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LOGICISM, INTUITIONISM, AND FORMALISM

WHAT HAS BECOME OF THEM?

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Preface

The present anthology has its origin in two international conferences that were arranged at Uppsala University in August 2004: “Logicism, Intuitionism and Formalism: What has become of them?” followed by “Symposium on Constructive Mathematics”. The first conference concerned the three major programmes in the foundations of mathematics during the classical period from Frege’s *Begriffsschrift* in 1879 to the publication of Gödel’s two incompleteness theorems in 1931: The logicism of Frege, Russell and Whitehead, the intuitionism of Brouwer, and Hilbert’s formalist and proof-theoretic programme. The main purpose of the conference was to assess the relevance of these foundational programmes to contemporary philosophy of mathematics. The second conference was announced as a satellite event to the first, and was specifically concerned with constructive mathematics—an active branch of mathematics where mathematical statements—existence statements in particular—are interpreted in terms of what can be effectively constructed. Constructive mathematics may also be characterized as mathematics based on intuitionistic logic and, thus, be viewed as a direct descendant of Brouwer’s intuitionism. The two conferences were successful in bringing together a number of internationally renowned mathematicians and philosophers around common concerns. Once again it was confirmed that philosophers and mathematicians can work together and that real progress in the philosophy and foundations of mathematics is possible only if they do. Most of the papers in this collection originate from the two conferences, but a few additional papers of relevance to the issues discussed at the Uppsala conferences have been solicited especially for this volume.

Many people have helped us in making the two conferences and this volume possible. The person who has meant the most from a scientific point of view is Professor Per Martin-Löf whose vision and good judgement inspired and accompanied us through the preliminary stages, and who through his personal involvement during the conferences contributed to the positive result.

The Conference on the Philosophy of Mathematics was organized by the Department of Mathematics and the Department of Philosophy at Uppsala University in cooperation with the Swedish National Committee for Logic, Methodology and Philosophy of Science. The Symposium on Constructive Mathematics was organized by the Department of Mathematics. We are grateful to the two departments for financial support and, especially, to the Department of Mathematics for supply-

ing the venue and for their generous organizational support. In this connection, the untiring organizational work of Zsuzsanna Kristófi was essential for the success of both conferences. We also wish to thank Ryszard Sliwinski for organizing things from the philosophy side. Moreover the help of PhD students Fredrik Dahlgren, Olov Wilander and Johan Granström with practical publicity issues was much appreciated. The latter also transformed Martin-Löf's manuscript into $\LaTeX$.

We are also very grateful to the Swedish Research Council (VR) and the Royal Swedish Academy of Sciences (KVA) for financial support. Sten Lindström's editorial work on this volume has been made possible by a research grant ("The Ontology and Epistemology of Mathematics") from the Bank of Sweden Tercentenary Foundation (RJ) and from a fellowship during 2007 – 08 at the Swedish Collegium for the Social Sciences (SCAS). Erik Palmgren's editorial work was supported by a grant from the Swedish Research Council.

All papers in this volume, except the few that are reprinted, were assigned anonymous referees by the editors. We wish to thank these referees for their excellent work. We are also grateful to Springer's anonymous referee of the entire volume for insightful and helpful comments.

Finally, we wish to thank Professor Vincent Hendricks, Editor-in-Chief of Synthese Library, for encouraging this project; and Floor Oosting and Ingrid van Laarhoven at Springer, and Indumadhi Srinivasan at Integra Software Services for all their help in connection with the production of this book."

Umeå, Sweden
Uppsala, Sweden
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Introduction: The Three Foundational Programmes

Sten Lindström and Erik Palmgren

1 Overview

The period in the foundations of mathematics that started in 1879 with the publication of Frege's *Begriffsschrift* [18] and ended in 1931 with Gödel's [24] *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*¹ can reasonably be called the classical period. It saw the development of three major foundational programmes: the logicism of Frege, Russell and Whitehead, the intuitionism of Brouwer, and Hilbert's formalist and proof-theoretic programme. In this period, there were also lively exchanges between the various schools culminating in the famous Hilbert-Brouwer controversy in the 1920s. The state of the foundational programmes at the end of the classical period is reported in the papers by Carnap, Heyting and von Neumann (Cf. Benacerraf and Putnam [1]) from the Conference on Epistemology of the Exact Sciences in Königsberg 1930. This was the very same symposium at which Gödel announced his First Incompleteness Theorem.²

A purpose of this volume is to review the programmes in the foundations of mathematics from the classical period and to assess their possible relevance for contemporary philosophy of mathematics. What can we say, in retrospect, about the various foundational programmes and the disputes that took place between them? To what extent do the classical programmes of logicism, intuitionism and formalism represent options that are still alive today?

A set of papers in constructive mathematics were specially solicited to the present anthology. This active branch of mathematics is a direct legacy of Brouwer's intuitionism. Today one often views it more abstractly as mathematics based on

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¹ *On formally undecidable propositions of Principia Mathematica and related systems*, reprinted in van Heijenoort [29].

² Many important original papers of the period are contained in van Heijenoort [29]. See Hesselning [31] for a recent historical account of the relation between the programmes at that time. See also Mancosu [41] where many of the major articles from the foundational debate between intuitionists and formalists in the 1920s appear in English translation.

intuitionistic logic. It can then be regarded as a generalisation of classical mathematics in that it may be given, firstly, the standard set-theoretic interpretation, secondly, algorithmic meaning, and thirdly, nonstandard interpretations in terms of variable sets (sheaves over topological spaces).

The volume will be of interest primarily to researchers and graduate students of philosophy, logic, mathematics and theoretical computer science. The material will be accessible to specialists in these areas and to advanced graduate students in the respective fields.

2 Logicism and Neologicism

2.1 Frege's Logicism and Dedekind's Analysis of the Natural Numbers

Kant claimed that our knowledge of mathematics is *synthetic a priori* and based on a faculty of *intuition*. Frege accepted Kant's claim in the case of geometry, i.e., he thought that our knowledge of Euclidian geometry is based on pure intuition of space. But he could not accept Kant's explanation of our knowledge of statements about numbers.

