Developments in Mathematics

V. Lakshmibai
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The Grassmannian Variety

Geometric and Representation-Theoretic Aspects

Springer

Developments in Mathematics

VOLUME 42

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ISSN 1389-2177 ISSN 2197-795X (electronic)
Developments in Mathematics
ISBN 978-1-4939-3081-4 ISBN 978-1-4939-3082-1 (eBook)
DOI 10.1007/978-1-4939-3082-1

Library of Congress Control Number: 2015947780

Mathematics Subject Classification (2010): 14M15, 14L24, 13P10, 14D06, 14M12

Springer New York Heidelberg Dordrecht London

Printed on acid-free paper

Springer Science+Business Media LLC New York is part of Springer Science+Business Media (www.springer.com)

Preface

This monograph represents an expanded version of a series of lectures given by V. Lakshmibai on Grassmannian varieties at the workshop on “Geometric Representation Theory” held at the Institut Teknologi Bandung in August 2011. While giving the lectures at the workshop, Lakshmibai realized the need for an introductory book on Grassmannian varieties that would serve as a good resource for learning about Grassmannian varieties, especially for graduate students as well as researchers who want to work in this area of algebraic geometry. Hence, the creation of this monograph.

This book provides an introduction to Grassmannian varieties and their Schubert subvarieties, focusing on the treatment of geometric and representation theoretic aspects. After a brief discussion on the basics of commutative algebra, algebraic geometry, cohomology theory, and Gröbner bases, the Grassmannian variety and its Schubert subvarieties are introduced. Following introductory material, the standard monomial theory for the Grassmannian variety and its Schubert subvarieties is presented. In particular, the following topics are discussed in detail:

- the construction of explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials in the Plücker coordinates (called standard monomials);
- the use of the standard monomial basis,
- the presentation of a proof of the vanishing of the higher cohomology groups of powers of the restriction of the tautological line bundle of the projective space (giving the Plücker embedding).

Further to using the standard monomial basis, the book discusses several geometric consequences, such as Cohen–Macaulayness, normality, unique factoriality, Gorenstein-ness, singular loci, etc., for Schubert varieties, and presents two kinds of degenerations of Schubert varieties, namely, degeneration to a toric variety, and degeneration to a monomial scheme. Additionally, the book presents the relationship between the Grassmannian and classical invariant theory. Included is a discussion on determinantal varieties—their relationship to Schubert varieties as well as to classical invariant theory. The book is concluded with a brief account of some topics related to the flag and Grassmannian varieties: standard monomial theory for a general G/Q, homology and cohomology of the flag variety, free resolutions of Schubert singularities, Bott–Samelson varieties, Frobenius splitting, affine flag varieties, and affine Grassmannian varieties.

This text can be used for an introductory course on Grassmannian varieties. The reader should have some familiarity with commutative algebra and algebraic geometry. A basic reference to commutative algebra is [21] and algebraic geometry [28]. The basic results from commutative algebra and algebraic geometry are summarized in Chapter 2. We have mostly used standard notation and terminology and have tried to keep notation to a minimum. Throughout the book, we have numbered theorems, lemmas, propositions, etc., in order according to their chapter and section; for example, 5.1.3 refers to the third item of the first section in the fifth chapter.

Acknowledgments: V. Lakshmibai thanks CIMPA, ICTP, UNESCO, MESR, MICINN (Indonesia Research School), as well as the organizers, Intan Muchtadi, Alexander Zimmermann of the workshop on “Geometric Representation Theory” held at the Institut Teknologi Bandung, Bandung, Indonesia, August 2011, for inviting her to give lectures on “Grassmannian Variety,” and also the Institute for the hospitality extended to her during her stay there.

Both authors thank Reuven Hodges for his feedback on some of the chapters. We thank the makers of ShareLaTeX for making this collaboration easier. We thank the referee for some useful comments, especially, pertaining to Chapter 11.

Finally, J. Brown thanks Owen, Evan, and Callen, for constantly reminding him just how fun life can be.

