Advanced Structured Materials

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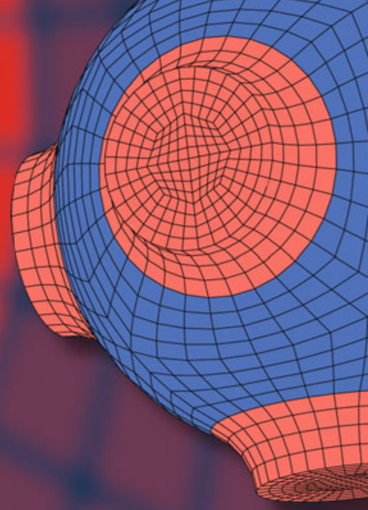

Wave Dynamics of Generalized Continua

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In Memoriam—Prof. Alexander G. Bagdoev

Ph.D., DSci., Professor, Corresponding Member of the Academy of Sciences of Armeni

Co-author of the monograph, our colleague Professor Alexander Bagdoev died on March 2, 2013, before he reached 4 months before his 80th anniversary.

He was a word-class scientist in the field of nonlinear wave processes in continuum mechanics, excellent teacher, friendly and sympathetic person.

Alexander Georgievich Bagdoev was born July 9, 1933 in Tbilisi (Georgia). After his family moved to Yerevan (Armenia) in 1950, he graduated from high school with a gold medal. In the same year, he entered the mechanics and mathematics faculty of Moscow State University named after
M.V. Lomonosov, where for outstanding studies he was awarded the Stalin scholarship.

After graduating from Moscow University, he entered for postgraduate education at the department of wave and gas dynamics and in 1959 under the supervision of Professor Arthur Soghomonyan, he defended his thesis devoted to the solution of problems of penetration of solids or shock waves in a compressible fluid.

After defending his Ph.D. dissertation A.G. Bagdoev returned to Yerevan, where he took a position in the Institute of Mechanics of Academy of Sciences of Armenia, where he worked until the end of his days.

In 1972 at Moscow state University, A.G. Bagdoev defended his doctoral thesis devoted to the problems of determining the peculiarities of the fronts of linear and nonlinear waves. In 1993, he was awarded the academic title Professor, and in 2000, he was elected a Corresponding Member of Academy of Sciences of Armenia.

The area of expertise of A.G. Bagdoev is quite broad and applies to a variety of problems of mechanics of deformable solids, aerohydromechanics. He is the author of three books and over 350 articles.

Great attention was paid to the study of nonlinear quasi-monochromatic modulation waves in various mechanical and optical media. From these nonlinear evolution equations that take into account the effects of dissipation, were derived nonlinear Schrödinger equations describing the behavior of the amplitude of the first harmonic with complex coefficients for which were analytically solved the problems of narrow beams. In the study of non-linear modulation waves in magnetoelastic plates, he proposed and developed a spatial approach to the determination of natural frequencies, investigated the modulation stability and sustainability of soliton solutions of evolution equations. A number of linear time-dependent problems of gas dynamics and dynamic elasticity theory, including the boundary, as well as nonlinear problem near a caustic were solved.

One of the major achievements made by A.G. Bagdoev in the development of mechanics is the generalization of the Poincare-Lighthill-Go method for the two-dimensional wave problems of diffraction and caustic.

In recent years, A.G. Bagdoev further expanded the range of his scientific interests, trying to apply the his experience of theoretical studies to deterministic and stochastic processes in economics, physics, sociology, biology, seismology, and, thus, bring together different areas of natural and humanitarian sciences. The results of his research in the field of practical philosophy and ethics have been published in popular and scientific literature.

A.G. Bagdoev was notable for quite active scientific, organizational, and teaching activities. He was the initiator and main organizer of the international conferences “Problems of dynamics of interaction of deformable media”, which for over 30 years were successfully carried out in the city of Goris (Armenia). Among his students, there are three doctors of sciences, 14 PhDs.

We will always remember Alexander Georgievich Bagdoev. We are proud that we were fortunate to work with him.

Nizhny Novgorod, Russia — Vladimir I. Erofeyev
Yerevan, Armenia — Ashot V. Shekoyan

Preface

Wave processes are observed in any field, where matter moves: in electrodynamics, plasma physics, optics, acoustics, fluid dynamics, complex two-phase media such as "gas-drip system", soils of various types, solids with pores filled with liquid, etc.

During wave propagation in various continuous media, the physical properties of matter play a very important role. The most important properties, which are present in majority of cases, are nonlinearity, dissipation, dispersion, diffraction, and heterogeneity.

Linear and nonlinear wave processes are also of special interest for their applications in various practical problems.

It is interesting to note that despite the difference in the physical nature of wave processes (acoustic, electromagnetic), they are described by similar equations. One of the powerful methods of mathematical study (especially, for nonlinear waves) is the method of evolution equation (or short-wavelengths) and the method of non-linear modulation equation, the latter equation is often referred to as a non-linear Schrödinger equation. There are two questions in this aspect: the first one is how to derive evolution equations from various complex systems of equations describing wave motion in a medium and the nature of the waves; and the second one is how to examine the obtained equations that in each case have different types of modification (different coefficients, order of equations, etc.).

For investigation of wave processes it is important to identify the laws of linear and nonlinear dispersion, to reveal types of modulation (amplitude, frequency, etc.), to study problems of stability (instability) of modulation and other types of waves, in particular, solitons. If wave beam propagation is studied, the important problems facing the researchers are focusing problems: it is necessary to determine the distance of focus formation, focal spots, the existence of self-focusing (defocusing), laws of variation of the beam radius in space and time.

In this monograph the original results are used and developed, which have been obtained by the authors in their research activity at the Mechanical Engineering Research Institute of the Russian Academy of Sciences (Nizhny Novgorod, Russia) and at the Institute of Mechanics of the National Academy of Sciences of Armenia

(Yerevan, Armenia), as well as in their joint research. Study of the self-modulation effects of elastic waves in media with complex physical and mechanical properties (interaction of deformation fields with electromagnetic fields, fields of defects, etc.) is also of great interest.

