LECTURE NOTES ON COMPOSITE MATERIALS

SOLID MECHANICS AND ITS APPLICATIONS
Volume 154

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Lecture Notes on Composite Materials

Current Topics and Achievements

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Preface

Composite materials are heterogeneous by nature, and are intended to be, since only the combination of different constituent materials can give them the desired combination of low weight, stiffness and strength. At present, the knowledge has advanced to a level that materials can be tailored to exhibit certain, required properties. At the same time, the fact that these materials are composed of various, sometimes very different constituents, make their mechanical behaviour complex. This observation holds with respect to the deformation behaviour, but especially with respect to the failure behaviour, where complicated and unconventional failure modes have been observed.

It is a challenge to develop predictive methods that can capture this complex mechanical behaviour, either using analytical tools, or using numerical methods, the finite element method being the most widespread among the latter. In this respect, developments have gone fast over the past decade. Indeed, we have seen a paradigm shift in computational approaches to (composite) material behaviour. Where only a decade ago it was still customary to carry out analyses of deformation and failure at a macroscopic level of observation only – one may call this a phenomenological approach – nowadays this approach is being progressively replaced by multiscale methods. In such methods it is recognized a priori that the overall behaviour is highly dependent on local details and flaws. For instance, local imperfections in spacing and direction of fibres can be detrimental to the overall bearing capacity of a structure that is composed of such a fibre-reinforced composite material. By upscaling, homogenization or methods that in a single calculation take into account the behaviour at different scales, an attempt is made to design numerical methods that have a wider range of applicability – by less reliance on adhoc assumptions – and are better rooted in the true physical behaviour of the constituent materials. Yet, few monographs have been published that present an account of recent developments in the analytical/numerical modelling of composite materials.

This volume – which has grown out of a series of lectures that has been given at Lublin University of Technology within the framework of the European

Community Marie-Curie Transfer-of-Knowledge project *Modern Composite Materials Applied in Aerospace, Civil and Sanitary Engineering: Theoretical Modelling and Experimental Verification* (contract MTKD-CT-2004-014058) – aims to fill this gap. It starts by a comprehensive account of methods that can be used at macroscopic level, followed by a précis of recent developments in modelling the failure behaviour of composites at a mesoscopic scale. Going down further, the third chapter treats fundamental concepts in micromechanics of composite materials, including the essential concept of the Representative Volume Element and Eshelby's method. As recognized widely, failure is seldom a consequence of pure mechanical loadings. Often, thermal effects and long-term effects for instance due to hygric or chemical actions play an important role as well. For this reason the ensuing two chapters are devoted to thermal shocks and the numerical treatment of diffusion phenomena in addition to mechanical loadings when describing failure in heterogeneous materials. The volume is completed by a review of fracture mechanics tools for use in the analysis of failure in composite materials.

Eindhoven and Lublin,
René de Borst and Tomasz Sadowski

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Contents

ANALYSIS OF HOMOGENEOUS AND NON-HOMOGENEOUS PLATES

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Abstract Plate theory is an old branch of solid mechanics – the first development of a general plate theory was made by Kirchhoff more than 150 years ago. After that many improvements were suggested; at the same time some research was focussed on the establishment of a consistent plate theory. Plate-like structural elements are widely used in classical application fields like mechanical and civil engineering, but also in some new fields (electronics, medicine among others). This paper gives a brief overview of the main theoretical directions in the theory of elastic plates. Additional information is available in the literature.

Keywords: structural analysis, plates, homogeneous and non-homogeneous cross-sections

1. Classification of structural models

Plates are structural elements with applications in various branches. The reason for this is that plates combine high bearing capacities with low weight (excellent specific stiffness properties). Modern plate structures are made from different materials – it is common to use classical structural materials like steel or concrete, but also modern composite materials like laminates. Increasing safety requirements dictate necessity of improving the analysis of plates. Since all commercial *Finite Element* codes allowing their analysis have special plate elements, but the manuals do give not enough theoretical background, an overview of the modeling approaches in the plate theory will be given.

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1.1 Introductional remarks

The basic problems in engineering mechanics are the analysis of strength, vibration behavior and stability of structures with the help of structural models. Structural models can be classified, for example, by their

- Geometrical (spatial) dimensions
- Applied loads
- Kinematical and/or statical hypotheses approximating the behavior

A complex structure can be built up of many individual structural elements; the behavior of the whole structure includes the interaction of all parts.