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Contents

Chapter 1
Introduction

This book is an expanded version of a series of lectures given by V. Lakshmibai on Grassmannian varieties at the workshop on "Geometric Representation Theory" held at the Institut Teknologi Bandung, Bandung, Indonesia, in August 2011. In this book, we have attempted to give a complete, comprehensive, and self-contained account of Grassmannian varieties and the Schubert varieties (inside a Grassmannian variety).

In algebraic geometry, Grassmannian varieties form an important fundamental class of projective varieties. In terms of importance, they are second only to projective spaces; in fact, a projective space itself is a certain Grassmannian. A Grassmannian variety sits as a closed subvariety of a certain projective space, the embedding being known as the celebrated Plücker embedding (as described in the next paragraph). Grassmannian varieties are important examples of homogeneous spaces; they are of the form $GL_n(K)/P$, P being a certain closed subgroup (for the Zariski topology) of $GL_n(K)$ (here, $GL_n(K)$ is the group of $n \times n$ invertible matrices with entries in the [base] field K which is supposed to be an algebraically closed field of arbitrary characteristic). In particular, a Grassmannian variety comes equipped with a $GL_n(K)$-action; in turn, the (homogeneous) coordinate ring (for the Plücker embedding) of a Grassmannian variety acquires a $GL_n(K)$-action, thus admitting representation-theoretic techniques for the study of Grassmannian varieties. Thus, Grassmannian varieties are at the crossroads of algebraic geometry, commutative algebra, and representation theory; their study is further enriched by their combinatorics. Schubert varieties in a Grassmannian variety form an important class of subvarieties, and provide a powerful inductive machinery for the study of Grassmannian varieties; in fact, a Grassmannian variety itself is a certain Schubert variety.

A Grassmannian variety (as a set) is the set of all subspaces of a given dimension d in K^n, for some $n \in \mathbb{N}$; it has a canonical projective embedding (the Plücker embedding) via the map which sends a d-dimensional subspace to the associated point in the projective space $\mathbb{P}(\Lambda^d K^n)$. A Grassmannian variety may be thought of

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as a partial flag variety: given a bunch of r distinct integers $\underline{d} := 1 \leq d_1 < d_2 < \ldots < d_r \leq n-1$, the *partial flag variety* $\mathcal{F}l_{\underline{d}}$, consists of partial flags of type $\underline{d}$, namely sequences $V_{d_1} \subset V_{d_2} \subset \ldots \subset V_{d_r}$, V_{d_i} being a K-vector subspace of K^n of dimension d_i. The extreme case with $r = 1$, corresponds to the *Grassmannian variety* $G_{d,n}$ consisting of d-dimensional subspaces of K^n. If $d = 1$, then $G_{1,n}$ is just the $(n-1)$-dimensional projective space $\mathbb{P}_K^{n-1}$ (consisting of one-dimensional subspaces in K^n). For $r = n-1$, we get the celebrated *flag variety* $\mathcal{F}l_n$, consisting of flags in K^n, where a (full) flag is a sequence $(0) = V_0 \subset V_1 \subset \ldots \subset V_n = K^n$, $\dim V_i = i$. The flag variety $\mathcal{F}l_n$ has a natural identification with the homogeneous space $GL_n(K)/B$, B being the (Borel) subgroup of $GL_n(K)$ consisting of upper triangular matrices.

Throughout the 20^{th} century, mathematicians were interested in the study of the Grassmannian variety and its Schubert subvarieties (as well as the flag variety and its Schubert subvarieties). We shall now mention some of the highlights of the developments in the 20^{th} century on the Grassmannian and the flag varieties, pertaining to the subject matter of this book.