Features of propagation and interaction of nonlinear strain waves in mechanical systems are being intensively investigated for the last three decades in many countries. This is explained, as already mentioned, by numerous physical, technical and technological applications of such systems. Some monographs on nonlinear waves in continuous media have been published (e.g., [24, 65, 83, 112, 133, 165, 166, 192, 193, 203, 214, 225, 237, 250, 281, 327, 331, 363, 378, 392, 400]).

This monograph is devoted, in the first place, to the study of wave processes in media, where interaction of deformation fields with fields of the physical nature is significant. The content of this monograph does not duplicate the content of the existing books, but is intended to supplement them, finding its "niche" in this research field.

The book is based on [9, 17, 18, 25–61, 108, 113, 115–128, 267–276, 286–300, 330–340, 388, 389, 398, 406, 407]. In one way or another, we could represent the results of our colleagues—"wave-researchers" belonging to different scientific schools of the former Soviet Union [2, 3, 6, 8, 10–14, 20–23, 62–65, 67–69, 71, 75–77, 80–84, 87–89, 92, 95, 98–101, 103, 107, 130–135, 143, 145, 147, 152–155, 162–164, 176–178, 181–185, 192–194, 196–202, 204, 208–210, 213, 215, 216, 218, 220, 221, 224, 227, 228, 230, 231, 233, 234, 239, 245, 246, 252, 277, 278, 282, 349, 361, 383, 402].

One of the authors (Erofeyev V.I.) recieved support from the Russian Science Foundation for the work (grant No 14-19-01637).

Contents

Chapter 1
Waves in a Viscous Solid with Cavities

1.1 Introduction

In the nature there are a lot of substances containing cavities, moreover, there are also artificially created materials, which are used in various devices, for example, in nanoengineering. In this connection, theoretical and practical interest arises to investigate physical processes in such media. In particular, it is possible to use results of studying of wave processes in them for nondestructive testing of properties of such media.

At present, propagation of waves in a liquid with gas bubbles [38, 202, 203, 281] has been enough well studied. A following physical model is used in these works: the acoustic wave travels in a liquid containing bubbles, under its influence the bubbles start to fluctuate. The equations of hydrodynamics and of fluctuation of bubbles are employed for theoretical research of such a process.

The analogous physical situation is observed, when the wave propagates in a solid with cavities. Hence, it is possible to use ideas of hydrodynamics. A similar attempt has been done in the book [203], where the equation of the theory of elasticity and the equation of fluctuations of cavities are derived. Only one-dimensional approach was considered there. In the chapter at issue, development of this theory in three-dimensional statement will be given using the mathematical methods developed by A.G. Bagdoev and A.V. Shekoyan [41].

This chapter has been written on the basis of the works [30, 106, 112–114, 216, 217].

1.2 Statement of the Problem and the Basic Equations

We shall consider a semi-infinite isotropic viscous medium (Voigt model) with cavities, in which the waves with final amplitude (i.e. nonlinear waves with account of the geometrical, physical and cavity nonlinearities) propagate. The matrix (the

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basic medium) is considered to be homogeneous. The distance between cavities, l, is assumed to be much more than the radius of cavities R_0 ($l \gg R_0$), but much less than the wavelength $\lambda\,(\lambda \ll R_0)$. It is supposed that the pressure in cavities is negligible and the quasilongitudinal wave propagates in the medium, so it is possible to assume that pressure upon a cavity is caused by the longitudinal stress $\sigma_{33} = (\lambda + 2\mu)\frac{\partial u_3}{\partial x_3} - z'\left(\lambda + q_j^u\right)$, where $z' = NV'$, N is a number of cavities in a volume unit, V' is the cavity volume, $V' = V_0 + V$, V_0 is the initial volume of a cavity, V is the volume of a cavity perturbed by a wave; and $\mu \ll \lambda$ is also supposed. Under the specified assumptions, on the base of works [41, 203], propagation of a quasilongitudinal nonlinear wave in terms of Lagrangian coordinates is described by the following equations:

$$\rho_0 \frac{\partial^2 u_{1,2}}{\partial t^2} = \mu \frac{\partial^2 u_{1,2}}{\partial x_3^2} + (\lambda + \mu)\frac{\partial^2 u_3}{\partial x_{1,2}\partial x_3}, \tag{1.1}$$

$$\begin{aligned} \rho_0 \frac{\partial^2 u_3}{\partial t^2} = {} & \mu\Delta_\perp u_3 + (\lambda + 2\mu)\frac{\partial^2 u_3}{\partial x_3^2} + (\lambda+\mu)\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) \\ & - N(\lambda + 2\mu)\frac{\partial V}{\partial x_3} + b\frac{\partial^3 u_3}{\partial x_3^2 \partial t} + P\frac{\partial u_3^2}{\partial x_3^2}\frac{\partial u_3}{\partial x_3}, \end{aligned} \tag{1.2}$$

$$\ddot{V} + \omega_0^2 V - \frac{R_0}{c_0}\dddot{V} - GV^2 + \beta_1\left(2V\ddot{V} + \dot{V}^2\right) = (2\mu + \lambda)\frac{4\pi R_0}{\rho_0}\left(\frac{\partial u_3}{\partial x_3} - NV\right), \tag{1.3}$$

where ρ_0 is initial density of the matrix, $\omega_0^2 = \frac{4\mu}{\rho_0 R_0}$ is a square of a resonant frequency, $c_0^2 = \frac{\lambda + 2\mu}{\rho_0}$, $G = (16\pi)^{-1}(9 + 2b_1)R_0^{-3}\omega_0^2$, $\beta_1 = \frac{1}{8\pi R_0^3}$, $P = (4\mu + 3\lambda + 2A + 6B + 2C)$, A, B, C are Landau nonlinear factors. Coordinates x_1 and x_2 are chosen in the plane tangent to the unperturbed mode, and x_3 is directed along the wave propagation. It is supposed that $u_1 = u_2 = 0$ in the plane $x_3 = 0$, i.e. the longitudinal wave is major and the weak transverse waves appear during the longitudinal wave propagation. The transverse waves, as consequence, are weak, therefore their equations are linearized.

