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IUTAM Symposium on Hamiltonian Dynamics, Vortex Structures, Turbulence

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Professor Keith Moffatt, Vice-President, IUTAM

WELCOME REMARKS

It is my great honour to welcome you on behalf of the Bureau of IUTAM to this Symposium on Hamiltonian dynamics, vortex structures and turbulence. The Symposium has been in preparation for two years, and I congratulate our hosts here at the Steklov Institute of the Russian Academy of Sciences for having prepared an excellent and wide-ranging programme, and for having succeeded in attracting such a distinguished gathering to debate problems in fluid dynamics many of which have a long history, yet still today present many challenges of a fundamental nature.

The letters IUTAM, as you all know, stand for the International Union of Theoretical and Applied Mechanics. This Union is one of the International Scientific Union members of ICSU, the International Council for Science, which this year celebrates its 75th anniversary. The roots of IUTAM itself go back to the early Congresses in Mechanics, the first of which was held in Delft in the Netherlands, in 1924. IUTAM was formally established as an International Union at the 7th Congress, which was held in London in 1948. The 13th Congress of Theoretical and Applied Mechanics was held here in Moscow in 1972, under the Presidency of the great Mushkhelishvili. The most recent 21st Congress was held in Warsaw in 2004, and the next will be held in Adelaide, South Australia, in 2008.

In addition to the Congresses, IUTAM also sponsors its Symposia, about 8 per year on average, covering all branches of fluid and solid mechanics, and rigid body dynamics. The present Symposium follows in a strong tradition of Symposia dealing with aspects of vortex dynamics and turbulence. I note just a few of the most relevant that have been held in the last few years, for all of which published Proceedings are now available:

1999, Sedona, Arizona, USA:
Laminar-Turbulent Transition

1999, Hayama, Japan:
Geometry and Statistics of Turbulence
2000, Limerick, Ireland:
Mathematical Modelling of Atmosphere and Ocean Dynamics
2000, Marseille, France:
Bluff-Body Wakes and Vortex-induced Vibration
2001, Kingston, Ontario, Canada:
Turbulent Mixing and Combustion
2001, Zakopane, Poland:
Tubes, Sheets and Singularities in Fluid Dynamics
2002, Princeton, NJ, USA:
Reynolds Number Scaling in Turbulent Flow
2004, Manchester, UK:
Non-Uniqueness of Solutions to the Navier–Stokes Equations and their Connection with Laminar-Turbulent Transition
2004, Kyoto, Japan:
Elementary Vortices and Coherent Structures: Significance in Turbulence Dynamics
2004, Bangalore, India:
Laminar–Turbulent Transition

This remarkable sequence of meetings reflects the great continuing challenge of the subject of vortex dynamics and turbulence, and its multifarious applications. For the present Symposium, it is the Hamiltonian aspects that have been singled out for special study, but I expect that we will not in fact be too constrained by this boundary condition, and it is evident from the programme that non-Hamiltonian aspects will play an equally prominent part in our discussions.

This is my fifth visit to this great city, although my first since the great changes wrought by perestroika in 1991. I came first in 1965 for the famous meeting on Atmospheric Turbulence and Radio Wave Propagation, hosted by A. M. Obukhov and Akiva Yaglom, and their colleagues of the Institute of Atmospheric Physics. I remember that we drank some excellent vodka on that occasion, and I look forward to renewing my acquaintance with that nourishing liquid. I was delighted to discover recently that the paper that I presented at that meeting on the interaction of turbulence with strong wind shear still has some relevance today.

On behalf of IUTAM, I welcome you to this Symposium, and I wish you all a happy and productive week here in Moscow.

VORTEX DYNAMICS: THE LEGACY OF HELMHOLTZ AND KELVIN

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Abstract. The year 2007 will mark the centenary of the death of William Thomson (Lord Kelvin), one of the great nineteenth-century pioneers of vortex dynamics. Kelvin was inspired by Hermann von Helmholtz's [7] famous paper "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen", translated by P. G. Tait and published in English [17] under the title "On Integrals of the Hydrodynamical Equations, which Express Vortex-motion". Kelvin conceived his "Vortex theory of Atoms" (1867–1875) on the basis that, since vortex lines are frozen in the flow of an ideal fluid, their topology should be invariant. We now know that this invariance is encapsulated in the conservation of helicity in suitably defined Lagrangian fluid subdomains. Kelvin's efforts were thwarted by the realisation that all but the very simplest three-dimensional vortex structures are dynamically unstable, and his vortex theory of atoms perished in consequence before the dawn of the twentieth century. The course of scientific history might have been very different if Kelvin had formulated his theory in terms of magnetic flux tubes in a perfectly conducting fluid, instead of vortex tubes in an ideal fluid; for in this case, stable knotted structures, of just the kind that Kelvin envisaged, do exist, and their spectrum of characteristic frequencies can be readily defined. This introductory lecture will review some aspects of these seminal contributions of Helmholtz and Kelvin, in the light of current knowledge.