The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic governs all that is numerable. This is the widest domain of all; for to it belongs not only the existent [das Wirkliche], not only the intuitable [das Anschauliche], but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought?³

Frege thought of numerical statements as being objectively true or false. Moreover, he interpreted these statements as literally being about abstract mathematical objects that do not exist in space or time. Now the question arose: *How can we have knowledge about numbers and their properties, if numbers are abstract objects?* Clearly we cannot interact causally with abstract entities. Neither is it plausible to explain our knowledge of them in terms of some kind of non-empirical intuition. For Frege it was evident that knowledge about numbers is possible only if it is conceptual and apriori, rather than based on experience or intuition. Thus his main question became:

How, then are numbers given to us, if we cannot have any ideas or intuitions of them?⁴

In order to show that apriori knowledge of arithmetic is possible, Frege thought it necessary and sufficient to establish *the logicist thesis* that arithmetic is reducible to logic. More precisely, he wanted to show that:

³ Frege [19], Section 14.

⁴ "Wir soll uns denn eine Zahl gegeben sein, wenn wir keine Vorstellung oder Anschauung von ihr haben können?", Frege, *Grundlagen der Arithmetik* [19], Section 62.

- (i) the concepts of arithmetic can be explicitly defined in terms of logical concepts;
- (ii) the truths of arithmetic can be derived from logical axioms (and definitions) by purely logical rules of inference.

The underlying epistemic idea seems to be that a rational subject could gain apriori knowledge of logic and the definitions of arithmetical concepts in logical terms. The same subject could then, in principle, infer the truths of arithmetic from the logical axioms and the definitions, and thereby gain apriori knowledge of arithmetical truths. In other words, for every arithmetical truth it is according to Frege possible for an ideally rational subject to gain apriori knowledge of that truth. Of course, a critic might question the assumption that the basic principles of Frege's logic are knowable apriori. Frege's reason for this assumption was presumably that he thought of these principles as conceptual truths. However, as it turned out, the principles of the logical system that Frege devised in *Grundgesetze* [20] were actually inconsistent.

The following four claims are implicit in Frege's logicist programme: (a) Logic is (or can be presented as) an interpreted *formal system* (a *Begriffsschrift*); (b) It can be known apriori that the axioms of logic are true and that the logical rules of inference preserve truth; (c) the concepts of arithmetic are logical concepts; and (d) the truths of arithmetic are provable in logic. From (a) and (b) it follows that the theorems of logic are true. Since a contradiction cannot be true, it follows that logic is consistent. Moreover, it seems to follow from (b) that we can gain apriori knowledge of the theorems of logic by proving them. In virtue of (d) then, arithmetic must be consistent and its truths knowable apriori.

Roughly at the same time as Frege conceived of his logicist program, Dedekind [12] was also arguing for a kind of logicism:⁵

In science nothing capable of proof ought to be accepted without proof. Though this demand seems so reasonable yet I cannot regard it as having been met even in . . . that part of logic which deals with the theory of numbers. In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number concept entirely independent of the notions of intuition of space and time, that I consider it an immediate result from the laws of thought. (Dedekind [12], quoted from the Engl. trans. *Essays on the Theory of Numbers*, p. 31)

Working within informal set theory (which he considered to be a part of logic), Dedekind gave—for the first time—an abstract axiomatic characterisation of the system of natural numbers. To be precise, he characterised the structure of the system of natural numbers (up to isomorphism) by means of the notion of a *simple infinite system*: A simply infinite system is a set X (representing the natural numbers)

⁵ Dedekind's version of logicism is discussed in great detail, both systematically and from a historical point of view, in Sieg and Schlimm [51]. See also Reck [47], who aptly refers to Dedekind's approach as "logical structuralism".

together with an element e (representing 0) and an operation S in X (the successor function), satisfying the conditions:⁶

- (a) S is a one-to-one mapping from X onto $X - \{e\}$.
- (b) for any set $Y \subseteq X$, if $e \in Y$ and $S(x) \in Y$, whenever $x \in Y$, then $Y = X$.
That is, X is the smallest set containing e and being closed under the successor operation.

In modern terms, such a system is a (standard) model of *Peano's axioms* (or more accurately the *Dedekind-Peano's axioms*⁷) for the natural numbers:⁸

- (P1) 0 is a natural number.
- (P2) Every natural number n has a unique successor $S(n)$.
- (P3) 0 is not the successor of any natural number.
- (P4) Two different natural numbers do not have the same successor.
- (P5) For every F , if the following two conditions hold: (a) $F(0)$, and (b) for every natural number n , if $F(n)$, then $F(S(n))$, then for every natural number n , $F(n)$. (*The Principle of Mathematical Induction*)

Dedekind proved that any two simply infinite systems are isomorphic. This means that second-order *Peano arithmetic*, with the standard semantics, is *categorical*, i.e., all its models are isomorphic,⁹ and hence it is negation-complete, i.e., for any sentence φ in the language of second-order arithmetic, either φ or $\neg\varphi$ is a *logically consequence* of Peano's axioms (i.e., true in all models of the axioms).

The natural number system Dedekind thought of as being obtained by a process of abstraction: Starting from any simply infinite system one abstracts from those features that distinguishes it from any other simply ordered system. One thereby obtain the abstract system of natural numbers that Dedekind describes as a “free creation of the human mind”. In “Letter to Keferstein” [13], Dedekind speaks of the natural number sequence as the “abstract type” of simply infinite systems:

⁶ Dedekind actually thought of the natural numbers as starting with 1 instead of 0, as is customary nowadays.

⁷ These axioms were presented independently by Dedekind [12] and Peano [46]. See also Dedekind's brilliant discussion in “Letter to Keferstein” [13] from 1890 of the basic ideas underlying the axioms.

⁸ We are here considering the second-order language of Peano arithmetic with its *standard* model-theoretic semantics. In each (standard) model, the unary predicate variables range over the entire power set of the domain D of individuals. And for $n > 1$, the n -ary predicate variables range over the power set of D^n . The standard semantics for second-order logic stands in contrast to the “non-standard” semantics devised by Henkin [30], which in addition to standard models, also allows for *generalised* or *Henkin models*. A *generalised model* is a structure where the unary predicate variables range over some non-empty subset of the full power set of D , and similarly for the n -ary predicate variables. Validity and logical consequence are defined relative to all generalised models. Second-order logic with the Henkin semantics is recursively axiomatizable as well as compact and satisfies Löwenheim–Skolem's theorem. Second-order logic with the standard semantics has none of these properties. See Shapiro [48] for a detailed development of second-order logic, as well as a defence of its use as a formal framework for mathematics.