1.3 Derivation of the Evolution Equation

First of all, we will simplify Eq. (1.3), supposing that the characteristic wave frequency, α, is much less than the resonant frequency ($\alpha \ll \omega_0$). Then, the nonlinear term with factor β_1 in (1.3) is negligible, and the main term of Eq. (1.3) is

$$V = \frac{F}{D}\frac{\partial u_3}{\partial x_3}, \tag{1.4}$$

where $F = 4\pi R_0(\lambda + 2\mu)\rho_0^{-1}, D = \omega_0^2 + FN$. Substituting (1.4) into small terms of Eq. (1.3), it is possible to receive an improved equation of a cavity

$$V = \frac{F}{D}\frac{\partial u_3}{\partial x_3} - \frac{F}{D^2}\frac{\partial^3 u_3}{\partial x_3 \partial t^2} + \frac{R_0 F}{c_0 D^2}\frac{\partial^4 u_3}{\partial x_3 \partial t^3} + \frac{GF^3}{D^3}\left(\frac{\partial u_3}{\partial x_3}\right)^2. \tag{1.5}$$

Taking into account (1.4) and (1.5), one can exclude V from (1.2), then an equation will yield:

$$\begin{aligned}\rho_0\frac{\partial^2 u_3}{\partial t^2} &= (\lambda+\mu)\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) + (\lambda + 2\mu)\left(1 - \frac{NF}{D}\right)\frac{\partial^2 u_3}{\partial x_3^2}\\ &+ \mu\Delta_\perp u_3 + (\lambda+2\mu)\frac{NF}{D^2}\frac{\partial^4 u_3}{\partial x_3^2\partial t^2} - \frac{R_0 NF}{c_0^2 D^2}(\lambda+2\mu)\frac{\partial^5 u_3}{\partial x_3^2 \partial t^3}\\ &+ \left[P - 2CF^2N(\lambda+2\mu)D^{-3}\right]\frac{\partial^2 u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3} + b\frac{\partial^3 u_3}{\partial x_3^2 \partial t}.\end{aligned} \tag{1.6}$$

So, Eq. (1.6) should be solved together with Eq. (1.1). We will pass to a new coordinate $\tau_1(l_1 - x_3)c_0^{-1} - t = \tau_1' - t$. As the layer with a thickness l_1 will be further considered, it is convenient to choose τ_1 in the form mentioned above, in a semi-infinite case $l_1 = 0$ and the x_3-axis is directed opposite to wave propagation. In the main order after transition to variable τ_1 it is possible to receive value of a wave velocity, c_1, with account of presence of cavities:

$$c_1^2 = c_0^2(1 - NFD^{-1}).$$

Entering a new function $\psi_1 = \frac{\partial u_3}{\partial \tau_1}$ characterizing the velocity of particles of a medium (matrix) in terms of variables x_1, x_2, x_3, and τ_1, after an exclusion of the transverse displacements u_1 and u_2 due to (1.1) with account of the accepted orders [225], the following evolution equation yields

$$\frac{\partial^2 \psi_1}{\partial \tau_1 \partial x_3} + L\Delta_\perp \psi_1 = \alpha_1 \frac{\partial}{\partial \tau_1}\left(\psi_1 \frac{\partial \psi_1}{\partial \tau_1}\right) + \delta\frac{\partial^3 \psi_1}{\partial \tau_1^3} + \beta\frac{\partial^4 \psi_1}{\partial \tau_1^4} + \gamma\frac{\partial^5 \psi_1}{\partial \tau_1^5}, \tag{1.7}$$

where

$$\begin{aligned} L &= -[\mu + (\lambda+\mu)^2(\rho_0 c_1^2 - \mu)^{-1}]M_1^{-1},\\ M_1 &= 2c_1^{-1}(1 - NFD^{-1})(\lambda+2\mu) = 2c_1\rho_0,\\ \alpha_1 &= M_1^{-1}c_1^{-3}[\rho - 2GF^2N(\lambda+2\mu)D^{-3}],\\ \delta &= b\, c_1^{-2}M^{-1},\\ \beta &= FN(\lambda+2\mu)c_1^{-2}D^{-2}M^{-1},\\ \gamma &= R_0FN(\lambda+2\mu)c_1^{-2}c_0^{-2}D^{-2}M^{-1}.\end{aligned} \tag{1.8}$$

As it is visible from (1.8), the term with factor β is concerned with dispersion and is caused by cavities, whereas δ and γ provide dissipation (δ is caused by viscosity and γ is related to cavities), thus, $\rho = \gamma = 0$ when $R_0 = 0$.

1.4 The Soliton Solution of the Evolution Equation of the Fifth Order

In Eq. (1.7) we shall pass to a new function $U = \frac{\alpha_1}{G}\psi_1$, then the obtained equation will be the same as (1.7), but U will be instead of ψ_1, and −6 instead of α_1. If in the obtained equation δ and γ are supposed to be equal to zero and $\beta = -\beta_2$, then, Kadomtsev-Petviashvili equation [152] will yield with accuracy up to coefficients. This equation has a soliton solution [249] in the following form:

$$U_0 = \frac{C}{2}\operatorname{sech}^2\left(\frac{\sqrt{C}\xi_1}{2a\sqrt{\beta_2}}\right), \tag{1.9}$$

where $\xi_1 = a\tau_1 + b_2x_1 + d_2x_2 - kx_3$, $C = \left[ak - L\left(b_2^2 + d_2^2\right)\right]a^{-2}$, and $C \geq 0$, $a > 0$, b_2 and d_2 show incline of the plane of the soliton front (ξ_1 = const) to the x_3-axis. The normal soliton velocity has the form:

$$V_c^2 = \frac{a^2}{\left(ac_1^{-1} - k\right)^2 + b_2^2 + d_2^2}.$$

Constants a and k are certain characteristic frequency and wavenumber of the wave process. We shall seek a solution of the equation for U in the form:

$$U = U_n(\xi_1) \tag{1.10}$$

After substitution of (1.10) into the equation for U and twice integration with account that U tends to zero, when ξ_1 tends to infinity, the following ordinary differential equation will yield:

$$a^2\beta_2\frac{d^2U}{d\xi_1^2} + 3U_n^2 - CU_n = \delta a\frac{dU_n}{d\xi_1} + a^3\gamma\frac{d^3U_n}{d\xi_1^3}, \tag{1.11}$$

For non-zero coefficients δ and γ, which are small in comparison with β (this fact means smallness of dissipation), a solution of Eq. (1.11) can be found by the method of slowly varying amplitude [91, 279]. Then a solution should be searched in the form

$$U_n = U_0(\xi_1)\,[1 + T_3(\xi_1)], \tag{1.12}$$

and inequalities

$$T_3 \ll 1, \frac{d^2T_3}{d\xi_1^2} \ll \frac{dT_3}{d\xi_1} \ll T_3 \tag{1.13}$$

must be valid.