Keywords: Knotted vortex tubes, vortex filaments, magnetohydrodynamics, magnetic flux tubes

1. The fluid dynamical origins of knot theory and topology

The origins of vortex dynamics lie in the seminal work of Hermann von Helmholtz [7], who (i) introduced the concepts of vortex line and vortex filament (the fluid bounded by the vortex lines passing through the points of an "infinitely small closed curve"), (ii) derived the vorticity equation for an ideal incompressible fluid, and (iii) demonstrated that vortex lines are

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transported with the fluid with intensification proportional to the stretching of its constituent line-elements. This work provided the basis for the bold, though ultimately erroneous, "vortex atom" conjecture of William Thomson (Lord Kelvin) [21, 22], Professor of Natural Philosophy at the University of Glasgow, who sought to explain the structure and spectra of atoms of all the known elements in terms of knotted and linked vortex filaments in a hypothetical background ideal fluid "ether" permeating the universe. It was this conjecture that led Peter Guthrie Tait, Kelvin's opposite number at the nearby University of Edinburgh, to develop techniques for the classification of knots of low crossing number (the minimum number of double points in any plane projection of a knot) [18–20] and thus to sow the seeds for the development of topology as a recognisable branch of modern mathematics. These developments of the period 1858–1885 have been discussed in depth by Epple [6], who conveys well the excitement and drama of this remarkable phase of Victorian science.

2. Tait's role in attracting Kelvin's interest

Helmholtz's work became more widely known when it was republished in English translation by Tait [17], who indicates in a concluding paragraph that his version "does not pretend to be an exact translation" but, following revisions that had been made by Helmholtz, "may be accepted as representing the spirit of the original". Tait had made this translation as soon as he received the German version in 1858, and, stimulated by Helmholtz's concluding remarks concerning the behaviour of vortex rings of small cross section, developed a technique for the experimental demonstration of vortex ring propagation, and of the "leap-frogging" of vortex rings propagating in succession along a common axis of symmetry. Although Kelvin had known of Helmholtz's work in 1858, it was only when Tait, in his Edinburgh laboratory in 1867, showed him his vortex ring demonstration that he was in turn stimulated to undertake his own extensive studies in vortex dynamics.

The second paragraph of Helmholtz's paper (in Tait's translation) deserves comment. He writes:

Yet Euler [Histoire de l'Académie des Sciences de Berlin 1755, p. 292] has distinctly pointed out that there are cases of fluid motion in which no velocity-potential exists, — for instance, the rotation of a fluid about an axis when every element has the same angular velocity. Among the forces which can produce such motions may be named magnetic attractions upon a fluid conducting electric currents, and particularly friction, whether among the elements of the fluid or against fixed bodies. The effect of fluid friction has not hitherto been mathematically defined; yet it is very great, and, except in the case of indefinitely small oscillations, produces most marked differences between theory and fact. The difficulty of defining this effect, and of finding expressions for *its measurement, mainly consisted in the fact that no idea had been formed of the species of motion which friction produces in fluids. Hence it appeared to me to be of importance to investigate the species of motion for which there is no velocity-potential.*

The mention of what amounts to the rotationality of the Lorentz force (*magnetic attractions upon a fluid conducting electric currents*) here shows remarkable foresight, as does recognition of the crucial role of internal friction (i.e. viscosity). It is evident however that Helmholtz was unaware of the epic work of Stokes [15,16] in which the effects of viscosity in a fluid continuum had been analysed in considerable detail. Tait adds a footnote to his translation in which he gently draws attention to this omission:

A portion of the contents of the paper had been anticipated by Professor Stokes in various excellent papers in the Cambridge Philosophical Transactions; but the discovery of the nature and motions of vortex-filaments is entirely novel, and of great consequence.

3. The analogy between vorticity and current as source fields

I was myself a student at the University of Edinburgh from 1953 to 1957 in the (then) Tait Institute for Mathematical Physics, and I recall seeing demonstrations with the "vortex ring generator" (sometimes known as a "Kelvin box" though perhaps more appropriately described as a "Tait box") in connexion with the third-year course on theoretical hydrodynamics given by Robin Schlapp that I attended exactly 50 years ago. The traditional style of presentation of this material, with Lamb's *Hydrodynamics* as the one and only recommended treatise, had been well maintained and cultivated since the time of Kelvin and Tait. We were taught a parallel course on Electromagnetism by Nicholas Kemmer (successor in 1953 to Max Born in the Edinburgh Chair of Natural Philosophy), in which context the name of James Clerk Maxwell, born and schooled in Edinburgh, and later first Cavendish Professor of Experimental Physics at the University of Cambridge (1871–1879), was equally venerated. The fact that the relationship between vortex filaments in fluid mechanics and the velocity field to which they gave rise (via the Biot–Savart Law) is the same as that between currents in conducting wires (i.e. "current filaments") and the magnetic field to which *they* give rise had been noted by Helmholtz and was equally familiar to Kelvin, who was in regular correspondence with Maxwell on this and related topics. We now know, as I shall discuss below, that such interdisciplinary analogies admit powerful exploitation in a manner that was not recognised until the development of magnetohydrodynamics nearly a century later. I propose to argue that, had Kelvin conceived of the ether as a perfectly conducting fluid medium supporting a tangle of magnetic flux tubes rather than as an ideal (inviscid) medium supporting a tangle of vortex filaments, then his theory would have been much more robust, and the development of natural philosophy (i.e. physics) in the early twentieth century might have followed a very different course.