⁹ On the other hand, among the Henkin models for *PA*, there are also models containing non-standard numbers greater than all the “natural” numbers.

After the essential nature of the simply infinite system, whose abstract type is the number sequence N , had been recognised in my analysis (articles 71 and 73), the question arose: does such a system exist at all in the realm of our ideas? Without a logical proof of existence it would always remain doubtful whether the notion of such a system might not contain internal contradictions. Hence the need for such proofs (articles 66 and 72 of my essay).

Dedekind defined a set X to be *infinite* if it can be mapped 1-1 and onto a proper subset of itself.¹⁰ He, then showed that any infinite set includes a simply infinite system. In other words, the existence of an infinite set is necessary and sufficient for the existence of a simply infinite system. And the existence of a simply infinite system is necessary for the semantic consistency (i.e., satisfiability) of second-order Peano arithmetic. As we see from the quote above, Dedekind wanted to give a logical proof of the existence of a simply infinite system (or equivalently of a Dedekind-infinite set). In modern set theory one simply takes it as an axiom, the *Axiom of Infinity*, that there is an infinite set. But this procedure is questionable from a logicist point of view, since it is far from obvious that such an axiom is logically true.

Dedekind in fact thought that he could prove that there exists an infinite set.¹¹ In his “proof” he considers his “own realm of thoughts”, i.e., “the totality S of all things, which can be object of my thought” and argues that it must be infinite. For any x in S , he defines the successor $s(x)$ as the thought that x can be an object of my thought. This thought, he maintains, can be an object of my thought, hence for any x , $s(x) \in S$. Moreover, there are elements in S that are not themselves thoughts “e.g., my own ego”. Furthermore, Dedekind argues the mapping s is one-to-one. Thus s is a one-to-one mapping from S into a proper subset of S . Hence, S is Dedekind-infinite.

Dedekind’s “proof” of the existence of an infinite set, lacks the stringency that one would ordinarily expect from a mathematical proof. As soon as one realises that every plurality cannot be assumed to form a set (e.g., there is presumably no set of all non-self-membered sets), then one sees that Dedekind has not proved that there actually *exists* a set S of all things that can be object of “my” thought. For all that Dedekind says, it is quite possible that there is no such set.¹² Hence, Dedekind’s

¹⁰ This is the notion of a set being *Dedekind-infinite*. The notion of Dedekind-infinite does not presuppose the notion of natural number. In Zermelo-Fraenkel set theory, **ZF**, it is provable that every Dedekind-infinite set is *infinite*, in the standard sense of not being finite, i.e., equinumerous to an initial segment of the natural numbers. The converse, i.e., that every infinite set is Dedekind-infinite is not provable in **ZF**, but it is provable in **ZFC** (Zermelo-Fraenkel set theory with the Axiom of Choice).

¹¹ Cf. Dedekind [12], Theorem 66.

¹² Indeed, the assumption that there is such a set is threatened by paradox. Suppose, namely, that the set S exists. Then, presumably, the subset R of S consisting of all the objects in S that are not members of themselves also exists. Thus, for any member x of S , x belongs to R just in case it does not belong to itself. But it seems that R itself can be an object of my thought. Hence, R is a member of S . Consequently, R belongs to R if and only if it does not. As a matter of fact, in “Letter to Dedekind” [9] Cantor gives the “totality of everything thinkable” as an example of an *absolutely infinite* or *inconsistent* multiplicity that cannot be “gathered together” into “one thing”, i.e., that does not form a set.

attempted proof that there is an infinite totality consisting of all things that can be objects of “my” thought must be considered a failure.

To prove that there are infinitely many objects—or for that matter, to prove that an infinite set exists—is a major stumbling block for any logicist program. From a modern perspective it is hard to see, for example, how the existence of infinitely many natural numbers could ever be provable within logic. According to the dominant contemporary view, logic should be topic-neutral and free of ontological commitment.¹³ As it seems, logicism demands a completely different conception of logic than the standard contemporary one.¹⁴

Before embarking on a logicist programme one ought to have some idea about how to answer questions of the following kind: (i) What are the criteria for distinguishing *logical concepts* from non-logical ones? (ii) What is it that makes an axiom or a rule *logical*? (iii) How can we recognise something as a logical axiom or rule? The mark of the logical concepts and the logical laws for Frege seems to have been their complete generality. This criterion, however, appears to be too vague to be workable. Even today the problem of characterising logical concepts and logical principles is wide open.

A sentence (of second-order arithmetic) is *true* (simpliciter) if and only if it is true in the intended model i.e., the structure $\langle N, 0, S \rangle$ consisting of the natural numbers N , the number 0, and the successor operation S on natural numbers. But this holds if and only if the sentence is true in all the standard models of second-order Peano arithmetic (since these models are all isomorphic). In other words, a sentence in the language of (second-order) Peano arithmetic is true if and only if it is a (standard) model-theoretic consequence of the Dedekind-Peano axioms. So it might appear that Frege’s logicist programme would be accomplished if one could define the basic concepts of Peano arithmetic within logic and derive the Dedekind-Peano axioms from logical axioms and definitions by means of logical rules of inference. This was in essence what Frege wanted to do. For this purpose he invented a formal system of higher-order logic (including a theory of extensions of concepts) and proved within it axioms equivalent to the Dedekind-Peano axioms.

Now, in hindsight, we know that Frege’s logicist programme, in its original form, *could not* have succeeded. It is not just that the logic he actually used turned out to be inconsistent. In view of Gödel’s first incompleteness theorem, there is no formal system that proves exactly those sentences of second-order arithmetic that are true. For any consistent formal logic the logicist may construct, there will be true arithmetical sentences that are not provable in it. The modern neo-Fregeans must, therefore, abandon Frege’s impossible dream of showing that all arithmetic truths are formally provable in logic. Instead they must be satisfied with something weaker.