Inequalities (1.13) mean that because of small dissipation the soliton shape varies a little and slowly, and function $T_3(\xi_1)$ is small. Substituting (1.12) into Eq. (1.11) and taking into account inequalities (1.13), one can obtain for T_3:

$$T_3 = \frac{a\delta}{3U_0}\frac{dU_0}{d\xi_1} + \gamma\frac{a^3}{3U_0}\frac{d^3U_0}{d\xi_1^3}. \tag{1.14}$$

After substitution of (1.9) into (1.14) the expression for function T_3 will take on the form:

$$T_3 = \frac{1}{3(C\beta_2)^{1/2}}\left[\frac{G\gamma C}{\beta_2}\mathrm{th}\left(\frac{\sqrt{C}\beta_2^{-1/2}}{2a}\xi_1\right) - \left(\delta + \frac{\gamma C}{\beta_2}\right)\mathrm{sh}\left(2\frac{\sqrt{C}}{2a\sqrt{\beta_2}}\xi_1\right)\right]. \tag{1.15}$$

In expression (1.15) T_3 tends to infinity, if ξ_1 tends to infinity, i.e. inequalities (1.13) are not satisfied. Therefore the solution (1.15) makes sense only near a soliton top, then for small ξ_1 the solution (1.15) can be rewritten in the form

$$T_3 = \xi_1\left[4\gamma C\beta_2^{-1} - 2\delta\right](6a\beta_2)^{-1} = T_2(6a\beta_2)^{-1}\xi_1. \tag{1.16}$$

From (1.16) follows that $T_3 > 0$, if ξ_1 and T_2 have the same signs; and $T_3 < 0$, if ξ_1 and T_2 have the opposite signs. Distortion of the soliton shape on account of dissipation is qualitatively shown in Figs. 1.1 and 1.2, where the dotted curve corresponds to function (1.9) at $T_3 = 0$, and the continuous one—to function U_n. It is necessary to note that the case $T_2 = 0$ is possible, which means that the same soliton can propagate in a dissipative medium as in a non-dissipative one.

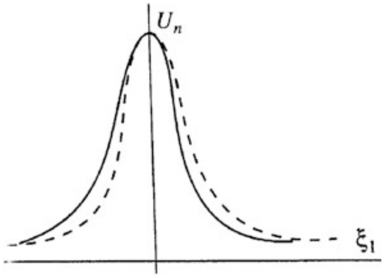

Fig. 1.1 The soliton profile for $T_2 > 0$

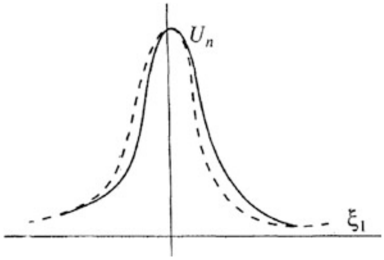

Fig. 1.2 The soliton profile for $T_2 < 0$

1.5 Derivation of the Modulation Equation for Diffraction and One-Dimensional Problems in the Case of Quasimonochromatic Waves

As both stationary and non-stationary problems are interesting to us, after substitution of $\frac{\partial}{\partial x_3} = c_1^{-1}\frac{\partial}{\partial t}$ into (1.7), we shall receive:

$$\frac{\partial^2\psi_1}{\partial t\partial\tau_1} + L\Delta_\perp\psi_1 = \alpha_1'\frac{\partial}{\partial\tau_1}\left(\psi_1\frac{\partial\psi_1}{\partial\tau_1}\right) + \delta'\frac{\partial^3\psi_1}{\partial\tau_1^3} + \beta'\frac{\partial^4\psi_1}{\partial\tau_1^4} + \gamma\frac{\partial^5\psi_1}{\partial\tau_1^5}, \tag{1.17}$$

where the factors with primes are derived from factors (1.8) due to multiplication by $-c_1$. As in the medium there are dispersion and dissipation, it is possible to search for the solution of Eq. (1.17) in the form of a quasimonochromatic wave

$$\begin{aligned}\psi_1 = \frac{1}{2}\{&A_1(\tau_1', x_1, x_2, t)\exp\left[i\alpha\tau_1 - (\nu + i\omega)\tau_1'\right] \\ &+ B_1(\tau_1', x_1, x_2, t)\exp\left[2i\alpha\tau_1 - 2(\nu + i\omega)\tau_1'\right] + C_1(\tau_1', x_1, x_2, t) + c.c.\},\end{aligned} \tag{1.18}$$

where A_1 and B_1 are the amplitudes, accordingly, of the first and second harmonics, C_1 is an absolute term, α is a carrying frequency, ω is a modulation frequency, and ν is an absorption factor. A monochromatic fluctuation is set on a border of the medium for $\tau_1 = 0$. In works [28, 42] it has been shown that dispersion and dissipation remain in exponents containing in solutions similar to (1.18), if $\tau_1 = 0$. This fact isn't so natural, though the definitive equations of modulation in the basic orders, as shown in calculations of article [30], are the same.