4. The (imperfect) analogy between vorticity and magnetic field

The curious thing is that the basic principles underlying magnetohydrodynamics (MHD) were already known by the mid-nineteenth century, well before Maxwell introduced the "displacement current" that was needed to guarantee charge conservation; this is neglected in MHD, current $\mathbf{j}$ being assumed instantaneously related to magnetic field $\mathbf{B}$ by Ampère's Law: $\mathbf{j} = \mathrm{curl}\,\mathbf{B}$ (in "Alfvén units" for which $\mathbf{B}$ has the dimensions of a velocity). When combined with Faraday's Law of Induction, and Ohm's Law in a medium of resistivity η moving with velocity $\mathbf{v}$, this yields the well-known "induction equation" for the evolution of magnetic field:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{1}$$

This bears an obvious superficial similarity to the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \mathrm{curl}(\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} \tag{2}$$

in a non-conducting medium of kinematic viscosity ν, superficial because whereas $\boldsymbol{\omega}$ is related to $\mathbf{u}$ in (2) by $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{u}$, $\mathbf{B}$ bears no such relation to the transporting velocity field $\mathbf{v}$ in (1). This imperfection in the analogy between $\mathbf{B}$ and $\boldsymbol{\omega}$ does not however vitiate an important conclusion: just as (2) implies that the $\boldsymbol{\omega}$-lines (i.e. vortex lines) are transported with the fluid when $\nu = 0$, so (1) implies that the $\mathbf{B}$-lines (i.e. Faraday's magnetic lines of force) are so transported when $\eta = 0$. Thus, conservation of topology of the $\mathbf{B}$-field in a perfectly conducting fluid could have provided an equally good starting point for Kelvin (rather than conservation of topology of the $\boldsymbol{\omega}$-field in an inviscid fluid) in formulating a theory of the structure and spectra of atoms, and indeed a more plausible one since, as was recognised early in the twentieth century, atoms do involve microscopic current circuits (conventionally pictured as electrons orbiting in their various shells around a nucleus) and their associated magnetic fields.

5. The long-delayed development of magnetohydrodynamics

Thus all the principles were available in the 1860s for such a complementary approach, but Kelvin's preoccupation was with vortices, while on the electromagnetic front, Maxwell's preoccupation was with providing a unified theory of electricity and magnetism. MHD was a subject waiting to be discovered, but it was not until the work of Alfven [1] that the subject was in the event developed to the point at which the crucial "frozen-in" property of the magnetic field in a perfectly conducting fluid was finally recognised. Soon after this, the analogy between vorticity and magnetic field referred to above was recognised and exploited by Batchelor [4] in a first investigation of the effect of turbulence on a random magnetic field. The explosive development of MHD in the 1950s and 1960s was greatly stimulated by technological problems associated with controlled thermonuclear fusion, as well as with an expanding recognition of its vital role in understanding fundamental processes in astrophysics and geophysics.

6. Helicity: the bridge between fluid mechanics and topology

Kelvin's vision of the role of knotted or linked vortex tubes in a hypothetical ether was largely qualitative in character. He correctly perceived that knots and linkages would be conserved by virtue of the frozen-in property of vortex lines, but he had no quantitative measure of such knottedness or linkage. The simplest such quantitative measure for any localised vorticity distribution is in fact provided by its helicity, the integrated scalar product of the vorticity field $\boldsymbol{\omega}$ and the velocity $\mathbf{u}$ to which it gives rise:

$$H = \int \mathbf{u} \cdot \boldsymbol{\omega} dV. \tag{3}$$

This quantity is an invariant of the Euler equations, either for an incompressible fluid or for a compressible fluid under the barotropic condition that pressure p is a function of density ρ alone: $p = p(\rho)$ [8, 13]. For the prototype linkage of two vortex tubes of circulation κ_1 and κ_2 (each having no internal twist), centred on unknotted but possibly linked closed curves C_1 and C_2, the helicity may be easily evaluated in the form

$$H = \pm 2n\kappa_1\kappa_2, \tag{4}$$

where the plus or minus sign is chosen according as whether the linkage is right- or left-handed, and n is an integer, actually the Gauss linking number of C_1 and C_2. It is here that the link between topology and fluid dynamics is at its most transparent.

7. Knotted vortex tubes

For a single vortex tube T of circulation κ whose axis C is in the form of a knot of type K, the situation is more subtle. The helicity in this case is given by

$$H = \kappa^2(Wr + Tw), \tag{5}$$

where Wr and Tw are respectively the writhe of C and twist of T [12]. The writhe is given by a double integral round C analogous to the Gauss integral, and admits interpretation as the sum of the (signed) crossings of the knot averaged over all projections. The twist can be decomposed in the form

$$Tw = \frac{1}{2\pi}\left(\int \tau(s)ds + N\right), \tag{6}$$

where $\tau(s)$ is the torsion of C as a function of arc-length s, and N represents the intrinsic twist of vortex lines around the axis C as they traverse the circuit round the tube (an integer if these vortex lines are closed curves). If the vortex tube is deformed through any configuration that instantaneously contains an inflexion point, then N jumps by an integer at this instant, but the jump is compensated by an equal and opposite jump in the total torsion, so that Tw varies in a continuous manner [12]. As shown by Calugareanu [5] in a purely geometric context, and as generalised to higher dimension by White [23], the sum [5] is indeed constant under arbitrary deformation of the tube.

8. Magnetic helicity and the lower bound on magnetic energy

In consequence of the analogy (albeit imperfect) between vorticity and magnetic field, there is an analogous topological invariant of a magnetic field $\mathbf{B}$ in a perfectly conducting fluid, namely the magnetic helicity

$$H_M = \int \mathbf{A}\cdot\mathbf{B}\, dV \tag{7}$$

where $\mathbf{A}$ is a vector potential for $\mathbf{B}$: $\mathbf{A} = \mathrm{curl}\,\mathbf{B}$ and note that the integral (7) is gauge-invariant provided the normal component of $\mathbf{B}$ vanishes on the boundary of the fluid domain). This invariant was discovered by Woltjer [24], but its topological interpretation was not recognised until some years later [8]. This invariant provides an important lower bound on the magnetic energy

$$M = \int \mathbf{B}^2/2\, dV, \tag{8}$$

namely [3]

$$M \geqslant q|H_M|, \tag{9}$$

where q is a constant (with the dimensions of $(\text{length})^{-1}$), which depends only on the domain topology, geometry and scale. There is no corresponding lower bound for the kinetic energy associated with a vorticity field in an ideal fluid, and it is here that there is great advantage in switching attention to the magnetic problem.

