$\vartheta^{\mathbb{X}}\alpha := \min\{\beta \in \mathbb{X} : \forall \xi < \alpha(K\xi < \beta \Rightarrow \vartheta^{\mathbb{X}}\xi < \beta)\,\&\, K\alpha < \beta\}$ $(\alpha \le \Lambda)$.
Theorem 5.1 now reads: $\vartheta\alpha = \vartheta^{\mathbb{E}}\alpha$ for $\alpha < \varepsilon_{\Omega+1}$.
Further, by Theorem 3.9 we have
(ϑ0) $\vartheta^{\mathbb{X}}(\Omega\alpha + \beta) = \overline{\phi}^{\mathbb{X}}_{\alpha}(\beta)$, if $\beta < \Omega$.

Therefore, properties of $\vartheta^{\mathbb{X}}$ can be proved by deriving them from corresponding properties of $\overline{\phi}$. But for various reasons it is also advisable to work directly from the above definition.

Let us first mention that for $\beta < \Omega$ the set $\{\xi < \alpha : K\xi < \beta\}$ is countable too, and therefore $\vartheta^{\mathbb{X}}\alpha < \Omega$. Moreover, directly from the definition of $\vartheta^{\mathbb{X}}$ we obtain:
(ϑ1) $K\alpha < \vartheta^{\mathbb{X}}\alpha \in \mathbb{X}$,
(ϑ2) $\alpha_0 < \alpha \,\&\, K\alpha_0 < \vartheta^{\mathbb{X}}\alpha \;\Rightarrow\; \vartheta^{\mathbb{X}}\alpha_0 < \vartheta^{\mathbb{X}}\alpha$,
(ϑ3) $\beta \in \mathbb{X} \,\&\, K\alpha < \beta < \vartheta^{\mathbb{X}}\alpha \;\Rightarrow\; \exists \xi < \alpha(K\xi < \beta \le \vartheta^{\mathbb{X}}\xi)$,
and then
(ϑ4) $\vartheta^{\mathbb{X}}\alpha_0 = \vartheta^{\mathbb{X}}\alpha_1 \Rightarrow \alpha_0 = \alpha_1$ [from (ϑ1), (ϑ2)],
(ϑ5) $\beta \in \mathbb{X} \,\&\, \beta < \vartheta^{\mathbb{X}}\Lambda \;\Rightarrow\; \exists \xi < \Lambda(\beta = \vartheta^{\mathbb{X}}\xi)$.
Proof of (ϑ5): If $\beta \le \omega$ then $\beta \in \{\vartheta 0, \vartheta 1\}$. Otherwise we have $K\Lambda < \beta < \vartheta^{\mathbb{X}}\Lambda$, and the assertion follows by transfinite induction from (ϑ3).

Note on Klammersymbols. As we mentioned above, Sects. 2, 3 remain valid if ϕ is replaced by $\phi^{\mathbb{X}}$. So by Theorem 2.8, for $A = \begin{pmatrix} \xi_0 & \dots & \xi_n \\ \alpha_0 & \dots & \alpha_n \end{pmatrix}$ and $\alpha = \Omega^{\alpha_n}\xi_n + \dots + \Omega^{\alpha_0}\xi_0$ we have $\phi^{\mathbb{X}}_0 A = \phi^{\mathbb{X}}\langle\alpha\rangle$ from which one easily derives $\overline{\phi^{\mathbb{X}}_0} A = \overline{\phi}^{\mathbb{X}}\langle\alpha\rangle$,[2] whence (by Theorem 3.9) $\overline{\phi^{\mathbb{X}}_0} A = \vartheta^{\mathbb{X}}\alpha$. Via Theorem 5.1 this fits together with Schütte's result $\overline{\phi^{\mathbb{E}}_0} A = \vartheta\alpha$ in [21].

The Function $\psi^{\mathbb{X}}$
In [9] (actually already in [8]) the author introduced the functions $\psi_\sigma : On \to \Omega_{\sigma+1}$ and proved, via an ordinal analysis of ID_ν, that $\psi_0\varepsilon_{\Omega_\nu+1} = \theta_{\varepsilon_{\Omega_\nu+1}}(0)$. In [12] ordinal analyses of several impredicative subsystems of 2nd order arithmetic are carried out by means of the ψ_σ's. The definition of ψ_σ in [12] differs in some minor respects from that in [9]; for example, $\lambda\xi.\omega^\xi$ is a basic function in [12] but not in [9]. In [18] Rathjen and Weiermann compare their ϑ with $\psi_0{\restriction}\varepsilon_{\Omega+1}$ from [12] which they abbreviate by ψ. In Sect. 6 we will present a refinement of this comparison which is based on Schütte's definition of the Veblen function φ (below Γ_0) in terms of ψ, given in Sect. 7 of [12].

Similarly as Theorem 5.1 one can prove

$\psi\alpha = \min\{\beta \in \mathbb{E} : \forall \xi < \alpha(K\xi < \beta \Rightarrow \psi\xi < \beta)\}$, for $\alpha < \varepsilon_{\Omega+1}$.

[2] $\overline{\varphi}A$ is the 'fixed-point-free version' of φA defined in [19, Sect. 4].