Substituting (1.18) into (1.17) for the highest orders, we will receive the following dispersion relations:

$$\omega = \alpha^3\beta', \nu = \alpha^2\delta' - \alpha^4\gamma'. \tag{1.19}$$

In the next approximations two problems are distinguished: a diffraction problem and a one-dimensional problem, for which the different orders of quantities take place. In both cases it is possible to obtain the following modulation equation for the first harmonics

$$\begin{aligned} & i\alpha\left(\frac{\partial A_1}{\partial t} + \frac{d\Omega}{d\alpha}\frac{\partial A_1}{\partial \tau_1'}\right) + \frac{\alpha}{2}\frac{d^2\Omega}{d\alpha^2}\frac{\partial^2 A_1}{\partial \tau_1'^2} + L'\Delta_\perp A_1 \\ & = (i\alpha - 3\nu - i\omega)^2\frac{\alpha'}{2}A_1^* B_1 \exp(-2\nu\tau_1') + \alpha'(i\alpha - \nu - i\omega)^2 C_1 A_1, \end{aligned} \tag{1.20}$$

where $\Omega = \alpha + \omega - i\nu$ is a complex linear frequency. In the one-dimensional problem the coefficient at $\frac{\partial^2 A_1}{\partial \tau_1'^2}$ is calculated using the equation

$$\frac{\partial}{\partial t} + \frac{d\Omega}{\partial \alpha}\frac{\partial}{\partial \tau_1'} = 0, \tag{1.21}$$

which has been obtained from the main term of Eq. (1.20).

The equation for the amplitude B_1 of the second harmonic after rejection of derivatives (it is possible, if $\omega\tau_1' \gg 1$) will yield for both problems in the form:

$$4(3\omega - i\nu - 6i\alpha^4\gamma')B_1 = \alpha'\alpha A_1^2. \tag{1.22}$$

In the diffraction problem the absolute term C_1 has an order ε^3 or $\nu\varepsilon^3$, where ε is a certain small parameter characterizing an order of ψ_1, and $\alpha \sim \varepsilon^{-1}$, $x_{1,2} \sim \varepsilon^{1/2}$, then the term with C_1 in (1.20) can be neglected.

If to exclude, in accordance with (1.21), the derivative with respect to t in the term $\frac{\partial^2 C_1}{\partial t \partial \tau_1'}$, it is possible to obtain an equation for C_1 in the one-dimensional problem:

$$\left(\frac{2i\nu}{\alpha} - \frac{3\omega}{\alpha} - 2i\alpha^3\gamma'\right)\frac{\partial^2 C_1}{\partial \tau_1'^2} = -\frac{\alpha}{2}\left(\nu\frac{\partial |A_1|^2}{\partial \tau_1'} - \frac{\partial^2 |A_1|^2}{\partial \tau_1'^2}\right)\exp(-2\nu\tau_1'). \tag{1.23}$$

Two cases are considered for interpretation of Eq. (1.23):

(1) $\nu\tau'_1 \gg 1$, i.e. $\exp(-2\nu\tau'_1) \approx 0$ and, according to (1.23), $C_1 \approx 0$. Hence, similarly to the diffraction problem, the absolute term doesn't give contribution into Eq. (1.20);

(2) $\nu\tau'_1 \ll 1$, that means $\nu\frac{\partial|A_1|^2}{\partial\tau'_1} \ll \frac{\partial^2|A_1|^2}{\partial\tau'^2_1}$ and from (1.23) follows

$$4\left(-\frac{3\omega}{\alpha}+\frac{2\,i\nu}{\alpha}-2\,i\alpha^2\gamma'\right)C_1=\alpha|A_1|^2. \tag{1.24}$$

In the one-dimensional problem, a linear equation appears from (1.20) in the first case. In the second case, excluding B_1 and C_1 from Eq. (1.20) by means of (1.22) and (1.24) and taking into account $\alpha \gg \omega$, ν, one can obtain the following equation:

$$\begin{aligned} & i\left(\frac{\partial A_1}{\partial t}+\frac{d\Omega}{d\alpha}\frac{\partial A_1}{\partial\tau'_1}\right)+\frac{1}{2}\frac{d^2\Omega}{d\alpha^2}\frac{\partial^2 A_1}{\partial\tau'^2_1} \\ & =\frac{\alpha'^2}{2}\left[2(-3\omega+i\nu-6i\alpha^4\gamma')^{-1}-(2i\nu-3\omega-2i\alpha^4\gamma')^{-1}\right]A_1|A_1|^2. \end{aligned} \tag{1.25}$$

It should be noted that, in contrast to the diffraction problem, contribution of C_1 into the nonlinear term is substantial.

In the diffraction problem from (1.20), taking into account smallness of $\frac{\partial^2 A_1}{\partial\tau'^2_1}$ and omitting C_1, one can find

$$i\alpha\left(\frac{\partial A_1}{\partial t}+\frac{d\Omega}{d\alpha}\frac{\partial A_1}{\partial\tau'_1}\right)+L'\Delta_\perp A_1=\frac{\alpha'^2\alpha|A_1|^2A_1\exp(-2\nu\tau'_1)}{8(i\nu-3\omega-6i\alpha^4\gamma')}. \tag{1.26}$$

1.6 Problem Statement about Wave Fields in the Case of a Layer

First of all, we consider a problem about acoustic waves in a resonator similarly to the optical problem about waves in a non-dissipative interferometer [184]. In this case it is supposed that there are two acoustic mirrors located symmetrically with respect to the plane $x_3 = 0$. In fact, these mirrors are surfaces of a constant phase for the waves propagating to the right and to the left sides, each of which satisfies a boundary condition on the appropriate mirror. In such a statement $u_3 = 0$ at $x_3 = 0$ due to symmetry. This problem corresponds to an acoustic interferometer [184], in which the left mirror is a source of oscillations, and there is a flat rigid reflector on the right. In this case the specified statement is reduced to the previous problem, in which there are two waves. Similarly, in the case when there is a layer, one end surface of which is free from stresses and oscillations are set at the other one, it is possible to consider that there are two waves propagating towards to each other. In this case, the wave running to the right satisfies the condition on a radiator, and the wave reflected from the free surface together with the falling wave satisfies the condition on the free surface. This conclusion follows from comparison of propagation of an elastic one-dimensional linear wave in a layer with the simple case. It should be noted that $\frac{\partial u_3}{\partial x_3} = 0$ at $x_3 = 0$ and $u_3 = \exp(-i\alpha t)$ at $x_3 = l_1$. The solution of the wave equation $\frac{\partial^2 u_3}{\partial t^2} = c_1^{-2} \frac{\partial^2 u_3}{\partial x_3^2}$ under these boundary conditions in the absence of initial conditions looks like

$$u_3 = \frac{1}{2}\left[\cos\frac{\alpha l_1}{c_1}\right]^{-1} \exp\left[\exp\left(i\frac{\alpha}{c_1}x_3 - i\alpha t\right) + \exp\left(-i\frac{\alpha}{c_1}x_3 - i\alpha t\right)\right]. \quad (1.27)$$

Thus, for a monochromatic wave the solution in the resonator represents two waves running towards to each other. Similarly, in the case of quasimonochromatic waves of type (1.18) there are two waves propagating towards to each other and their amplitudes will be slowly varying functions on account of dispersion, dissipation, nonlinearity and diffraction.