Let us introduce three basic classes of structural elements. The *first* one is the class of three-dimensional structural elements which can be defined as follows:

> *A three-dimensional structural element has three spatial dimensions of the same order; there is no predominant dimension.*

Typical examples of geometrically simple, compact structural elements in the theory of elasticity are shown in Fig. 1.

The *second* is the class of two-dimensional structural elements which can be defined as follows:

> *If two spatial dimensions have the same order, and the third, which is related to the thickness, is much smaller, one has a two-dimensional structural element.*

Typical examples of two-dimensional structural elements in civil engineering/structural mechanics are shown in Fig. 2.

The *last* class is related to one-dimensional structural elements which can be defined as follows:

> *Two spatial dimensions, which can be related to the cross-section, have the same order. The third dimension, which is related to the length of the structural element, has a much larger order in comparison with the cross-section dimension orders.*

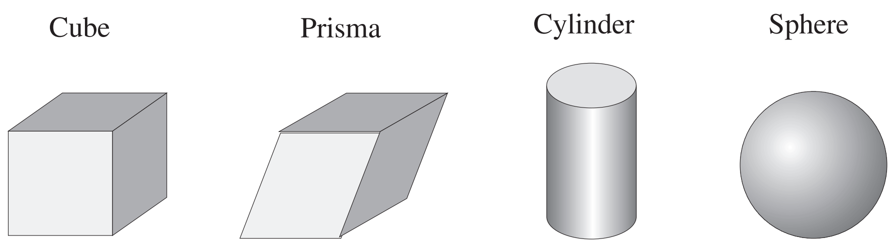


Figure 1. Examples of simple three-dimensional structural elements

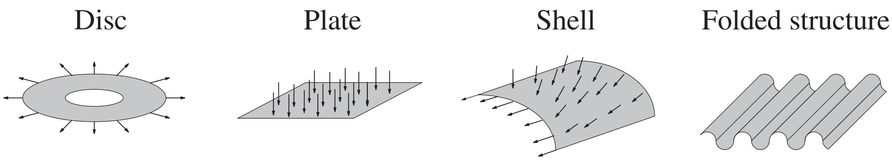


Figure 2. Examples of simple two-dimensional structural elements

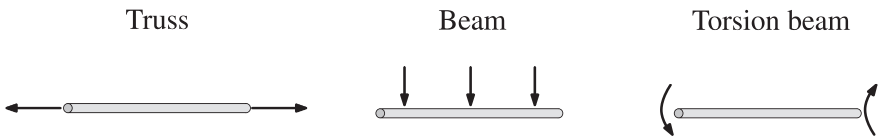


Figure 3. Examples of simple one-dimensional structural elements

Typical examples in engineering mechanics are shown in Fig. 3.

It is possible to introduce other classes. For example, in shipbuilding, thin-walled structural elements are often used. These are thin-walled light-weight structures with a special profile, and they require an extension of the classical one-dimensional structural models:

> *If the spatial dimensions are of significantly different order and the thickness of the profile is small in comparison to the other cross-section dimensions, and the cross-section dimensions are much smaller in comparison to the length of the structure one can introduce quasi-onedimensional structural elements.*

Suitable theories for the analysis of quasi-onedimensional structural elements are:

- Thin-walled beam theory (Vlasov-Theory; Vlasov, 1958) and
- Semi-membrane theory or generalized beam theory (Altenbach et al., 1994)

Typical thin-walled cross-section profiles are shown in Fig. 4.

1.2 Two-dimensional structures – definition, applications, some basic references

Let us introduce the definition of a two-dimensional structure:

> *A two-dimensional load-bearing structural element is a model for analysis in Engineering/Structural Mechanics, having two geometrical dimensions which are of the same order and which are significantly larger in comparison with the third (thickness) direction.*