¹³ Standard predicate logic is ontologically committed to the existence of at least one object since the domain of quantification is required to be non-empty. This relatively harmless ontological commitment can be eliminated at the price of some loss of elegance.

¹⁴ See, for example, Goldfarb [23] concerning the differences between Frege’s conception of logic and the contemporary one.

2.2 Frege's Logic and Russell's Paradox

The formal logic that Frege actually used in [20] to carry out his programme consisted of the following ingredients:

- (i) A *language* of higher-order predicate logic, where the individual variables are assumed to range over the collection of absolutely all objects and the variables of higher types are taken to range over “unsaturated” entities, i.e., *functions* and *concepts* (i.e., functions from entities of some kind to truth-values).
- (ii) *Axioms* and *rules of inference* for higher-order predicate logic.
- (iii) *Principles of Comprehension* for functions and concepts. For instance, there is a comprehension schema:

$$\exists F \forall x (F(x) \leftrightarrow A(x))$$

to the effect that every formula $A(x)$ (in which the variable F does not occur free) defines a first-level concept, i.e., a concept taking objects as arguments. The formula $A(x)$ is here allowed to be *impredicative*, i.e., it may involve quantification over all (first-level) concepts, including the concept that the formula defines.

- (iv) *Frege's Basic Law V* according to which every concept F is associated with an object $\{x : F(x)\}$ (called the *extension* of F) satisfying the requirement that any two concepts F and G have the same extension just in case they are true of the same objects. That is:

$$(\text{Basic Law V}) \{x; F(x)\} = \{x : G(x)\} \leftrightarrow \forall x (F(x) \leftrightarrow G(x)).$$

The object $\{x : F(x)\}$ may be understood as the class of all object that fall under the concept F .

At first appearance, it might seem that the statement that every concept has an extension is a conceptual truth. After some hesitation, Frege also came to regard it as such. However, Basic Law V leads to a contradiction within Frege's system. The relation of coextensionality between concepts is an equivalence relation, and hence partitions the concepts into mutually exclusive equivalence classes. Intuitively, these equivalence classes represent classes of objects. In view of Cantor's theorem, there are more equivalence classes of concepts than there are objects. But Basic Law V is tantamount to assuming that there is a one-to-one mapping from equivalence classes of concepts to objects. Hence, there cannot be more equivalence classes of concepts than there are objects. We have a contradiction. In Fine's terminology [17], Basic Law V is *inflationary*, since it demands that there be more abstract objects representing concepts than there are objects. Now, Russell showed that this paradoxical argument can in fact be represented within Frege's system (*Russell's paradox*).

This is how Russell's paradox arises in Frege's system. In terms of the class abstraction operator $\{x : F(x)\}$, which Frege took as a primitive, one can define the membership relation $\in$:

$$x \in y \leftrightarrow \exists F[F(x) \wedge y = \{x : F(x)\}],$$

i.e., x is a member of the class y if and only if there is a concept F such that x falls under F and y is the class of all objects falling under F .

Within Frege's higher order logic, one can infer the following *Naive Comprehension Principle* for classes from Frege's Basic Law V:

$$\forall F \exists y \forall x (x \in y \leftrightarrow F(x)).$$

We say that an object x is *Russellian* (in symbols, $R(x)$) iff x is not a member of x (i.e., it is not the case that x is the extension of a concept F under which x itself falls). The existence of the concept R is guaranteed by the strong comprehension principles that are built into Frege's logic. Notice also that the definition of the concept R is *impredicative*, since it involves quantification over all concepts.

Since every concept has an extension, there is a class r (the *Russell class*) of all the objects that fall under the concept R , i.e., $r = \{x : x \notin x\}$. We can now easily derive a contradiction in Frege's system:

$$r \in r \leftrightarrow r \notin r.$$

Hence, Frege's logical system in [20] is inconsistent and thereby useless as a foundation for arithmetic.

Although Frege's programme in its original form cannot be carried out, it may be instructive to analyse what went wrong and see what can be done to repair Frege's system (Cf. Burgess [8]). One can obtain consistent subsystems of Frege's logic by modifying one or more of the ingredients of Frege's system that led to Russell's paradox. Thus, one can:

- (i) Replace classical logic with a weaker one. One radical alternative is to choose a paraconsistent logic that tolerates contradictions by preventing them from trivialising the system. In such a system, not every sentence follows from a contradiction.
- (ii) Replace Frege's impredicative principles of comprehension for concepts by weaker predicative ones. Then, one cannot prove that the Russell concept R or other impredicatively defined concepts exist. The mathematics that one obtains in this way is however very weak, even if one adopts Frege's axiom V of class abstraction (Cf. Burgess [8], Chapter 2).
- (iii) Abandon or weaken Frege's theory of extensions. Here there are various alternatives: (a) base mathematical theories on a Neo-Fregean theory of extensions, where certain concepts are not assumed to have corresponding extensions; or

(b) base mathematical theories on Fregean abstraction principles (e.g., *Hume's principle* (see below) as a basis for Peano arithmetic).

Fregean approaches are characterised by an effort to make only minimal changes to Frege's original system in order to obtain consistency. One can then investigate how much of mathematics can be captured in the resulting systems. Fregean approaches should be contrasted with approaches that use, for example, (ramified or simple) type theory, axiomatic set theory, or category theory, as foundational frameworks for mathematics.¹⁵

2.3 Neo-Fregean Logicism

The new development started with Wright's book *Frege's Conception of Numbers as Objects* [56], where a proof is outlined of what has become known as *Frege's theorem*, namely that the standard axioms of arithmetic are provable in second-order logic extended with Hume's principle (so-called *Frege arithmetic*).¹⁶ Hume's principle says that two concepts F and G have the same cardinal number iff they are equinumerous, i.e., iff there is a one-to-one correspondence between the objects falling under F and the objects falling under G . In symbols:

$$(HP) \forall F \forall G [Nx Fx = Nx Gx \leftrightarrow F \approx G],$$

where $F \approx G$ means that there exists a one-to-one correspondence between the objects that fall under F and G respectively.