The same conclusion can be made for the resonator, when $u_3\ (x_3 = 0) = 0$. In this case it is necessary to change a sign before the second term in the formula for u_3 and to divide by $i \sin\frac{\alpha}{c_1} l_1$ instead of cosine.

Formula (1.27) for u_3 can be also represented in the form:

$$u_3 = \left\{\exp\left[i(x_3 - l_1)\frac{\alpha}{c_1} - i\alpha t\right] + \exp\left[i(x_3 + l_1)\frac{\alpha}{c_1} - i\alpha t\right]\right\} \sum_{n=0}^{\infty} (-1)^n \exp\left(2i\frac{\alpha n l_1}{c_2}\right). \quad (1.28)$$

For the high-frequency waves ($\alpha \gg c_1 l_1^{-1}$), only the terms in the brace give the contribution to asymptotic of a solution. These terms correspond to the falling wave and to the wave reflected from the plane, i.e. eikonals τ_1 and τ_2, where $\tau_2 = (x_3 + l_1)c_1^{-1} - t$.

In this case, as follows from (1.28), the boundary condition at $x_3 = l_1$ satisfies only the first terms in square brackets, and the remaining terms will cancel in pairs. The condition at the free surface is automatically satisfied by the first two terms. Thus, the boundary conditions are satisfied by the first two terms in (1.28), which can be taken as the waves propagating to the left and to the right, and the remaining terms (up to sign for the free boundary problem and precisely for the problem of the reflector) periodically repeat the first two terms of (1.28) and can be included in these two waves, that leads to the problem statement mentioned above.

In works [28, 30, 42, 153] the solution of the quasi-linear systems of equations is given for high-frequency asymptotics in the form of two functions, each of which depends on its eikonal. Under the assumption that average values of the unknown functions in their eikonals are equal to zero in the major orders of infinitesimal, the set of equations describing the waves propagating to the right (once primed, the eikonal τ_1) and to the left (double primed, the eikonal τ_2) break up into two independent nonlinear equations. The values of the functions, averaged on eikonals, are equal to zero. This condition is valid both for diffraction problems, where $c_{1,2}$ are negligible, and for one-dimensional problems, where $c_1 = c_2$ (c_i is a constant term of the reflected wave). Equation (1.7) will be for the falling wave. For the reflected waves it is necessary in Eq. (1.7) to change τ_1 by τ_2 and ψ_1 by $\psi_2 = \frac{\partial u_3}{\partial \tau_2}$. Equations (1.18)–(1.26) should be written similarly—replacing subscript "1" with "2" in the amplitudes and eikonals.

1.7 A Diffraction Problem for Narrow Beams

Considering $\frac{d\Omega}{d\alpha} = 1$ in Eq. (1.26) and $\frac{\partial A_1}{\partial t} = 0$ for the stationary problem, one can obtain equation

$$i\,\alpha\frac{\partial A_1}{\partial \tau_1} + L\Delta_{\perp}A_1 = (\chi_1 + \chi_2)\,|\,A_1|^2 A_1, \tag{1.29}$$

where

$$\xi = \frac{\alpha'^2\alpha}{8}\left[9\omega^2 + (\nu - 6\alpha^4\gamma')^2\right]^{-1}\exp(-2\nu\tau_1'),$$
$$\chi_1 = 3\omega\xi,\ \chi_2 = \left(\nu - 6\alpha^4\gamma'\right)\xi.$$

In the case of resonator the same equation is derived for the falling wave A_1', and for the reflected wave in (1.29) A_1 should be changed by A_1'' and τ_1' by $\tau_2' = (x_3 + l_1)\,c_1^{-1}$. Therefore, we will further write solutions for Eq. (1.29).

Taking

$$A_1 = a_1\exp(i\omega_1), \tag{1.30}$$

where φ_1 is an excited eikonal and a_1 is a real amplitude, we shall substitute (1.30) into Eq. (1.29), separate imaginary and real parts, pass to cylindrical coordinates for an axisymmetric problem, we will receive the equation for a_1 and φ. Substituting (1.30) into Eq. (1.29), separating imaginary and real parts, passing to cylindrical coordinates for an axisymmetric problem, we will receive the equation for a_1 and φ. They have the form

$$-\alpha a_1\frac{\partial \phi_1}{\partial \tau_1'} + L\frac{\partial^2 a_1}{\partial r^2} - a_1\left(\frac{\partial \phi_1}{\partial r}\right)^2 + \frac{L}{r}\frac{\partial a_1}{\partial r} = \chi_1 a_1^3, \tag{1.31}$$

$$\alpha \frac{\partial a_1}{\partial \tau_1} + 2L \frac{\partial a_1}{\partial r}\frac{\partial \omega_1}{\partial r} + La_1 \frac{\partial^2 \omega_1}{\partial r^2} + L \frac{a_1}{r}\frac{\partial \omega_1}{\partial r} = \chi_2 a_1^3. \tag{1.32}$$