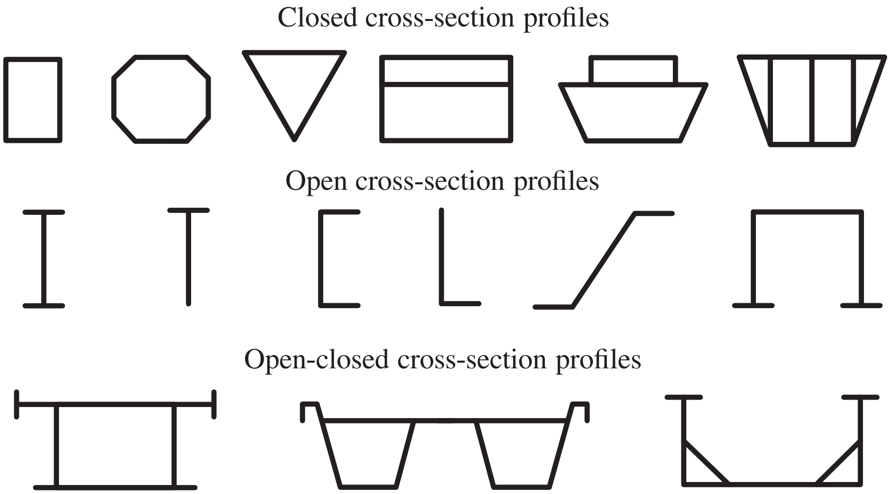


Figure 4. Various profiles of thin-walled structures

This definition does not contain any restriction on the type of loading (in-plane, transverse, etc.). In addition, the thinness hypotheses (see, e.g., Başar and Krätzig, 1985) is not specified.

The mathematical consequence is obvious: instead of a three-dimensional problem, which is represented by a system of coupled partial differential equations with respect to three spatial coordinates, one can analyze a two-dimensional problem, which is described by a system of coupled partial differential equations with respect to two spatial coordinates. The two coordinates represent a surface; the behavior in the thickness direction is approximated mostly by use of engineering assumptions. The transition from a three-dimensional to a two-dimensional problem is non-trivial, but the solution effort decreases significantly and the possibility of solving problems analytically increases.

Two-dimensional structures have many applications in various branches: thin homogeneous plates, thin inhomogeneous plates (laminates, sandwiches), plates with structural anisotropy, moderately thick homogeneous plates, folded plates, membranes, biological membranes, etc. The main industrial branches for plate applications are aeronautics and aircraft industries, automotive industries, shipbuilding industries, vehicle systems, civil engineering, medicine, ...

During the last few years many scientific papers, textbooks, monographs and proceedings about the state of the art and recent developments in plate theory have been published. Some of the most important publications are listed here without comment.

- Review articles: Naghdi, 1972; Grigolyuk and Kogan, 1972; Grigolyuk and Seleznev, 1973; Reissner, 1985; Noor and Burton, 1989a,b; Noor and Burton, 1990a,b; Reddy, 1990; Irschik, 1993; Burton and Noor, 1995; Noor et al., 1996
- Monographs and textbooks: Panc, 1975; Kączkowski, 1980; Girkmann, 1986; Timoshenko and Woinowsky Krieger, 1985; Ambarcumyan, 1987; Gould, 1988; Reddy, 1996; Altenbach et al., 1998; Woźniak, 2001; Zhilin, 2007
- Actual conferences like EUROMECH 444 (Kienzler et al., 2004), Shell Structures Theory & Applications (Pietraszkiewicz and Szymczak, 2005), IUTAM Symposium Relation of Shell, Plate, Beam and 3D Models (Jaiani and Podio-Guidugli, 2008)

1.3 Formulation principles, historical remarks

The plate equations can be deduced as follows (Altenbach, 2000b; Wunderlich, 1973):

- Starting from a 3D continuum and
- Starting from a 2D continuum

If one starts from the 3D continuum there are two possibilities:

- The use of hypotheses to approximate the three-dimensional equations and
- The use of mathematical approaches (series expansions, special functions, etc.) to develop a set of two-dimensional equations

All these methods have advantages and disadvantages and it is difficult to say in advance which is the best method for derivating the governing equations. In addition, it can often be shown that different methods lead to identical sets of equations.