9. Magnetic relaxation

Let us then conceive of a perfectly conducting incompressible fluid contained in a fixed domain Δ with surface S, containing a magnetic field $\mathbf{B}_0(\mathbf{x})$ of nonzero magnetic helicity, the fluid being at rest at time $t = 0$. In general, the associated Lorentz force $\mathbf{j} \times \mathbf{B}$ is rotational, and the fluid will move under the action of this force; as it moves, it transports the magnetic field, whose topology is conserved. If we suppose that the fluid has nonzero viscosity, then, for so long as the fluid is in motion, energy (magnetic M plus kinetic K) is dissipated through the agency of viscosity, and is therefore monotonic decreasing; it is however constrained by the inequality (9), which implies that ultimately $M + K$ tends to a constant, and so the rate of dissipation of energy tends to zero. It is at least reasonable then to conjecture that the velocity field must tend to zero identically in Δ, and that we must arrive at an equilibrium state that is stable within the framework of perfect conductivity because magnetic energy is then minimal under frozen-field perturbations; this magnetostatic equilibrium is described by the force balance

$$\mathbf{j} \times \mathbf{B} = \nabla p, \tag{10}$$

where p is the fluid pressure. The asymptotic field $\mathbf{B}$ results from deformation of $\mathbf{B}_0(\mathbf{x})$ by a velocity field $\mathbf{v}(\mathbf{x}, t)$ which dissipates a finite amount of energy over the whole time interval $0 < t < \infty$ in this sense, it may be said to be “topologically accessible” from $\mathbf{B}_0$. This process has been described in detail by Moffatt [9]. One important feature is that, in general, tangential discontinuities of $\mathbf{B}$ (i.e. current sheets) may develop during the relaxation process. The prototype configuration for which this happens is that consisting of two unknotted, untwisted, linked magnetic flux tubes which, under relaxation, contract in length and expand in cross section (volume being conserved) until they make contact on an open surface which is then necessarily such a surface of tangential discontinuity. Actually, in this situation, one tube then spreads round the other, the ultimate magnetostatic equilibrium being axisymmetric and the current sheet (asymptotically) a torus.

10. Relaxation of knotted flux tubes

A flux tube of volume V, carrying magnetic flux Φ (the analogue of κ) and knotted in the form of a knot of type K, has magnetic helicity the analogue of (5), i.e.

$$H_M = h\Phi^2, \tag{11}$$

where $h = Wr + Tw$ is the conserved writhe-plus-twist of the tube. This tube will relax under the procedure outlined above to a minimum energy state of magnetostatic equilibrium, in which the minimum energy $M_{\min}$ is determined by the three characteristic properties of the initial field that are conserved during relaxation, namely Φ, V, and h; on dimensional grounds, this relationship must take the form

$$M_{\min} = m_K(h)\Phi^2 V^{-1/3}, \tag{12}$$

where $m_K(h)$ is a dimensionless function of the dimensionless helicity parameter h, whose form is determined solely by the knot type K [11]. Moreover, this state, being stable, will be characterised by a spectrum of real frequencies ω_n, which, again on dimensional grounds, are given by

$$\omega_n = \Omega_{Kn}(h)\Phi V^{-1}, \tag{13}$$

where the $\Omega_{Kn}(h)$ $(n = 1, 2, 3, \ldots)$ are again dimensionless functions of h, determined solely by the knot type K. I suspect that it was just such relations as (12) and (13) that Kelvin was seeking in relation to knotted vortex tubes. He was unsuccessful because there is no known relaxation procedure in three dimensions analogous to that described above that conserves *vorticity* topology and minimises *kinetic* energy.

11. The analogous Euler flows

There is nevertheless a second analogy (and this time it is perfect!) which is an extension of the analogy already recognised by Helmholtz and Kelvin, and touched on in §3 above. This is the analogy between $\mathbf{B}$ and $\mathbf{u}$ (and consequently between $\mathbf{j} = \operatorname{curl}\mathbf{B}$, and $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$). The analogue of (10) is then

$$\mathbf{u} \times \omega = \nabla H, \tag{14}$$

where $H = p_0 - p$, for some constant p_0. Equation (14) may be immediately recognised as the steady form of the Euler equation with H the total head. Thus, to each magnetostatic equilibrium satisfying (10), there corresponds a steady Euler flow, obtained by simply replacing $\mathbf{B}$ by $\mathbf{u}$, $\mathbf{j}$ by $\boldsymbol{\omega}$, and p by $p_0 - H$. Note here that, through this analogy, a magnetic flux tube corresponds not to a vortex tube in the Euler flow, but to a streamtube! So a knotted flux tube corresponds to a knotted streamtube, a somewhat curious concept within the context of the Euler equations. However, although the analogy is perfect as far as the steady state is concerned, it does not extend to the stability of the steady state: stability of the minimum energy knotted flux configurations does not imply stability of the analogous Euler flows. The reason is that under perturbation of the magnetostatic equilibrium, the $\mathbf{B}$-field must be frozen in the fluid, whereas under perturbation of the Euler flow satisfying the time-dependent Euler equation, it is not the "analogous" $\mathbf{u}$-field, but rather the $\boldsymbol{\omega}$-field, that is frozen in the fluid. This subtle distinction completely changes the stability criterion for steady states [10]. One should in fact expect all the analogous Euler flows to be in general unstable if only because they will generally contain vortex sheets (the analogue of the current sheets referred to above) and these will be generically subject to Kelvin–Helmholtz instability. It has in fact been shown by Rouchon [14] that steady Euler flows that are nontrivially three-dimensional fail to satisfy Arnold's [2] sufficient condition for stability: the constant-energy trajectories on the "isovortical" folium through a fixed point in the space of divergence-free velocity fields of finite energy are in general hyperbolic in character, so that the perturbed flow is not constrained by conservation of energy to remain near the fixed point. This does not imply instability, but it makes it very likely!