In Eqs. (1.31) and (1.32) r is a cylindrical radial coordinate. We shall seek a solution of these equations in the form

$$\begin{aligned} a_1 &= b_1 f_1^{-1} \exp\left[-\frac{r^2}{2}(r_1 f_1)^{-2}\right], \\ \varphi_1 &= \sigma_1(\tau_1) + \frac{r^2}{2} R_1^{-1}(\tau_1), \end{aligned} \tag{1.33}$$

where f_1 is a dimensionless width of the beam, σ_1 is a wave phase incursion on the axis of the beam, $\alpha R_1 c_1^{-1}$ is a variable radius of curvature of the wave front, b_1 and r_1 are the amplitude and radius of the beam on border $x_3 = l_1$. Substituting (1.33) into the equations for a_1 and φ_1, we will receive, by the ordinary way [28, 41], the following equations

$$R_1^{-1} = \frac{\alpha}{2Lf_1}\frac{d f_1}{d \tau_1'} + \frac{\chi_2 b_1^2}{2Lf_1^2} \tag{1.34}$$

$$\frac{d \sigma_1}{d \tau_1'} = -2\left(\alpha L r_1^2 f_1^2\right)^{-1} - \chi_1 b_1^2 (\alpha f_1^2)^{-1} = G f_1^{-2} \tag{1.35}$$

$$\frac{d^2 f_1}{d\tau_1'^2} = \frac{M}{f_1^3} + \frac{\chi_2 \nu b_1^2}{\alpha L f_1} \tag{1.36}$$

where

$$M = \alpha^{-2}\left[L^2 r_1^{-4} + 2\chi_1 b_1^2 L r_1^{-2} - \chi_2^2 b_1^4\right]. \tag{1.37}$$

For the reflected wave, Eqs. (1.34)–(1.37) are valid, where subscript “1” should be replaced by “2” for R_1, σ_1, f_1, b_1, and r_1. The other quantities must be with primes.

1.8 Boundary Conditions

As a statement of problems for an interferometer and free border are similar, we will start from the free border. For mechanical quantities it is necessary to set conditions at the end surfaces of a layer ($x_3 = 0$ and $x_3 = l_1$). The first of them in a plane ($x_3 = l_1$) or $\tau_1' = 0$ relates to the falling wave. It is supposed that in this plane the beam with a Gaussian profile is given and following conditions are satisfied:

$$f_1(0) = 1, \frac{d f_1(0)}{d \tau_1'} = F, \tau_1(0) = 0, \; F \; = \frac{2L}{\alpha}\left[R_1^{-1}(0) - \frac{\chi_2}{2} b_1^2 L\right]. \tag{1.38}$$

We shall solve Eqs. (1.34)–(1.36) with boundary conditions (1.38). For the reflected wave, boundary conditions are set in the plane $x_3 = 0$, in which it is supposed that $\sigma_{32} = \sigma_{31} = \sigma_{33} = 0$. In the highest order these equations are split, as we study only a beam of quasilongitudinal waves. The condition $\sigma_{33} = 0$ gives in the highest order

$$\frac{\partial u_3}{\partial x_3} = 0. \tag{1.39}$$

In the highest order, condition $\sigma_{32} = \sigma_{31} = 0$ is automatically satisfied. Substituting into (1.39) $u_3 = u_3' + u_3''$, where u_3' corresponds to the falling wave and u_3''—to the reflected one [28, 35, 42], passing in expressions $\psi_1 = -\frac{\partial u_3'}{\partial \tau_1}$ and $\psi_2 = -\frac{\partial u_3''}{\partial \tau_2}$ from coordinates τ_1 and τ_2 to x_3, taking into account $\frac{\partial}{\partial x_3} = \pm c_1^{-1} \frac{\partial}{\partial \tau_{1,2}}$, we shall obtain the following boundary condition for $x_3 = 0$:

$$\psi_1 = -\psi_2. \tag{1.40}$$

Substituting solution (1.18) for $\tau_{1,2}' = \frac{l_1}{c_1}$ into (1.40) and taking into account only the first harmonics, one can obtain $A_1 = -A_2$, where A_2 is the reflected wave amplitude. After substitution of eikonal solutions (1.30) and, then, relations (1.33), into the last equation, the following conditions can be received for the beam parameters in the plane $x_3 = 0$, $\tau_1' = l_1 c_1^{-1}$:

$$\begin{gathered} b_1 = -b_2, f_1\left(\frac{l_1}{c_1}\right) = f_2\left(\frac{l_2}{c_1}\right), R_1\left(\frac{l_1}{c_1}\right) = R_2\left(\frac{l_1}{c_1}\right), \\ \sigma_1\left(\frac{l_1}{c_1}\right) = \sigma_2\left(\frac{l_1}{c_1}\right), \frac{df_1(l_1 c_1^{-1})}{d\tau_1'} = \frac{df_2(l_1 c_1^{-1})}{d\tau_2'}. \end{gathered} \tag{1.41}$$

Conditions (1.34)–(1.36) for the reflected wave should be solved with boundary conditions (1.41). From the second condition (1.41) follows that $r_1 = r_2$ everywhere.

In the case of interferometer, condition (1.38) takes place for the falling wave and relation $u_3 = 0$ will be instead of condition (1.39). Conditions (1.41) remain valid, but the first equality will be changed by $b_1 = b_2$.

1.9 The Equation of Dimensionless Width of a Beam for Nonparaxial Rays

Equations (1.34)–(1.36) have been received for paraxial rays by equating of zero and second powers of the radial coordinate. The more general approach for non-paraxial rays consists in a choice of Eqs. (1.34) and (1.35), taking place on a beam axis, and the integrated law of conservation following from Eq. (1.29) is taken instead of Eq. (1.36). In the case $\chi_2 = 0$, this method has been used in [41], where it was shown that the solution has the same form, as for paraxial beams, but the factor χ_1 is replaced by $\frac{\chi_1}{4}$, that better displays the nature of the numerical solution of the Schrödinger equation. For $\chi_2 \neq 0$, when the nonlinear absorption is taken into account, we multiply Eq. (1.29) by $\frac{\partial A_1^*}{\partial \tau_1'}$, where A_1^* is a complex-conjugate quantity to A_1. We will multiply by $\frac{\partial A_1}{\partial \tau_1'}$ the equation conjugated to (1.29) and after summarizing these two equations we will integrate them on cylindrical coordinates r and θ. Then, for a case of an axisymmetric problem we shall obtain:

$$-\frac{L}{2}\frac{d}{d\tau_1'}\left\{\int_o^\infty\left[\left|\frac{\partial A_1}{\partial r}\right|^2+\frac{\chi_1}{2}|A_1|^4\right]rdr\right\} = i\chi_2\int_o^\infty |A_1|^2\left(A_1\frac{\partial A_1^*}{\partial \tau_1'}-A_1^*\frac{\partial A_1}{\partial \tau_1'}\right)rdr. \tag{1.42}$$