According to Hume's principle, the concept of (*cardinal*) *number* is obtained by (Fregean) *abstraction* from the concept of *equinumerosity* between concepts (or properties). The latter concept is definable in second-order logic and is therefore, according to Frege and the neo-Fregeans, a logical concept. In second-order logic, Hume's principle becomes:

$$\begin{aligned} \forall F \forall G [Nx Fx = Nx Gx \leftrightarrow \exists R \forall x ((Fx \rightarrow \exists! y (Gy \wedge R(x, y))) \\ \wedge (Gx \rightarrow \exists! y (Fy \wedge R(y, x))))]. \end{aligned}$$

¹⁵ The logicism of Whitehead and Russell's *Principia Mathematica* [55] differs radically from Frege's in abandoning the idea that classes and numbers are objects. Propositions which, on the face of it, speak of classes or numbers are analyzed as really being about higher-order entities, namely, in Russell's terminology, *propositional functions*. See Linsky [40] for a thorough analysis of the logical framework of *Principia Mathematica*, i.e., Russell's Ramified Theory of Types, with its three characteristic axioms of Choice, Infinity, and Reducibility.

¹⁶ That Frege's theorem is implicit in Frege's own work was first pointed out in 1965 by Charles Parsons [45, 14]. Subsequent reconstructions of Frege's logic by Boolos and Heck (Boolos [3, 4], Heck [27, 14]) confirm that Frege's theorem was indeed proved by Frege himself in [20], although he did not state it explicitly.

Wright and Hale avoid using Frege's inconsistent theory of classes or axiomatic set theory as a foundation for mathematics. Instead they use abstraction principles like Hume's principle to introduce various domains of mathematical entities. Hume's principle is taken as an implicit definition of the cardinal number operator: the number of x such that: $F(x)$, or in symbols: $NxF(x)$. Intuitively, $NxF(x)$ is the cardinal number associated with the concept F , i.e., the number of objects falling under the concept F . From this operator, one can define the natural numbers: $0 = Nx(x \neq x)$, i.e., the number of objects that are not self-identical. $1 = Nx(x = 0)$, $2 = Nx(x = 0 \vee x = 1)$, etc. Frege shows how one can define the concepts of successor and natural number (finite cardinal) in Frege arithmetic. Using Frege's definitions and Hume's principle, one can prove the Dedekind-Peano axioms for the natural numbers including the (second-order) *Principle of Mathematical Induction*.

The idea of using Hume's principle as a logical basis for arithmetic goes back to Frege's *Grundlagen der Arithmetik* [19], but was immediately rejected by Frege with the motivation that Hume's principle is not strong enough to uniquely identify the cardinal numbers. Hume's principle answers to the question under what conditions two cardinal numbers are identical. But it does not tell us what objects are cardinal numbers. For instance, it does not say whether the number 2 is identical to the class of all classes that are equinumerous to $\{0, 1\}$ as Frege thought, or whether it is identical to the class $\{0, 1\}$ itself, as von Neumann suggested. It cannot even answer the question whether Julius Caesar is a number (The *Julius Caesar Objection*).¹⁷ For this reason, Frege preferred defining the natural numbers explicitly in terms of classes rather than basing arithmetic on Hume's principle. Another reason for Frege not to base arithmetic on Hume's principle, was presumably that he did not think that it could be taken as a fundamental principle of logic, since its subject matter was too specific for that purpose.

Wright and Hale ([56], [26]) advocate Hume's principle as a basis for arithmetic and argue that the principle is analytically true. The truth of the principle in turn implies the existence of infinitely many natural numbers. This means that they abandon Frege's view that arithmetic is in a strict sense reducible to logic, while retaining the idea that our knowledge of arithmetic is apriori and based on analytic truths. The neo-Fregean programme also aims at defining other domains of mathematical entities by means of abstraction principles, for instance, the domain of real numbers, and the domain of sets. A difficult problem is to differentiate between "bad" abstraction principles (cf. Frege's Basic Law V) that are inconsistent and "good" ones that can be the basis of mathematical concept formation. If there is no principled way of making such a distinction, it might seem that all abstraction principles are more or less suspect ("the Bad Company Objection").¹⁸

¹⁷ Cf. Heck [28].

¹⁸ Boolos [3] has given a relative consistency proof to the effect that Frege arithmetic is consistent provided that second-order Peano arithmetic is. See Fine [17], for a comprehensive study of the philosophical and logical aspects of Fregean abstraction and "the Bad Company Objection".

2.4 Fregean Abstraction Principles

Basic Law V and HP are examples of (Fregean) abstraction principles, i.e., principles of the form:

$$\$F = \$G \leftrightarrow F \sim G,$$

where $\sim$ is an equivalence relation between concepts and $\$F$ and $\$G$ are objects representing “equivalence classes” of concepts with respect to the relation $\sim$. A Fregean abstraction principle may be viewed as postulating the existence of a mapping $\$$ from equivalence classes of concepts to objects. In particular, Frege’s Basic Law V postulates the existence of such a mapping from equivalence classes of concepts to objects that is one-to-one. This means that there must exist at least as many objects as there are concepts. On the other hand, the strong axioms of comprehension for concepts imply that there are more concepts than there are objects. Thus, we get a contradiction. HP is also a strong assumption, but it does not, as far as one knows, lead to inconsistency. In the context of second-order logic with unlimited comprehension principles for concepts, HP implies that there are infinitely many objects, i.e., any model of Frege arithmetic has to be at least denumerably infinite. Thus, Fregean abstraction principles can be very powerful as HP, or even inconsistent as Frege’s Basic Law V.

Wright and Hale argue that Frege’s theorem is of great philosophical importance. That is, they think that HP can be viewed as an implicit definition of the concept of a cardinal number, and therefore as an analytic truth. Given the truth of HP, one can prove that infinitely many cardinal numbers exist: Consider the empty concept $[x:x \neq x]$ of being an object which is not identical with itself. This concept exists by concept comprehension. Let E be this concept. Then, the formula $E \approx E$ is logically true. Hence, by HP the following is true: $NxE(x) = NxE(x)$. Now, according to the Fregean semantics, adopted by Hale and Wright, this sentence can be true only if the singular term $NxE(x)$ refers to an object. Hence there must exist some object that is the number of things that are not self-identical. But this object is by definition the cardinal number 0. Once 0 has been proved to exist, one can prove the existence of $1 = NxE(x = 0)$, in a completely analogous way, Thus, Wright and Hale claim that it is analytically true and apriori that there are infinitely many cardinal numbers.