12. Conclusions

Kelvin was frustrated in his vortex ambitions on two accounts: first in failing to find steady non-axisymmetric solutions of the Euler equations having knotted vortex lines; and second in being unable to demonstrate the stability of even the simplest vortex ring configurations. His investigations of the 1870s and 1880s laid the basis for many subsequent investigations of problems of vortex structure and stability that remain very much alive today; but his initial concept of the "vortex atom" failed to gain ground because of these two fundamental barriers to progress. If instead one adopts the complementary scenario of magnetic flux tubes in a perfectly conducting fluid, then the natural technique of magnetic relaxation, as described above, leads in principle to stable equilibria of magnetic flux tubes knotted in an arbitrary manner. The actual realisation of the relaxation process, and the determination of the frequency spectra of these stable equilibria, present computational challenges that should be within the power of current super-computers. I hope that someone may soon be able to rise to these challenges, and thus revive the vision and spirit of the great nineteenth-century pioneers of the subject of this Symposium.

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VORTEX DYNAMICS OF WAKES

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Abstract. Several problems related to the dynamics of vortex patterns as observed in wake flows are addressed. These include: The universal Strouhal–Reynolds number relation. The Hamiltonian dynamics of point vortices in a periodic strip, both the classical two-vortices-in-a-strip problem, which gives the structure and self-induced velocity of the traditional vortex street, and the three-vortices-in-a-strip problem, which is argued to be relevant to the wake behind an oscillating body. The bifurcation diagram for wake structure found experimentally by Williamson and Roshko is addressed theoretically.

Keywords: Vortex streets, wakes; Strouhal–Reynolds number relation

1. Introduction

Vortex street wakes are ubiquitous. We can create them in the laboratory and we observe them in Nature. We see them in planetary atmospheres. Thus, in recent years spectacular vortex street wakes at very high Reynolds number have been observed "behind" certain islands in satellite images (cf. Fig. 1). We realize their profound effect from instances such as the collapse of the Tacoma Narrows Bridge on 7 November 1940.

While the phenomenon of vortex streets had been observed qualitatively for many years, it was not until the seminal work of T. von Kármán in 1911–1912 [13–15] that the first theory of these structures was produced. So important was this contribution of von Kármán that the Hungarian postage stamp commemorating him (issued in 1992) shows his portrait on a background of the streamline pattern (in the co-translating frame) of the particular staggered vortex street that he identified as being not linearly unstable (see Fig. 2). I shall return to von Kármán's contributions in Section 4. Let me first mention another very important result that has emerged, mostly from experiment, namely the well-known relation between the *Strouhal number* for vortex

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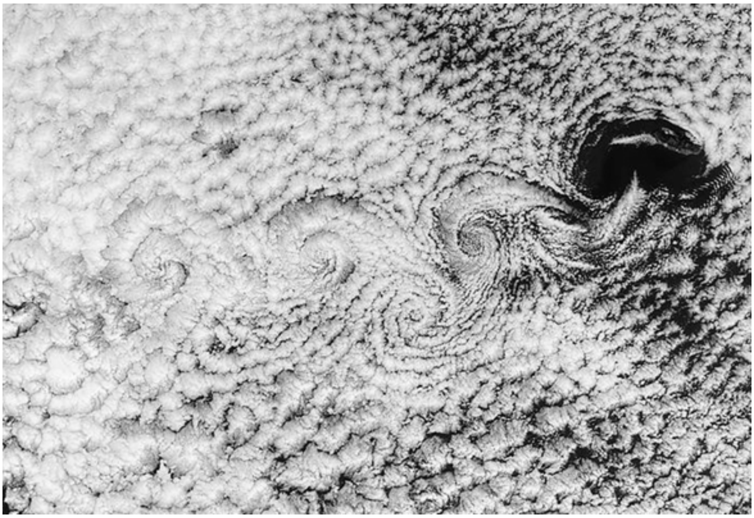

Fig. 1. NASA satellite image of 26 April 2002 showing a well-developed vortex street behind Madeira island.

Fig. 2. Hungarian postage stamp memorializing von Kármán. In the background the streamline pattern for a staggered point vortex street.

shedding into the wake and the *Reynolds number* of the wake-generating flow (see Fig. 3). Let us first ask: How might one think about such a relation theoretically?

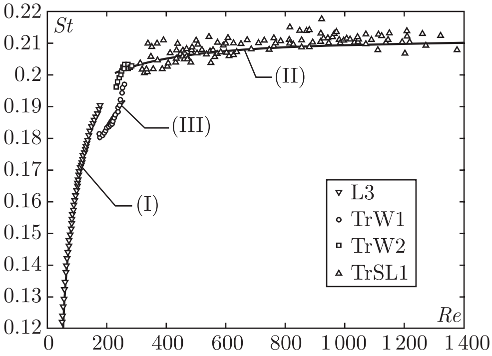

Fig. 3. The empirical Strouhal–Reynolds number relation for flow behind a cylinder. Different symbols have been used for different vortex shedding regimes abbreviated L3, TrW1, etc. The overall division into three regimes, I, II and III, is described in the text.