In 1934, *Ehresmann* (cf. [20]) showed that the classes of Schubert subvarieties in the Grassmannian give a $\mathbb{Z}$-basis for the cohomology ring of the Grassmannian, and thus established a key relationship between the geometry of the Grassmannian varieties and the theory of characteristic classes. This result of Ehresmann was generalized by *Chevalley* (cf. [14]) in 1956. Chevalley showed that the classes of the Schubert varieties (in the *generalized flag variety* G/B, G a semisimple algebraic group and B a Borel subgroup) form a $\mathbb{Z}$-basis for the Chow ring of the generalized flag variety. The results of Ehresmann and Chevalley were complemented by the work of *Hodge* (cf. [31, 32]). Hodge developed the *Standard Monomial Theory* for Schubert varieties in the Grassmannian. This theory consists in constructing explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials (called standard monomials) in the Plücker coordinates. Hodge's theory was generalized to G/B, for G classical by *Lakshmibai, Musili,* and *Seshadri* in the series Geometry of G/P I-V (cf. [49, 50, 53, 56, 82]) during 1975–1986; conjectures were then formulated (cf. [55]) by Lakshmibai and Seshadri in 1991 toward the generalization of Hodge's theory to exceptional groups. These conjectures were proved by *Littelmann* (cf. [65, 67, 68]) in 1994–1998, thus completing the standard monomial theory for semisimple algebraic groups. This theory has led to many interesting and important geometric and representation-theoretic consequences (see [44, 46, 48, 52, 54, 57, 65, 67, 68]).

In this book, we confine ourselves to the Grassmannian varieties and their Schubert subvarieties, since our goal is to introduce the readers to Grassmannian varieties fairly quickly, minimizing the technicalities along the way. We have attempted to give a complete and comprehensive introduction to the Grassmannian variety — its geometric and representation-theoretic aspects.

This book is divided into three parts. Part I is a brief discussion on the basics of commutative algebra, algebraic geometry, cohomology theory, and Gröbner bases. Part II is titled "Grassmann and Schubert Varieties." We introduce the Grassmannian variety and its Schubert subvarieties in Chapter 5. In Chapter 5, we also present the standard monomial theory for the Grassmannian variety and its Schubert subvarieties, namely construction of explicit bases for the homogeneous coordinate ring of the Grassmannian and its Schubert varieties (for the Plücker embedding) in terms of certain monomials in the Plücker coordinates (called standard monomials); we present a proof of the vanishing of the higher cohomology groups of powers of the restriction of the tautological line bundle of the projective space (giving the Plücker embedding), using the standard monomial basis. In Chapter 6, we deduce several geometric consequences—such as Cohen–Macaulayness, normality, a characterization for unique factoriality for Schubert varieties—in fact, these properties are established even for the cones over Schubert varieties (for the Plücker embedding). In addition, we describe the singular locus of a Schubert variety. In Chapter 7, we show that the generators for the ideal of the Grassmannian variety (as well as a Schubert variety) given by the Plücker quadratic relations give a Gröbner basis. We have also presented two kinds of degenerations of Schubert varieties, namely degeneration of a Schubert variety to a toric variety and degeneration to a monomial scheme. We also give a characterization for Gorenstein Schubert varieties.

Part III is titled "Flag Varieties and Related Varieties," and begins with Chapter 8, where we have included a brief introduction to flag varieties and statement of results on the standard monomial theory for flag varieties, as well as degenerations of flag varieties. In Chapter 9, we present the relationship between the Grassmannian and classical invariant theory. In Chapter 10, we present determinantal varieties, their relationship to Schubert varieties as well as to classical invariant theory. In Chapter 11, we give a brief account of some topics related to the flag and Grassmannian varieties: standard monomial theory for a general G/Q, homology and cohomology of the flag variety, free resolutions of Schubert singularities, Bott–Samelson varieties, Frobenius splitting, affine flag varieties, and affine Grassmannian varieties.

Part I
Algebraic Geometry: A Brief Recollection

Chapter 2
Preliminary Material

This chapter is a brief review of commutative algebra and algebraic geometry. We have included basic definitions and results, but omitted many proofs. For details in commutative algebra, we refer the reader to [21, 72] and in algebraic geometry to [28, 75].