Engineers prefer theories which are based on hypotheses. For example, there are many theories which are based on displacement approximations. Let us consider the plate geometry as shown on Fig. 5. The three displacements u_i in the classical three-dimensional continuum are now split into in-plane displacements u_α and the deflection w. The first theory of plates based on displacement assumptions, was presented by Kirchhoff, 1850. The modern form of the basic assumptions, which can be used for homogeneous and non-homogeneous plates, is

$$\begin{aligned} u_1(x_1,x_2,z) &\approx u_1^0(x_1,x_2) - z w_{,1}(x_1,x_2), \\ u_2(x_1,x_2,z) &\approx u_2^0(x_1,x_2) - z w_{,2}(x_1,x_2), \\ w(x_1,x_2,z) &\approx w(x_1,x_2) \end{aligned} \tag{1}$$

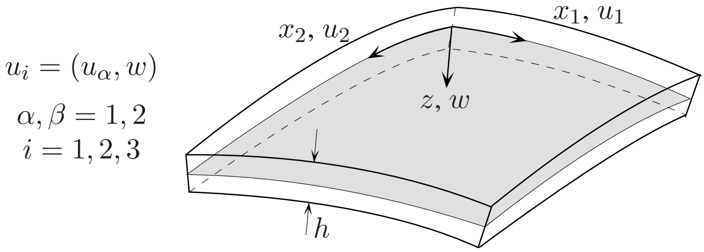


Figure 5. Plate geometry and displacements

or using the Einstein's summation convention

$$u_\alpha(x_\beta, z) \approx u^0_\alpha(x_\beta) - zw_{,\alpha}(x_\beta), \qquad w(x_\beta, z) \approx w(x_\beta), \tag{2}$$

where $u_\alpha(x_\beta, z)$ are the three-dimensional displacements, $u^0_\alpha(x_\beta)$ are the displacements of the reference surface (usually this surface is assumed to be the mid-plane) and $w(x_\beta, z)$ are the three-dimensional deflections which are approximately equal to the two-dimensional deflections $w(x_\beta)$. $(\ldots)_{,\alpha}$ denotes the derivative with respect to the in-plane coordinates x_α.

Approximately 100 years later this theory was improved (see, for example, Hencky, 1947; Mindlin, 1951)

$$u_\alpha(x_\beta, z) \approx u^0_\alpha(x_\beta) + z\varphi_\alpha(x_\beta), \qquad w(x_\beta, z) \approx w(x_\beta) \tag{3}$$

In this equations $\varphi_\alpha(x_\beta)$ are the cross-section rotations. Comparing both approaches, one can see that the improvement was realized by introducing additional degrees of freedom. Calculating the strains as usual in the theory of elasticity, one gets

- Neither theory takes into account the thickness changes and
- The Kirchhoff theory leads to zero transverse shear, while the improved theory considers transverse shear in an approximate sense

The introduction of independent rotations is in some cases not enough, since it is assumed that any cross-section will be plane before and after deformation. For example, for plates made from rubber-like materials, the assumption of plane cross-section is not valid. A weaker assumption was proposed by Levinson (1980) and Reddy (1984) among others

$$u_\alpha(x_\beta, z) \approx u^0_\alpha(x_\beta) - [w_{,\alpha}(x_\beta) + \varphi_\alpha(x_\beta)]\, \frac{4z^3}{3h^2}, \qquad w(x_\beta, z) \approx w(x_\beta) \tag{4}$$

The latter representation and the Kirchhoff or Mindlin plate equations can be discussed from the point of view of introducing additional degrees of freedom.

On the other hand all equations of this type can be understood as some part of a power series. The first suggestion of this type was made by Lo et al., 1977a. Some kind of generalization of the power series approach was given in Meenen and Altenbach, 2001

$$\begin{aligned} u_\alpha(x_\beta, z) &= \sum_{q=0}^{K_1} u_\alpha^q(x_\beta)\phi^q(z) + \sum_{q=0}^{K_2} w_{,\alpha}^q(x_\beta)\psi^q(z), \\ w(x_\beta, z) &= \sum_{q=0}^{K_2} w(x_\beta)^q \chi^q(z) \end{aligned} \tag{5}$$

The disadvantage of this approach is that, with an increasing number of terms in the series the physical interpretation of all terms is impossible.

In addition, the method of hypotheses for the stress and/or the strain (displacement) states was applied in Reissner (1944, 1945, 1947), Bollé (1974a,b) and Kromm (1953). It is easy to show that, for example, Mindlin's and Reissner's theories contain partly identical equations, but the coefficients differ slightly, and the interpretations are not the same.