2. The Strouhal–Reynolds number relation

Empirically one finds that the Strouhal number, which is the non-dimensional shedding frequency, depends on the Reynolds number of the wake-producing flow as

$$St = 0.2175 - \frac{5.1064}{Re}, \tag{1}$$

for the "laminar" regime (regime I in Fig. 3; up to $Re \approx 200$), and by

$$St = 0.212 - \frac{2.7}{Re}, \tag{2}$$

for large values of Re, say 400 and higher. The latter fit includes the famous limiting value (0.212) of the Strouhal number at high Reynolds number. Of course in this second regime (regime II in Fig. 3) the flow does not just respond with one frequency but the Strouhal number corresponds to the frequency with most of the energy.

There is a "transition" regime (regime III in Fig. 3) where the curve seems to break. This regime is related to three-dimensional vortex motion in the wake. I shall not have anything to say about this regime in this paper.

We have approached this problem in the following way: First, we recall that the Navier–Stokes equations only give us that St is a function of Re, i.e., that there must exist some functional relation $St = f(Re)$ where f is (somehow) to be determined from the equations of motion and the shape of the body. Second, we assume — based on an analogy to the phenomenology of phase transitions or general ideas from bifurcation theory — that close to the bifurcation the Strouhal number depends as a power law on the deviation of $1/Re$ from its "critical" value at the bifurcation, i.e., we should expect for $Re \approx Re_{crit}$ that

$$St = A\left(\frac{1}{Re_{crit}} - \frac{1}{Re}\right)^{\alpha}, \tag{3}$$

where Re_{crit} is the bifurcation Reynolds number (which is non-universal), A is a non-universal coefficient, but α is a universal exponent. Experiment further suggests that $\alpha = 1$ which points to a mean-field theory of the phenomenon.

So, what equation should one try to apply a "mean-field analysis" to? We [24] thought the two-dimensional vorticity equation,

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{V} \cdot \nabla\zeta = \nu\nabla^2\zeta, \tag{4}$$

was a natural candidate. In the paper just cited we estimate the terms in this equation as follows:

$$\frac{\partial \zeta}{\partial t} \approx f\Delta\zeta, \quad \boldsymbol{V} \cdot \nabla\zeta \approx U\frac{\Delta\zeta}{d}, \quad \nu\nabla^2\zeta \approx \nu\frac{\Delta\zeta}{d^2}. \tag{5}$$

Here f is the shedding frequency, which sets a natural time scale for the flow, U is the free stream velocity and d the diameter of the cylinder. The quantity $\Delta\zeta$ gives the scale of vorticity fluctuations in the emerging wake. There are points of large vorticity, primarily in the vortices that are forming to make up the vortex street, and there are points of smaller vorticity in sheets and other "background" flow structures that will ultimately be swept up into the vortices.

In our paper [24] we argue, based on careful examination of the vortex wake formation process in a numerical simulation [26], that the viscous term acts exclusively to spread out and impede vortex formation, i.e., in the vorticity balance in the near wake this term should be viewed as a sink when writing the vorticity balance. We also argue that part of the advective term on the right hand side acts to assemble the vortices (the rest simply advects the vorticity downstream). This is a source term for vortex generation and should enter the vorticity balance with a positive sign. Based on this kind of order of magnitude estimates and physical reasoning to determine the signs of the various contributions, we recast the vorticity equation in the following form (in terms of orders of magnitude with signs):

$$f\Delta\zeta = k_a U\frac{\Delta\zeta}{d} - k_d\nu\frac{\Delta\zeta}{d^2}, \tag{6}$$

where k_a and k_d are two dimensionless parameters that require a more comprehensive analysis to determine. It is easily seen that this relation, after cancellation of $\Delta\zeta$ from all terms and multiplication by d/U, is precisely of the form of the empirical Strouhal–Reynolds number relation.

There are a number of questions one can ask of this simple "derivation", e.g., whether it is satisfactory that $\Delta\zeta$ cancels out of all the terms[1]. In particular, the crude estimate for the advective term may seem dubious. We will not

[1] I am indebted to T. Bohr for raising this point.

enter into a discussion of these issues here (for more detail see the paper cited) but simply state the suggestion that the correct and fully rigorous approach to this problem requires finding a similarity solution of the vorticity equation that, somehow, applies to vortex shedding. We leave this as a challenge to the reader!

3. Hamiltonian dynamics of point vortex dynamics

The point vortex model originated with Helmholtz's seminal 1858 paper on vortex dynamics [17]. The most elegant statement arises if one concatenates the x- and y-coordinates of the vortices into complex positions $z_\alpha = x_\alpha + \mathrm{i} y_\alpha$, $\alpha = 1, 2, ..., N$. Then the equations of motion take the form

$$\dot{z}_\alpha^* = \frac{1}{2\pi \mathrm{i}} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \tag{7}$$

Here the Γ_β are the circulations of the vortices, invariant in time by Helmholtz's theory — even better, maybe, by Kelvin's circulation theorem — the asterisk on the left-hand side denotes complex conjugation, the dot differentiation with respect to time, and the prime on the summation symbol reminds us to skip the singular term $\beta = \alpha$.