The “Scottish neo-logicists” Hale and Wright do not argue that Hume’s principle is a logical truth. Instead they claim that it (or, some modified version of it) is an analytic truth concerning the concept of a cardinal number. Hence, they give up Frege’s idea of a strict reduction of arithmetic to logic, while keeping the Fregean doctrine that arithmetic has a foundation that is analytical and apriori. It is part of their neo-logicist programme to try to show that also other areas of mathematics can be logically based on analytically true abstraction principles. The programme is based on the conviction that substantial portions of mathematics can in this way be shown to be analytically true and apriori.

2.5 Contributions to this Volume

This book contains four papers, by John Burgess, Øystein Linnebo, Neil Tennant, and Stewart Shapiro, discussing various aspects of logicism and neo-logicism.

Burgess investigates the attenuated form of logicism that was introduced by Richard Jeffrey under the name *logicism lite*. According to this view, mathematics, although not reducible to logic, is a theoretical superstructure built upon logic and testable against logical data that are given in the form of logically valid schemata. Thus, “logicism lite” can be seen as a kind of instrumentalism: mathematical theories are seen as instruments for efficiently drawing logically valid conclusions. On such a view, mathematical objects like numbers and sets are “useful fictions” and mathematical theories are neither true nor false. Although finding “logicism lite” technically interesting, Burgess rejects it as a general philosophy of mathematics, arguing that the type of instrumentalism and fictionalism that it represents is contrary to the realist attitude of the working mathematician.

Linnebo gives an explanation of our reference to the natural numbers along broadly Fregean lines, starting out from Frege’s famous *Context Principle*, according to which a word has meaning only in the context of a proposition. On Linnebo’s approach, the natural numbers are presented via numerals and questions about natural numbers are reducible to questions about numerals. Hence, the metaphysical status of natural numbers, Linnebo argues, is “thinner” than that of numerals.

Tennant’s paper is concerned with completing the constructive logicist programme, which he started in [52], of deriving the Dedekind-Peano axioms within a theory of natural numbers that also accounts for their role in counting finite collections of objects. According to this approach, the primitive concepts of arithmetic are introduced via Gentzen-style introduction and elimination rules within a system of natural deduction for intuitionistic logic. In the present paper, the goal is to show how the axioms of addition and multiplication can be introduced in a conceptually satisfying way within such a constructive logicist approach. For this purpose, Tennant develops a natural deduction system for the logic of orderly pairing. Orderly pairing is here treated as a logical primitive with its own introduction and elimination rules. This notion is then used to formulate introduction and elimination rules for addition and multiplication.

Shapiro, finally, considers various motivations that have been given for logicist programmes in the foundations of mathematics: the rationale of such a programme may be (i) mathematical, i.e., to prove anything that is capable of being proved; (ii) to provide mathematics with a foundation that is *epistemologically secure*, (iii) to determine the *epistemological source* of our mathematical knowledge; or (iv) to find a foundation for mathematics that is *not in need of further justification*. Shapiro then uses these aims as “yardsticks” to take the measure of Wright’s and Hale’s programme of Fregean neo-logicism.

3 Intuitionism and Constructive Mathematics

We give a brief sketch of the emergence of intuitionism and constructivism in mathematics. For a comprehensive account of the history, see Hesselning [31], and for the mathematical aspects we refer to Troelstra and van Dalen [54] and Dummett [16]. Kronecker's criticism in the 1870s of Cantor's transfinite set theory is often considered as the starting point of the development. His finitistic standpoint regarding mathematical objects is condensed in his famous dictum "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk".¹⁹ This view was of course at odds with set theory which had stipulated the existence of an abundance of abstract infinite objects. The success of set-theoretic methods in the theory of functions soon became undisputed and they were vigorously employed by Weierstrass and others. In 1904 Zermelo surprised the mathematical world by proving that the set of real numbers could be well-ordered (cf. van Heijenoort [29]). He had used his version of the axiom of choice, a principle that was already implicit in Cantor's set theory. Among the critics of his method were the so-called French semi-intuitionists Baire, Borel and Lebesgue. Borel and Baire contended that only countable choices were admissible in mathematics. Such choices could be performed one by one using induction on natural numbers. Lebesgue went further in the criticism and claimed that choices were only admitted if they followed some law. Borel emphasised mathematics as a human activity rather than a formal one, subjective but communicable. Poincaré objected to another aspect of set-theoretic methods, namely the use of impredicative definitions. He also argued against Russell and the logicist reduction of mathematics to logic.

Brouwer started in his thesis from 1907 *Over de Grondslagen der Wiskunde (On the Foundations of Mathematics)*, to analyse the situation in the foundations of mathematics.²⁰ He came to the conclusion that the use of logic as the basis of mathematics is unreliable. Instead it should be founded on (wordless) mental constructions and the intuition of time. Many of the views of the semi-intuitionists were shared by Brouwer. His analysis further concluded that it was not abstract mathematical objects that were the problem, but the unheeded application of the principle of excluded middle, in particularly when dealing with infinite objects (Brouwer [6]). He did not use formal logical language in his own writing. Intuitionistic logic was only formalised later by Heyting [32] in 1930; and partially already by Kolmogorov [37] in 1925. Also Brouwer's notion of mental construction was analysed and made precise in what is now called the *Brouwer-Heyting-Kolmogorov (BHK) interpretation*.

Brouwer's next important idea for intuitionism was his notion of a *choice sequence* (around 1917), which was obtained by a reflection on the intuition of time.

¹⁹ *God made the integers, everything else is Man's work.*

²⁰ English translations of several of Brouwer's papers can be found in van Heijenoort [29] and Mancosu [41]. See also Brouwer's *Collected Works* [7] edited by Heyting.

One may argue that a decision based on a given infinite proceeding sequence of data, integers or the like, must be taken after a finite length of time, unless one knows that the sequence obeys a certain law which makes it predictable. This has the consequence that any function of the totality, or “spread”, of sequences must depend continuously on the data. A remarkable result is then that all functions from real numbers to real numbers are continuous. This contradicts classical mathematics, where a step function provides an immediate counter example. Brouwer would object that such a function is not defined at the actual step point, as this would require a survey of an entire infinite sequence of rationals that defines the point on the real line.