Purely mathematical approaches are mostly based on power series, trigonometric functions, or special functions, etc. (see, e.g., Lo et al., 1977a,b; Kienzler, 1982; Preußer, 1984; Touratier, 1991). The mathematical approaches are very helpful if one wants to check the accuracy of the given approximation. A nice comparison of the different approximations in the series approach is given in Kienzler (2002).

The direct approach is based on the *a priori* introduction of an two-dimensional deformable surface. This approach was applied by Günther (1961), Green et al. (1965), Naghdi (1972), Rothert (1973), Zhilin (1976, 1982), Palmow and Altenbach (1982), Robin (2000), etc. This approach is still under discussion since the application is not trivial; but the direct theories are mathematically and physically so strong and as exact as three-dimensional continuum mechanics.

2. Classical plate theories

Below, we discuss some aspects of classical plate theories. Classical plate theories are theories based on hypotheses and which are mostly used in engineering practice.

2.1 Small deflections

In contrast to other classifications, here we first discussing two types of models: small and large deflection models. There are two basic theories used in practice: the Kirchhoff and the Mindlin theories. In the simplest case, both are

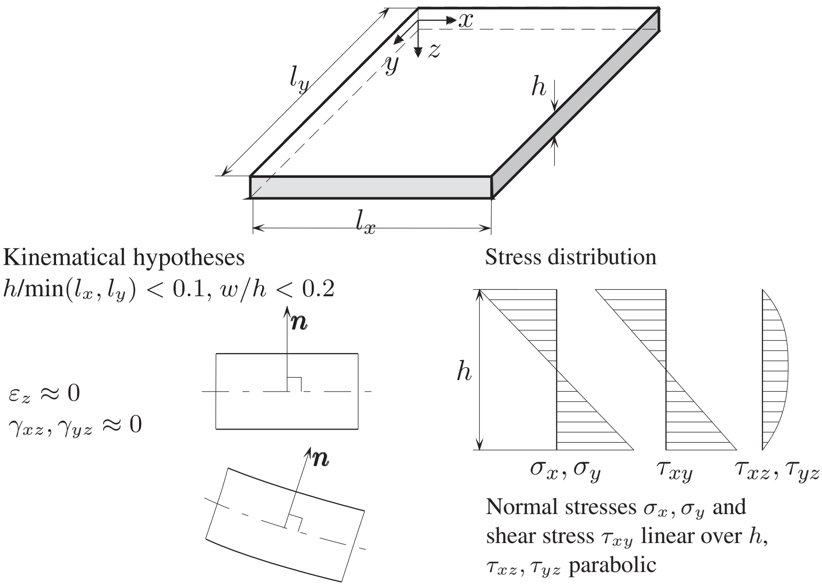


Figure 6. Kirchhoff plate – basic assumptions

restricted to small deflections (that means less than 0.2 of the plate thickness). The basic assumptions of Kirchhoff plate theory are shown in Fig. 6. They can be summarized as follows: no thickness changes, no transverse shear, linear distributions of the in-plane stresses, and parabolic distribution of the transverse shear stresses. The model can be applied to plates made of classical isotropic materials, and for small deflections; it is assumed that any cross-section it must be plane and orthogonal to the mid-plane before and after deformation.

Since the Kirchhoff model omits transverse shear, the Mindlin model is a suitable improvement. Now the assumption of plane cross-sections before and after deformation holds valid, but the cross-sections are no longer orthogonal after deformation. This assumption leads to additional degrees of freedom – two independent rotations. The basic assumptions of the Mindlin plate theory are shown in Fig. 7. They can be summarized as follows: no thickness changes, constant transverse shear, linear distributions of in-plane stresses, and constant distribution of the transverse shear stresses. The model can be applied to plates made of composite materials (e.g. sandwiches) and to relatively thick plates. The Mindlin theory is a transverse shear deformable theory.