Helmholtz gave the solution of the two-vortex problem, where he showed that two vortices would have orbits on concentric circles, which in the special case of a vortex pair degenerate to translation along parallel lines.

A major formal development of the theory was provided by Kirchhoff [18], who in his lectures on theoretical physics, published in several editions starting in 1876, showed that the point vortex equations could be recast in Hamilton's canonical form:

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \qquad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \tag{8}$$

where the Hamiltonian, H, is

$$H = -\frac{1}{4\pi} \sum_{\alpha,\beta=1}^{N}{}' \Gamma_\alpha \Gamma_\beta \log |z_\alpha - z_\beta|. \tag{9}$$

Again we exclude the singular terms $\alpha = \beta$ and remind ourselves to do so by placing a prime on the summation. A complete correspondence with Hamilton's form of the equations of motion is obtained by choosing the generalized coordinates to be $q_\alpha = x_\alpha$ and the generalized momenta to be $p_\alpha = \Gamma_\alpha y_\alpha$. This also shows that for vortices *phase space is configuration space*, a feature that has profound consequences for both the statistical physics of point vortices and for the phenomenon of chaotic advection [2]. Many of these aspects were covered by other speakers at the symposium.

The Hamiltonian nature of the point vortex equations immediately leads to important insights about the availability of integrals of the motion and, in turn, about integrability of the N-vortex problem. Thus, the invariance of H to translation and rotation of coordinates, and its independence of time, leads to the integrals X, Y and I given by

$$X + \mathrm{i}Y = \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha, \qquad I = \sum_{\alpha=1}^{N} \Gamma_\alpha |z_\alpha|^2, \tag{10}$$

and, of course, H itself. The quantities X and Y are the two components of the *linear impulse*. The quantity I is the *angular impulse*.

Pursuing the formalism of classical dynamics a bit further, we introduce the *Poisson bracket*

$$[f, g] = \sum_{\alpha=1}^{N} \frac{1}{\Gamma_\alpha} \left(\frac{\partial f}{\partial x_\alpha} \frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha} \frac{\partial g}{\partial x_\alpha} \right). \tag{11}$$

The *fundamental brackets* may be written

$$[z_\alpha, z_\beta] = 0, \qquad [z_\alpha, z_\beta^*] = -\frac{2\mathrm{i}}{\Gamma_\alpha} \delta_{\alpha\beta}. \tag{12}$$

We now obtain the key results

$$[X, Y] = \sum_{\alpha=1}^{N} \Gamma_\alpha, \qquad [X.I] = 2Y, \qquad [Y, I] = -2X, \tag{13}$$

from which the very important result

$$[X^2 + Y^2, I] = 2X[X, I] + 2Y[Y, I] = 0 \tag{14}$$

follows. These results show (a) that no new integrals arise by taking Poisson bracket of the known integrals, and (b) that the problem always has three independent integrals *in involution*, namely $X^2 + Y^2$, I and H.

Poincaré realized as much in his lectures of 1891–1892 [23] and concluded (from what we today call Liouville's theorem) that the three-vortex problem on the unbounded plane is always integrable. Apparently this was not of sufficient interest to him and he never returned to the problem. The general formalism given above was pursued by the Italian E. Laura in a number of papers early in the 20th century [20] but then lay dormant for decades.

Actually some 15 years before Poincaré's work the three-vortex problem had been completely solved by a young Swiss mathematician W. Gröbli whose 1877 thesis [12] was for some reason overlooked[2] for about a century. Even

[2] This happened in spite of references to it in Kirchhoff's lectures (2nd ed.) [18] and in Lamb's well-known text [19].

the revival of Gröbli's work in an important paper [31] by J. L. Synge for the inaugural issue of the *Canadian Journal of Mathematics* in 1949, an issue that contained a seminal paper in general relativity by Einstein and Infeld, failed to introduce the solution of this three-body problem into the mainstream of fluid mechanics. For a review of this history see [5].

It turns out that there is a bit of a "hole" in the treatments of Gröbli and the later work by Synge, Novikov and the author [1, 22, 31] concerning the special case $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$. While being covered in principle by the general analysis, it admits of a much more complete discussion. This was provided by Rott [28] and the author [3]. In essence what our treatment of the problem shows is that the relative separation of two of the vortices, say vortices 1 and 2, i.e., $Z = z_1 - z_2$, evolves as if it were the position of a fictitious passive particle in the field of three *fixed* vortices. The strengths and locations of the three fixed vortices are given by the strengths of the original three vortices and the linear impulse of the original three-vortex system. Thus, if the original three vortices have strengths Γ_1, Γ_2, Γ_3, the three fixed vortices in the advection problem have strengths Γ_1^{-1}, Γ_2^{-1}, Γ_3^{-1}. (All that matters is really the proportion of the vortex strengths — the absolute value can be absorbed in a rescaling of space and time.)

This reduction of the problem — from three points corresponding to the three original vortices, to one point corresponding to an advected particle — is somewhat akin to what happens in the Kepler problem of celestial mechanics, where the motion of two interacting mass points is decomposed into a trivial center-of-mass motion and a relative motion. It leads to the following scenario: There is the *physical plane* where the motion of the three vortices takes place, i.e., the vortex positions z_1, z_2, z_3 "live" in this plane. There is a *phase plane* where the advection of the fictitious particle takes place, i.e., Z evolves in this plane.

For three vortices on the infinite plane the advection problem in the phase plane is relatively simple. There are four distinct regimes of motion. Three of these arise in the obvious way through two of the vortices being closer to one another than to the third vortex, and hence moving as if in a "bound state". The fourth regime corresponds to truly "collective states" where all three vortices interact continuously.