Substituting value of A_1, like in (1.30), and using (1.34) and (1.35), one can receive the following equation instead of (1.36):

$$f_1'' = \left(f_1' + \frac{b_1^2\chi_2}{4\alpha f_1}\right)^{-1}\left\{f_1'\left[\left(\frac{L^2}{\alpha^2 r_1^4}+\frac{b_1^2\chi_1 L}{2\alpha^2 r_1^2}+\frac{3b_1^4\chi_2^2}{2\alpha^2}\right)f_1^{-3}+\frac{\chi_2 b_1^2}{\alpha}f_1^{-2}\right] + \frac{\chi_2 b_1^2}{\alpha f_1}\left[2\nu+\left(f_1'\right)^2(4f_1)^{-1}\right]+\frac{5\chi_2^2 b_1^4\nu}{2\alpha^2 f_1^2}+\left(\frac{L}{r_1^2}+\chi_1 b_1^2\right)L\chi_2 b_1^2 f_1^{-4} r_1^{-2}\alpha^{-3}\right\}. \tag{1.43}$$

The received Eq. (1.43) with boundary conditions (1.38) and (1.41) should be solved numerically. As the numerical solving of Eqs. (1.43) and (1.36) have identical difficulty, it is preferable to solve more precise Eq. (1.43). Under the assumptions of small and high dissipations it is possible to put χ_2 equal to zero in the whole of the brace. The result will be the same as for Eq. (1.36), but χ_1 should be changed by $\chi_1/4$.

1.10 The Solution of the Equation for Dimensionless Width of a Beam for Paraxial Rays

We will search for the solution of Eq. (1.36) in the cases of weak and strong absorption. In the first case $\nu\tau_1$ and $\nu\tau_2$ are small, and it is possible to consider exponents entering in $\chi_{1,2}$ equal to one. In the case of strong absorption it is possible to consider exponents as zero, and the problem will be linear. In both cases of strong and weak absorption the second term in the right-hand side of Eq. (1.36) can be rejected.

In accordance with the aforesaid, solution of (1.36) for M < 0 and M > 0 with account of (1.38) looks like

$$f_1^2 = \frac{MF}{F^2 + M} + (F^2 + M)\left(\tau' + \frac{F}{F^2 + M}\right)^2. \tag{1.44}$$

For the reflected wave with account of boundary conditions (1.41) solution (1.36) has the form

$$f_2^2 = \left[F_1^2 + \frac{M}{f_1^2(0)}\right]\left[\tau_1'' + F_1 f_1(0)\right]^2 + \frac{M}{F_1^2 + M' f_1^{-2}(0)}, \tag{1.45}$$

where $F_1 = \frac{df_{1(0)}}{d\tau'}, \tau' = -x_3 c_1^{-1}, \tau'' = x_3 c_1^{-1}$.

Thus, the solutions of narrow beams in wave guides have been obtained that enable one to study their focusing.

1.11 The Analysis of Solutions for Narrow Beams

We shall consider only the case of focal spots, which corresponds to M > 0, F < 0. The received formula is suitable both for $\tau' < \tau_0'$ and for $\tau' > \tau_0'$, where

$$\tau_0' = -\frac{l_1}{c_0} - \frac{F}{F^2 + M}. \tag{1.46}$$

Formula (1.46) yields from the condition $\frac{df_1}{d\tau'} = 0$.

At the value of l_1, for which $\tau_0' < 0$, the focal stain is inside the layer, in the case $\tau_0' > 0$ it is out of the layer and, at last, if $\tau_0' = 0$, the focal stain is on the layer border. The last case will be for $l = -c_1 F\,(F^2 + M)^{-1}$, then formula (1.44) becomes simpler and takes on the form:

$$f_1^2 = \frac{M}{F^{\,2} + M} + (F^{\,2} + M)\,(\tau')^2. \tag{1.47}$$

For the reflected wave, we shall consider only the case M′ > 0. Formula (1.44) can be also written in the form

$$f_1^2 = \frac{M}{F^2 + M} + (F^2 + M)\left(\tau' - \tau_0'\right)^2. \tag{1.48}$$

One can find from (1.47): $\frac{d f_1(0)}{d\tau'} = -\tau_0' \frac{F\ + M}{f_1(0)}$, then the sign of $\frac{df_1(0)}{d\tau'}$ is determined by the sign of τ_0'. If $\tau_0' < 0$, then $\frac{df_1(0)}{d\tau'} > 0$ and $\frac{df_2(0)}{d\tau''} > 0$, and the sign "plus" should be taken in (1.45). $\frac{df_1(0)}{d\tau'} < 0$ and $\frac{df_2(0)}{d\tau''} < 0$ for $\tau_0' > 0$, then the sign "minus" is chosen in (1.45). In both cases the second square bracket in formula (1.45) can be written in the form

$$[\tau'' + F_1 f_1(0)].$$

The focal stain of the reflected wave can be found from the condition $\frac{d f_2}{d\tau''} = 0$. Then, equating (1.47) to zero, one can get

$$\tau_0'' = -F_1 f(0). \tag{1.49}$$

If $F_1 < 0$, τ_0'' is located inside the layer, whereas τ_0' is situated out of the layer. And vise versa for $F_1 > 0$: τ_o'' is located out of the layer, whereas τ_0' is situated inside the layer.

In the case, when $\tau_0' = \frac{df_1(0)}{d\tau'} = \frac{df_2(0)}{d\tau''} = 0$, formula (1.45) with account of $f_1^2(0) = M(F^2 + M)^{-1}$ can be written in the form:

$$f_2^2 \frac{M'}{f_1^2(0)}(\tau'') + f_1^2(0) = \frac{M'(F^2 + M)}{M}(\tau'')^2 + M(F^2 + M)^{-1}. \tag{1.50}$$

So, $\tau_o' = 0$ and $\tau_o'' = 0$, i.e. both focal points are located on a free border of a medium.

1.12 Transition to an One-Dimensional Case. The Analysis of Dispersion Properties of Plane Waves

Propagation of a longitudinal wave in a porous material along x_3-axis can be described by the following set of two nonlinear equations (as a one-dimensional equation will be further considered in the chapter, for convenience, the designations for coordinate x_3 and for the + u_3 are changed, accordingly, by x and u):