2.2 Large deflections

For large deformations, we have two special cases which are important for engineering praxis: the membrane model and the von Kármán model. The basic

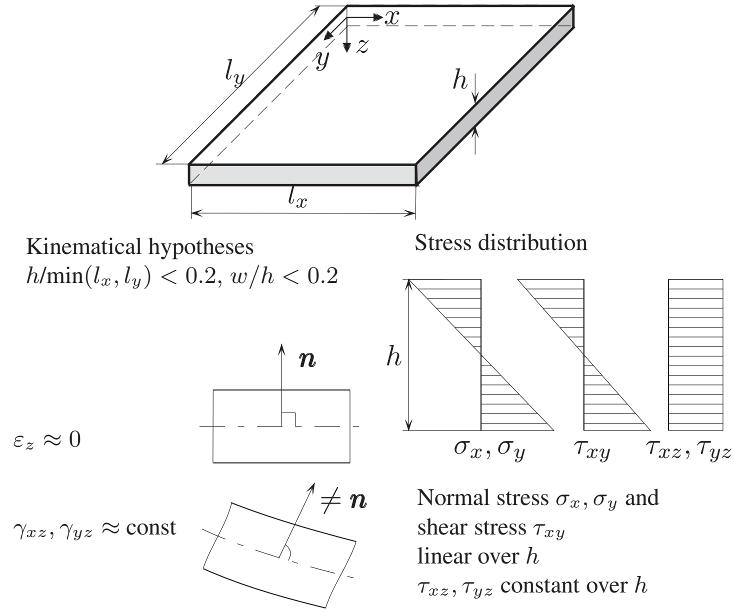


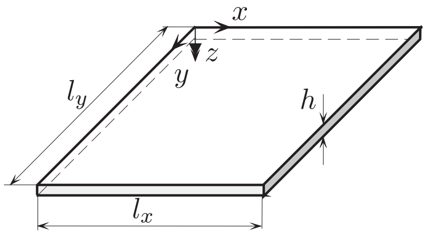
Figure 7. Mindlin plate – basic assumptions

assumptions for the first are given in Fig. 8; there are no changes across the thickness. This model cannot describe shear stresses. Note that the membrane model is a specific structural model, since we assume a similar behavior under tension and compression. In practice a membrane is unable to respond compression.

Another way to describe large deflections is through the von Kármán model. The basic assumptions are shown in Fig. 9. The von Kármán plate theory was introduced as a engineering theory. The possibility of deducing the basic equations from the three-dimensional non-linear continuum mechanics is still under discussion, see, e.g., Ciarlet, 1990. A possible solution is given in Meenen and Altenbach, 2001.

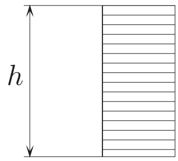
2.3 Kirchhoff plate

Let us discuss briefly some basic features of the classical theories. As we mentioned earlier the first set of equations was given within the framework of the Kirchhoff plate theory. The possible loading cases in plate theory are shown in Fig. 10. The basic kinematic relations of the Kirchhoff plate are illustrated in Fig. 11. In the Mindlin theory, the rotations are independent entities; in Kirchhoff theory they are derivatives of the deflection, and so have simple geometrical interpretation.



Assumptions
$h \ll \min(l_x, l_y),\ w/h \geq 0.5$
$\tau_{xy}, \tau_{xz}, \tau_{yz}, \sigma_z \approx 0$

Stress distribution



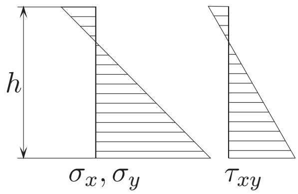
σ_x, σ_y
(tension stresses)

No shear stresses!

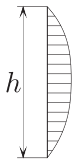
Figure 8. Membrane – basic assumptions

Assumptions
$h/\min(l_x, l_y) < 0.1$,
$0.2 < w/h < 5$

Stress distribution

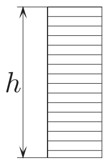


shear rigid
$\varepsilon_z, \gamma_{xz}, \gamma_{yz} \approx 0$



shear rigid model
τ_{xz}, τ_{yz}

shear deformable
$\varepsilon_z \approx 0, \gamma_{xz}, \gamma_{yz} \approx$ const