4. Point vortex modeling of wakes

It turns out that the solution method for three vortices on the infinite plane can be extended to the problem of three vortices in a domain with periodic boundary conditions as was first shown by Aref and Stremler [6, 30]. In the case of vortices in a periodic strip, which is the case that is most immediately applicable to vortex wakes, one has to stipulate that $\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$ just as on the infinite plane. (In the case of vortices in a periodic parallelogram the periodicity of the flow assures that the sum of the "base" vortices in the basic parallelogram is zero.) The equations of motion for vortices in a periodic strip of width L are

$$\dot{z}_\alpha^* = \frac{1}{2L\mathrm{i}} \sum_{\beta=1}^{N}{}' \Gamma_\beta \cot\left[\frac{\pi}{L}(z_\alpha - z_\beta)\right]. \tag{15}$$

These equations appear first to have been written down in 1928 by Friedmann and Poloubarinova [11]. See also [27].

With the wisdom of hindsight one may say that von Kármán's theory of the structure of the vortex street follows from (15) with $N = 2$ and $\Gamma_1 = -\Gamma_2 = \Gamma$ and, thanks to later work by Domm [10], his theory of the stability of vortex streets follows almost entirely, although not quite, from (15) with $N = 4$ and $\Gamma_1 = \Gamma_2 = -\Gamma_3 = -\Gamma_4 = \Gamma$. (Probably the most accessible account of von Kármán's theory for the modern reader is the exposition in [19].)

In brief, von Kármán's theory of the vortex street shows, first, that the only two-vortex-per-strip configurations to propagate downstream are the symmetric and the staggered configuration. From the two-vortex version of (15) one easily deduces that a $\pm\Gamma$ pair in a periodic strip propagates with velocity

$$U - \mathrm{i}V = \frac{\Gamma}{2L\mathrm{i}} \cot\left[\frac{\pi}{L}(z_+ - z_-)\right]. \tag{16}$$

For the velocity to be real, i.e., in order to have $V = 0$ in (16), the cotangent must be pure imaginary. This implies $\Re(z_+ - z_-) = 0$ or $\Re(z_+ - z_-) = L/2$. The first possibility corresponds to symmetric vortex streets, the second to staggered vortex streets.

Von Kármán next considered the stability of these two types of configurations. He did, in essence, two stability calculations, in both cases working with infinite rows of vortices. In the first he simply perturbed one vortex keeping all the others fixed. This calculation showed that the symmetric configuration was always linearly unstable and the staggered configuration was linearly unstable unless the ratio of $b = \Im(z_+ - z_-)$ and the intervortex distance in each row, h, has a certain value. (To avoid confusion we use a new symbol, h, for the distance between vortices in either row because for, say, four-vortices-in-a-strip the period of the strip, L, is related to the intervortex distance by $L = 2h$, whereas $L = h$ for the two-vortices-per-strip case.) In fact, in his first attempt von Kármán produced the erroneous result $\sinh(\pi b/h) = \sqrt{2}$. (The reason for this "error" is that when perturbing just one vortex one is adding linear momentum and kinetic energy to the system being perturbed. The appropriate criterion arises from perturbations that do not add linear momentum or energy.) The correct result, which von Kármán quickly produced as well, and which is today known as his famous stability criterion for vortex streets is

$$\sinh\frac{\pi b}{h} = 1. \tag{17}$$

The main thrust of our work on more complicated vortex wakes — we have used the term "exotic" — is to apply the solution for three-vortices-in-a-strip

Fig. 4. "Exotic" vortex street wake behind an oscillating cylinder (courtesy of C. H. K. Williamson).

that we have found to model these in the same spirit that von Kármán modeled steady vortex streets by the two-vortices-in-a-strip solutions. An example of an "exotic" wake with three vortices shed per cycle is show in Fig. 4. It is a tenet of vortex wake dynamics, apparently true but difficult to prove, that the total circulation of all vortices shed during one cycle is zero. This applies also to such cases as a cylinder oscillating normally to an oncoming uniform flow.

A recent paper by Ponta, Stremler and the author [4] gives a rather thorough exposition of our ideas so we shall be content with a brief summary here.

In the extension of the solution for three vortices with sum of circulations equal to zero to periodic boundary conditions [6, 30] one finds, once again, that the problem can be "reduced" to an advection problem for the relative position of two of the vortices, say again $Z = z_1 - z_2$. This time, however, the advecting system of vortices consists of three rows of advecting vortices, not just three vortices. The vortices in each of the three rows are identical, and their circulations are, respectively, Γ_1^{-1}, Γ_2^{-1}, Γ_3^{-1} (modulo rescaling of the time). Indeed, the position of the "base vortex" in each row is given exactly as in the unbounded plane case in terms of the linear impulse of the system and the circulations. It turns out that if the ratio of the circulations is rational (and because the sum is zero, if the ratio of two circulations is rational, the ratio of any two circulations is rational), the three rows of advecting vortices fit into a periodic strip with a width that is a multiple of the period L of the strip in the physical plane. If the ratios are irrational, the three rows of advecting vortices have no common period and we are faced with advection by an infinite system of stationary vortices.

Again an advection problem in the phase plane arises but this time with a more complicated structure of the various *regimes of motion* than in the unbounded plane case. There are, in general, many more regimes for Z to wander through and thus many more regimes for the vortex motion itself. (To find z_1, z_2 and z_3 from Z requires an additional quadrature.) This provides the first qualitative conclusion: Vortex wakes with three (and, thus,