2.1 Commutative Algebra

In this section we list the necessary definitions and preliminary results from commutative algebra.

Definition 2.1.1. A ring R is *Noetherian* if every ideal is finitely generated, or equivalently, if every increasing chain of ideals terminates (this is called the *ascending chain condition*).

Theorem 2.1.2 (Hilbert Basis Theorem, cf. Theorem 1.2 of [21]). *If a ring R is Noetherian, then the polynomial ring $R[x_1, \ldots, x_n]$ is Noetherian.*

Note that any field is Noetherian (since a field has no nontrivial, proper ideals).

An ideal P of a ring R is *prime* if $P \neq R$ and $xy \in P$ implies $x \in P$ or $y \in P$. Given an ideal I of R, the *radical* of I is the set $\{r \in R \mid r^n \in I, \text{ for some } n \in \mathbb{N}\}$ and is denoted $\sqrt{I}$. We have $\sqrt{(0)}$ equals the intersection of all prime ideals of R. If $\sqrt{(0)} = (0)$, then R is *reduced*.

We say that $S \subset R$ is a multiplicative set if $0 \notin S$, $1 \in S$, and if $a, b, \in S$ then $ab \in S$. If P is a prime ideal of R, then $R \setminus P$ is a multiplicative set. The *ring of fractions* $S^{-1}R$ is constructed using equivalence classes of pairs $(r, s) \in R \times S$ such that $(r_1, s_1) \sim (r_2, s_2)$ if there exists $s \in S$ such that $s(r_1 s_2 - r_2 s_1) = 0$. One denotes the element (r, s) as the fraction $\frac{r}{s}$. Addition and multiplication in $S^{-1}R$ are defined as

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$$\frac{r_1}{s_1}+\frac{r_2}{s_2}=\frac{r_1s_2+r_2s_1}{s_1s_2}, \quad \frac{r_1}{s_1}\cdot\frac{r_2}{s_2}=\frac{r_1r_2}{s_1s_2}.$$

We have the natural ring homomorphism $R \to S^{-1}R$, $r \mapsto \frac{r}{1}$.

Theorem 2.1.3 (cf. Theorem 4.1 of [72]). *The prime ideals of $S^{-1}R$ correspond bijectively to the prime ideals of R disjoint from S.*

Definition 2.1.4. A *zero divisor* of a ring R is a nonzero element $a \in R$ such that there exists a nonzero element $b \in R$ such that $ab = 0$. A nonzero element $a \in R$ that does not satisfy this definition is called a *nonzero divisor*.

When S is the set of all nonzero divisors in R, we call $S^{-1}R$ the *full ring of fractions of R*. If R is an integral domain, then the full ring of fractions is in fact a field, which we call the *field of fractions of R*.

As mentioned above, if P is a prime ideal, then $S = R \setminus P$ is a multiplicative set, and we will denote $S^{-1}R$ as R_P; we call this the *localization* of R with respect to P. Note that R_P is a *local ring*, meaning it has a unique maximal ideal. In this case the maximal ideal is PR_P, the set of all elements in R_P without multiplicative inverses. The prime ideals of R_P correspond to the prime ideals in R contained in P. (Throughout, we use the notation PM when P is an ideal in R and M is an R-module to denote $\{pm \mid p \in P,\ m \in M\}$.) If R is Noetherian, then so is R_P.

For an R-module M, $S^{-1}M$ is defined by using classes of pairs $(m, s) \in M \times S$, where $(m_1, s_1) \sim (m_2, s_2)$ if there exists $s \in S$ such that $s(s_1m_2 - s_2m_1) = 0$. Then $S^{-1}M$ is an $S^{-1}R$-module (in a natural way), and is naturally isomorphic to $S^{-1}R \otimes M$. The functor $M \to S^{-1}M$ from R-modules to $S^{-1}R$-modules is exact, i.e., it takes exact sequences to exact sequences.