shear deformable model
τ_{xz}, τ_{yz}

Figure 9. Von Kármán plate – basic assumptions

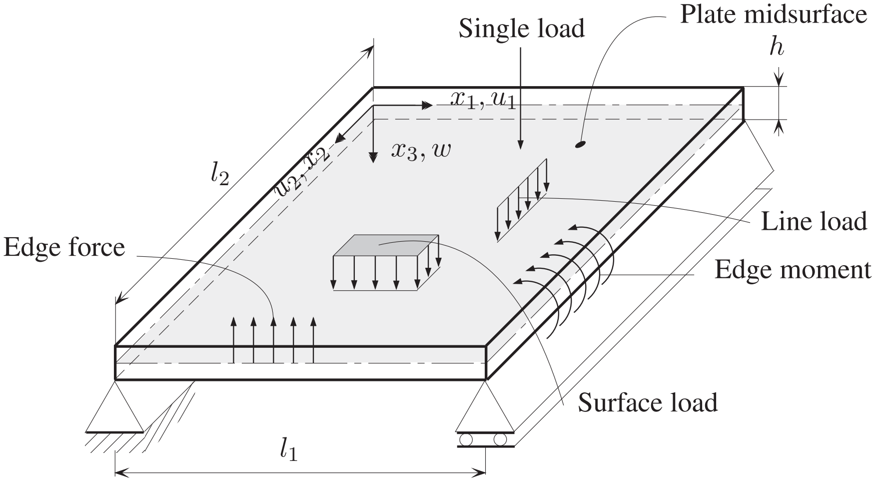


Figure 10. Loading of a plate

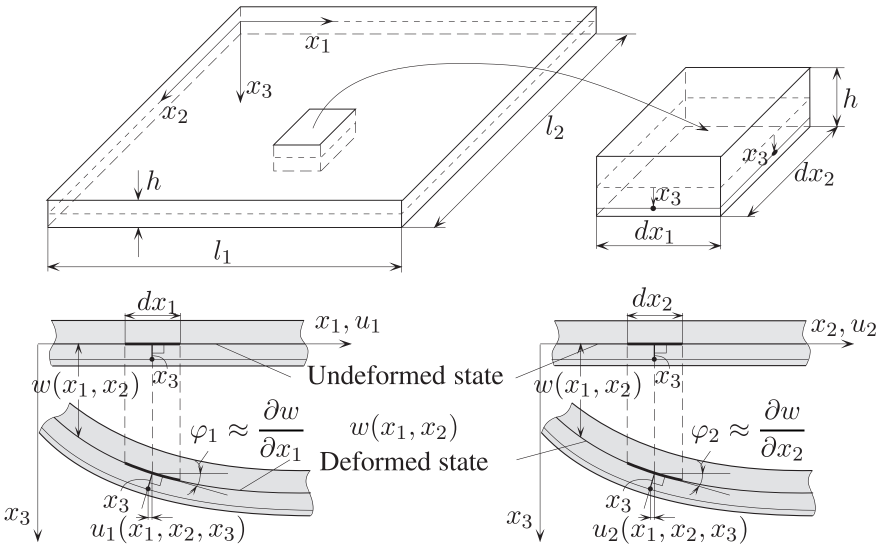


Figure 11. Kinematics of the Kirchhoff plate

In addition, in plate theory stress resultants are used instead of stresses. Such resultants are known from the strength of materials. For the Kirchhoff plate there are bending and torque moments and shear forces. They are shown in Fig. 12 for the equilibrium state.

The formulation of the boundary conditions for the Kirchhoff plate is non-trivial and widely discussed in the literature, see Altenbach et al. (1998) among

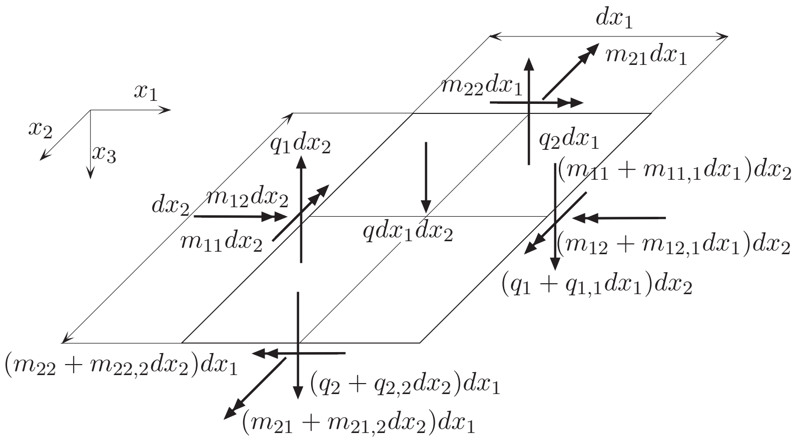


Figure 12. Stress resultants of the Kirchhoff plate

others. The reason for this is that the deflections of the Kirchhoff plate are described by partial differential equation of the fourth order, but in the general case one has to prescribe three boundary conditions at each edge. This problem is better solved in the framework of the Mindlin or Reissner theory (both are based on sixth order equation for the deflection). In addition, the Kirchhoff theory is characterized by a special description of the edge and corner forces.