Other implications of his continuity principles are the *Fan Theorem* (FT) and the *Bar Theorem* (BT). The former is a reformulation of König’s lemma and the second is a transfinite induction principle. Both are consistent with classical mathematics.

A computational and formalised model of the BHK-interpretation was provided by Kleene [36] with his so-called recursive realisability interpretation. This model refuted both FT and BT and showed that they had no immediate computational content in the sense of Church and Turing. On the other hand there are indirect constructive interpretations of choice sequences.

Kreisel and Troelstra (cf. Troelstra [53]) proved a conservation result that eliminated the need to use choice sequences in many circumstances. Models of choice sequences using topos theory were constructed by van der Hoeven and Moerdijk [34]. The use of the principles FT and BT in topology, e.g. as in the Heine-Borel theorem, can often be eliminated by the use of point-free spaces or locales. A crucial ingredient in the construction of such spaces is the inductive generation of covers, indeed also present in Kreisel’s first sketch of the elimination theorem in 1968; see also (Martin-Löf [42], pp. 77–78).

Further precise mathematical versions of the BHK-interpretation were obtained by introducing various constructive type theories. Curry found a similarity between axioms for propositional logic and types for certain combinators in his combinatory logic (Curry and Feys [11]), Howard in 1969 (Howard [35]), building on this observation, constructed a type theory for predicate logic. A type theory suitable for a full development of constructive mathematics was devised by Martin-Löf ([43, 44]). The correspondence between types and formulas (or propositions) goes under the name of the *Curry-Howard isomorphism*.

A different and very influential interpretation of intuitionistic arithmetic, the so-called *Dialectica interpretation*, is due to Gödel and was published in the paper *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*²¹ from 1958 (Cf. Gödel [25]). This interpretation came to have a great influence on much later development of the revised Hilbert programme. The Dialectica interpretation was also based on a type theory. Type theory provided a new and more fundamental justification of the intuitionistic logical principles, and provides in a sense an

²¹ On an extension of finitary mathematics which has not yet been used.

embodiment of Brouwer’s mental constructions. In any case, the philosophy behind type theory clearly puts mathematical constructions before formal logic.²²

Topos theory grew out of the work of Grothendieck on algebraic geometry in the late 1950s. The connection between this theory and model theory of intuitionistic logical systems became gradually clearer with the introduction of the elementary topos by Lawvere and Tierney 1969–1970 and Joyal’s generalisation of Kripke- and Beth-semantics. Category theory, the basis of topos theory, reduces logic—just as type theory does—to very simple and basic mathematical constructions. This is not accidental as there are strong connections to the Curry-Howard isomorphism (Cf. Lambek and Scott [38]). Indeed, Lawvere suggested in [39] that category theory could be made the foundation of mathematics on par with set theory.

Prior to the introduction of abstract set-theoretic notions and the modern notion of function, there was no pressing need to consider a separate realm of constructive mathematics. Deliberate attempts to distinguish such a realm came only after these notions became widely used. Constructive mathematics in a wide sense includes finitism and French semi-intuitionism, where the logic is still classical, and Brouwer intuitionism and Markov school constructive mathematics (from 1950s), in which the logic is intuitionistic. See [54] and [5] for surveys. An important legacy of intuitionism is so called *Bishop-style constructive mathematics*, which has developed into a largely independent mathematical field. The pioneer, Errett Bishop, shared much of Brouwer’s anti-formalist and anti-logicist stance, in that mathematical constructions are considered to be prior to logic. Constructive mathematics, in a narrow sense, tends nowadays to be identified with the development initiated in Bishop’s book *Foundations of Constructive Analysis* [2] from 1967. It has its roots firmly in Brouwer’s intuitionism, and is indeed building on much of the results of his school. However it makes certain generalisations and more modest assumptions on the mathematical ontology. As a result it is intelligible from a classical set-theoretic viewpoint, as well from the viewpoint of computability, via recursive realisability or through generalised inductive definitions as in Martin-Löf type theory, or Aczel-Myhill constructive set theory.

3.1 Contributions to this Volume

We now turn to the individual contributed articles on intuitionism of the present volume. Martin-Löf analyses Zermelo’s axiom of choice through an interpretation in a classical extension of a basic constructive type theory and identifies the cru-

²² Modern constructive type theories such as the *Calculus of Constructions* [10] may also be considered as examples realising the logicist programme. For instance natural numbers may be defined by writing down a single type $N = (\Pi X)(X \rightarrow (X \rightarrow X) \rightarrow X)$, where ΠX denotes quantification over all types. All natural numbers and their properties can be constructed or derived, by applying abstraction and application to that type.

cial point where its justification is non-constructive. However, in the basic theory, the countable axiom of choice can actually be proved due to the meaning of the existential quantifier.

The mentioned axioms FT and BT make it possible to work freely with point-wise continuity, and derive, for instance, the Heine-Borel Theorem about uniform continuity on compact intervals. This volume contains two papers studying the role of FT. Berger and Schuster show in their paper that FT, in a restricted form, is equivalent to Dini's theorem. Veldman shows in his paper that FT is equivalent to a two-dimensional approximate version of Brouwer's fixed point theorem. These results are instances of so called reverse mathematics: deriving axioms from specific important mathematical theorems over a weak and constructive base theory. This makes it possible to judge exactly which axioms are needed to obtain a certain theorem.

The lack of numerical (or computational) meaning of the FT axiom led Errett Bishop, along with others, to consider uniformly continuous functions as the basic continuous maps. An obvious obstacle with this choice is that there must be a way of expressing uniformity, as indeed there is in metric or uniform spaces. Going beyond such spaces, various ideas of furnishing spaces with more information has been developed. In apartness spaces, a fundamental idea is that the assertion that points are apart is more informative than an assertion about equality: equality may in such spaces be defined from a basic apartness relation, whereas the converse is usually not possible constructively. Bridges and Vîță, in their contribution to this anthology, give an overview of research in this area. Another approach is point-free topology, where the covering relation between neighbourhoods is the basic information. This allows for good notions of compactness and continuity, in fact, eliminating the need for axioms FT or BT. Two contributions on locale theory are included here. Aczel gives a constructive version of Lusin's separation theorem that avoids use of the BT axiom, employed in early intuitionistic versions of the theorem, with the help of point-free topologies. In the paper of Palmgren an application of point-free methods to the foundations of homotopy theory is given.