Let $R_1 \subset R_2$ be an extension of rings. An element $r \in R_2$ is *integral* over R_1 if r is a zero of some monic polynomial with coefficients in R_1; i.e.,

$$r^n + a_{n-1}r^{n-1} + \ldots + a_1r + a_0 = 0$$

where $a_i \in R_1$. The ring R_2 is *integral over* R_1 if every element of R_2 is integral over R_1. The *integral closure* of R_1 in R_2 is the subring of R_2 consisting of all the elements of R_2 integral over R_1. If R is an integral domain such that R contains all elements of $K(R)$ that are integral over R (here, $K(R)$ is the field of fractions of R, as defined above), then we say R is *integrally closed*.

Definition 2.1.5. If R is a ring such that R_P is an integrally closed domain for every prime ideal $P \subset R$, then R is a *normal ring*.

Example 2.1.6. The ring of integers $\mathbb{Z}$ is a normal ring.

Let $K_1 \subset K_2$ be an extension of fields. The elements $a_1, \ldots, a_d \in K_2$ are *algebraically independent over* K_1 if there does not exist any nonzero polynomial $f(x_1, \ldots, x_d)$ over K_1 such that $f(a_1, \ldots, a_d) = 0$. A maximal subset of algebraically independent elements over K_1 in K_2 is called a *transcendence basis* of K_2/K_1. If K_2

is a finitely generated extension of K_1, then any two transcendence bases of K_2/K_1 have the same cardinality called the *transcendence degree* of K_2 over K_1, and is denoted $\mathrm{tr.deg}_{K_1} K_2$.

Let $R = K[a_1, \ldots, a_n]$ be a finitely generated K-algebra, where R is an integral domain with field of fractions denoted by $K(R)$. Then define $\mathrm{tr.deg}_K R = \mathrm{tr.deg}_K K(R)$.

Definition 2.1.7. For a Noetherian ring R, $\dim R$ (called the Krull dimension of R) is the maximal length t of a strictly increasing chain of prime ideals

$$P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_t \subset R.$$

For example,

1. $\dim \mathbb{Z} = 1$.
2. $\dim K[x_1, \ldots, x_n] = n$, where K is a field.

Definition 2.1.8. If P is a prime ideal of the Noetherian ring R, the *height* of P is defined to be $\dim R_P$, and is denoted $\mathrm{ht}(P)$; the *coheight* of P is defined to be $\dim R/P$.

For the following definitions, let R be a Noetherian ring, and let M be a finitely generated R-module. Our primary examples of R-modules will be ideals of R, localizations of R, or R itself.

Definition 2.1.9. An element $r \in R$ is a *zero divisor* in M if there exists a nonzero $m \in M$ such that $rm = 0$.

Definition 2.1.10. A sequence $r_1, \ldots, r_s \in R$ is an *M-regular sequence* if

1. $(r_1, \ldots, r_i)M \neq M$, and
2. r_i is not a zero divisor in $M/(r_1, \ldots, r_{i-1})M$ for $2 \leq i \leq s$, and r_1 is not a zero divisor in M.

If $r_1, \ldots, r_s$ is an M-regular sequence contained in an ideal I of R, we call it an M-regular sequence in I. If, for any $r_{s+1} \in I$, $r_1, \ldots, r_s, r_{s+1}$ is not an M-regular sequence, we say that $r_1, \ldots, r_s$ is a maximal M-regular sequence in I.

Theorem 2.1.11 (cf. Theorem 16.7 of [72]). *Let R be a Noetherian ring, I an ideal in R, and M a finitely generated R-module. Then any two maximal M-regular sequences in I have the same length.*

Definition 2.1.12. For a local Noetherian ring R with maximal ideal $\mathfrak{m}$ and R-module M, the *depth of M* is defined as the length of a maximal M-regular sequence in $\mathfrak{m}$, and is denoted depth M.

Remark 2.1.13. If R is a local Noetherian domain, then depth $R \geq 1$.

Proposition 2.1.14 (cf. Prop. 18.2 of [21]). *Let R be a ring and P a prime ideal, then depth $R_P \leq ht(P)$.*