Let us summarize the basic equations of the Kirchhoff theory for a rectangular plate with constant bending stiffness K (Δ is the two-dimensional Laplace operator)

- Bending equation for simply supported plate

$$K\Delta\Delta w(x_1, x_2) = q(x_1, x_2)$$

- Bending equation for elastically supported plate

$$K\Delta\Delta w(x_1, x_2) = q(x_1, x_2) - cw(x_1, x_2)$$

- Bending vibration equation for simply supported plate

$$K\Delta\Delta w(x_1, x_2, t) + \rho h\ddot{w}(x_1, x_2, t) = q(x_1, x_2, t)$$

- Bending vibration equation for elastically supported plate

$$K\Delta\Delta w(x_1, x_2, t) + \rho h\ddot{w}(x_1, x_2, t) = q(x_1, x_2, t) - cw(x_1, x_2, t)$$

In these equations ρ, h, q, c are the density of the plate material, the plate thickness, the distributed external transverse load and the Winkler foundation property, respectively. The dot denotes the time derivative. The bending stiffness can be calculated in the simplest case of constant thickness and elastic properties as $K = Eh^3/12(1-\nu^2)$ with E as the Young's modulus and ν as the Poisson ratio. As shown in Altenbach et al. (1996) and Altenbach et al. (2004) this approach can be easily extended to laminates (**C**lassical **L**aminate **T**heory).

2.4 Mindlin plate

The kinematics of the Mindlin plate differ from the Kirchoff kinematics: now two additional rotational degrees of freedom are introduced (Fig. 13). The basic equations of the Mindlin theory can be presented as follows:

$$Gh_S(\Delta w + \Phi) + q = \rho h \ddot{w},$$

$$\frac{K}{2}[(1-\nu)\Delta\psi_1 + (1+\nu)\Phi_{,1}] - Gh_S(\psi_1 + w_{,1}) = \frac{\rho h^3}{12}\ddot{\psi}_1,$$

$$\frac{K}{2}[(1-\nu)\Delta\psi_2 + (1+\nu)\Phi_{,2}] - Gh_S(\psi_2 + w_{,2}) = \frac{\rho h^3}{12}\ddot{\psi}_2$$

Here the abbreviation $\psi_{1,1} + \psi_{2,2} = \Phi$ is used. In addition, G is the shear modulus and h_S is the corrected plate thickness (see, e.g., Altenbach et al.,

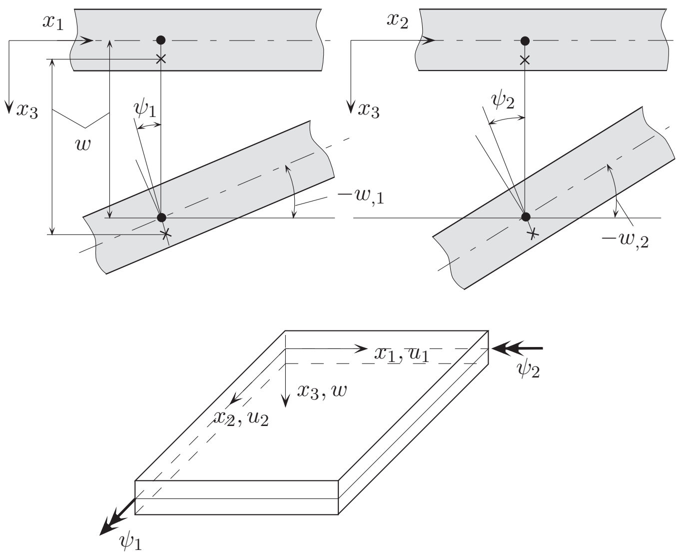


Figure 13. Kinematics of the Mindlin plate