Various sharpenings of the constructive position in mathematics have been suggested over the years. One is to take into account not only computability, but also computational complexity. Ishihara's paper shows how a theory of real numbers can be developed on the assumption that their fundamental sequences of rational approximations are computable in a certain complexity class. Related to this is the approach in Schwichtenberg's paper, which refines notions of constructive analysis to make explicit witnesses to existence theorems. In addition he shows that a computer program for computing roots of functions may be extracted from a proof of the intermediate value theorem.

Dummett [15] considered the philosophical justifications for classical and intuitionistic logic from the meaning-as-use perspective. His conclusion is that the *principle of bivalence*, i.e. that every proposition is either true or false, has to be rejected as a logically valid principle, thus undermining the realist justification of classical logic. The debate of his anti-realist argument has been going on since then, with contributions by Prawitz among others. Dummett has argued that the rejec-

tion of bivalence—for meaning-theoretic reasons—has metaphysical consequences. According to this view, there are meaning-theoretic reasons for giving up a realist metaphysics.

Pagin has criticised Dummett's argument against metaphysical realism and argued that it contains a gap. The fact that bivalence has to be given up as a principle of logic—for meaning-theoretic reasons—does not imply that the principle cannot still be true. According to this line of reasoning, a realist metaphysics is not refuted by Dummett's meaning-theoretic arguments against the validity of bivalence. What is needed according to Pagin to close the gap between meaning theory and metaphysics is the principle:

(P) If A is true, then A is provable.

If (P) is provable, metaphysical realism (the *truth* of bivalence) will entail its provability. So, Dummett's meaning-theoretic argument against bivalence as a logical principle, would have metaphysical consequences after all. Pagin, however, has argued that the principle (P) is not intuitionistically acceptable and cannot be proved. Prawitz disagrees and has actually tried to prove a formal counterpart of (P). In his contribution to this volume, Pagin criticises Prawitz' argumentation. If there is no reason for holding on to (P), Pagin argues, there is no reason coming from meaning theory to doubt the truth of bivalence (i.e., metaphysical realism) either.

4 Formalism

A milestone in mathematics is Hilbert's *Grundlagen der Geometrie* [33] from 1899. Its importance for the conceptual development of modern mathematics is difficult to overstate. Here Hilbert gave, for the first time, a fully precise axiomatization of Euclidean geometry. The entities like point, line and plane are defined only implicitly by their mutual relations. Generalising this method of implicit definitions it became possible to work also with complicated mathematical systems characterised axiomatically up to structural equivalence or isomorphisms. Hilbert's structuralist approach, of course, goes back to Dedekind's characterisation in [12] of the natural number system in terms of simply infinite systems. It was also foreshadowed by Felix Klein's classification of geometries using group invariants (the Erlangen programme).

At the time of writing *Grundlagen der Geometrie*, Hilbert subscribed to the view that mathematical truth and existence simply means consistency. In a famous letter to Frege of December 29 1899, he wrote: "As long as I have been thinking, writing and lecturing on these things, I have been saying ... if the arbitrary given axioms do not contradict each other with all their consequences, then they are true and the things defined by them exist. This is for me the criterion of truth and existence." This structuralist approach of Hilbert made it possible for him to be indifferent to

the ontological questions about the nature of mathematical objects that were of such great concern to Frege.²³

The formalistic view of mathematics was shaken by the inconsistencies discovered in naive set theories, e.g. the Russell paradox and the Burali-Forti paradox. There were various cures proposed for the childhood illness of modern mathematics. One was to forbid impredicative definitions as prescribed by Poincaré, Borel and Russell. Another was Brouwer's, to restrict the use of classical logic. There were of course deep contentual considerations behind these suggestions.

4.1 Hilbert's Proof-Theoretic Programme

Throughout his career, Hilbert had a deep interest in foundational questions, although his views went through many changes. Around the turn of the century, Hilbert's foundational point of view was close to Dedekind's brand of logicism. Like Dedekind he thought that he could prove the consistency of fundamental mathematical theories like analysis and Euclidian geometry by constructing logical (or set-theoretic) models. This belief was shattered, however, by the discovery of the set-theoretic paradoxes. After that, Hilbert turned to the idea of providing fundamental mathematical theories, like arithmetic, analysis and set theory, with direct meta-mathematical proofs of consistency. In this way Hilbert thought that he could defend abstract infinitistic mathematics from constructivist critics like Kronecker, Brouwer and Weyl. In particular, he wished to respond to Brouwer's criticism of the unheeded use of the law of excluded middle in arguments about infinite objects. However, *Hilbert's proof-theoretic programme* took a long time to reach its mature form in the beginning of the 1920s. By taking a constructivist, or finitist, position, Hilbert attempted to provide a justification for abstract mathematics going way beyond the constructive basis. In spite of the heated "Grundlagenstreit" between Brouwer and Hilbert, and their followers, in the 1920s, Hilbert's program is best viewed as an attempt at mediating between classical and constructive mathematics.²⁴

According to the formalist view, a consistent system, formalising a sufficiently rich ontology, was all that was required to carry out abstract mathematics, with all its ideal objects. Hilbert's idea was to prove the consistency of such a formal system in a finitistic system that Brouwer, or other constructivists, could not object to. This would save classical mathematics from constructivist criticism. For this purpose Hilbert devised his proof theory, which studies proofs of a formal system, and aim to show that no absurdity could ever be derived according to the rules of the system.

²³ See Frege [21] for the correspondence between Frege and Hilbert. See also Shapiro's paper "Categories, structures, and the Frege-Hilbert controversy: the status of metamathematics" in this volume for a philosophical analysis of Frege's and Hilbert's respective views on the role of mathematical axioms and the relationship between truth, consistency and mathematical existence.

²⁴ See Sieg's contribution to this volume [50] as well as Sieg [49] for a detailed analysis the development of Hilbert's foundational views. In this connection Zach [57] is also useful.