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Optimization in Economics and Finance

Some Advances in Non-Linear, Dynamic, Multi-Criteria and Stochastic Models

by

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Table of Contents

Preface

Many optimization questions arise in economics and finance; an important example of this is the society's choice of the optimum state of the economy (which we call a social choice problem). This book,

Optimization in Economics and Finance,

extends and improves the usual optimization techniques, in a form that may be adopted for modelling optimal social choice problems, and other related applicastions discussed in section 1.2, concerning new[3] economics. These types of optimization models, based on welfare economics, are appropriate, since they allow an explicit incorporation of social value judgments and the characteristics of the underlying socio-economic organization in economic and finance models, and provide realistic welfare maximizing optimal resource allocation and social choices, and decisions consistent with the reality of the economy under study. The methodological questions discussed include:

- when is an optimum reached, and when is it unique?
- relaxation of the conventional convex (or concave) assumptions on an economic or financial model,
- associated mathematical concepts such as *invex* (relaxing *convex*) and *quasimax* (relaxing *maximum*),
- multiobjective optimal control models, and
- related computational methods and programs.

These techniques are applied to models of economic gropwth and development, including

- small stochastic perturbations,
- finance and financial investment models (and the interaction between financial and production variables),
- modelling sustainability over long time horizons,
- boundary (transversality) conditions, and
- models with several conflicting objectives.

Although the applications are general and illustrative, the models in this book provide examples of possible models for a society's social choice for an allocation that maximizes welfare and utilization of resources. As well as using existing computer programs for optimization of models, a new computer program, named SCOM, is presented in this book for computing social choice models by optimal control.

This book contains material both unpuhlished and previously published by the authors, now rearranged in a unified framework, to show the relations between the topics and methods, and their applicability to questions of social chnoice and decision making.

This book provides a rigorous study on the interfaces between mathematics, computer programming, finance and economics. The book is suitable as a reference book for researchers, academics, and doctoral students in the area of mathematics, finance, and economics.

The models and methods presented in this book will have academic and profesional application to a wide range of areas in economics, finance, and applied mathematics, including optimal social choice and policy planning, use of optimal models for forecasting, market simulation, developmenjt planning, and sensitivity analysis.

Since this is an interdisciplinary study involving mathematics, economics, finance and computer programming, readers of this book are expected to have some familiarity with the following subjects: Mathematical Analysis, Optimal Control, Mathematical Finance, Mathematical Economics, Mathematical Programming, Growth Economics, Economic Planning, Environmental Economics, Economics of Uncertainty, Welfare Economics, and Computational Economics.

The various SCOM computer programs listed in this book may also be downloaded from the web site: *http://bdc.customer.netspace.net.au* .

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Chapter 3 is based (with some additional material) on *Computing Optimal Control on MATLAB - The SCOM Package and Economic Growth Models,* Chapter 5 in *Optimisation and Related Topics,* Eds, A. Rubinov et al. Volume 47 in the Series *Applied Optimization,* Kluwer Academic Publishers, 2001. Kluwer has given permission to reproduce this article.

Chapter 4 is (with minor changes) the paper *Computation of Non-Linear Continuous Optimal Growth Models: Experiments with Optimal Control Algorithms and Computer Programs,* 2001, *Economic Modelling: The International Journal of Theoretical and Applied Papers on Economic Modelling,* Vol. 18, pp. 551-586, North Holland Publishing Co.

Chapter 6 is (with minor modifications) the paper *Measuring Sustainable Growth and Welfare: Computational Models, Methods,* Editors: Thomas Fetherston and Jonathan Batten, *"Governance and Social Responsibility"*, Elsevier-North Holland, 2003.

Chapter 9 is the paper *Transversality Conditions for Infinite Horizon Optimal Control Models in Economics and Finance*, submitted to Journal of Economic Dynamcs and Control, an Elsevier publication.

Elsevier have stated that an Elsevier author retains the right to use an article in a printed compilation of works of the authors.

The book also includes excerpts from other papers by the authors, often rearranged to show relevance and relation to other topics; these items are acknowledged where they occur. The material in this book is now presented in a unified manner, with a focus on applicability to economic issues of social choice.

Chapter 1

Introduction : Optimal Models for Economics and Finance

1.1. Introduction

This book is concerned with applied quantitative welfare economics, and describes methods for specification, analysis, optimization and computation for economic and financial models, capable of addressing normative social choice and policy formulation problems. Here *social choice* refers to the optimal intertemporal allocation of aggregate and disaggregate resources. The institutional and organizational aspects of achieving such allocation in a society are not discussed here. Zahedi (2001) has surveyed other methods for social choice. The book aims to provide some extensions and improvements to the traditional methods of optimization, as applied to economics and finance, which could be adopted for social decision making (*social choice*) and related applications. The mathematical techniques include nonlinear programming, optimal control, stochastic modelling, and multicriteria optimization.

Many questions of optimization and optimal control arise in economics and finance. An optimum (maximum or minimum) is sought for some objective function, subject to constraints (equalities or inequalities) on the values of the variables. The functions describing the system are often nonlinear. For a time-dependent system, the variables become functions of time, and this leads to an optimal control problem. A control function describes a quantity (such as consumption, or investment) that can be controlled, within some bounds. A state function (such as capital accumulation) takes values determined by the control function(s) and the dynamic equation(s) of the system.

Some recent developments in the mathematics of optimization, including the concepts of invexity and quasimax, have not previously been applied to models of economic growth, and to finance and investment. Their applications to these areas are shown in this book. Some results are presented concerning when an optimal control model has a unique optimum, what happens when the usual convexity assumptions are weakened or absent, and stability to small disturbances of the model or its parameters. A new computational package called SCOM, for solving optimal control problems on MATLAB, is introduced. It facilitates computational experiments, in which there are changes to model features or parameters.

These developments are applied, in particular, to:

- models of optimal (welfare maximizing) intertemporal allocation of resources.

• economic growth models with a small stochastic perturbation.
• models for finance and investment, including some stochastic elements, and especially considering the interaction between financial and production variables.
• modelling sustainability over a long (perhaps infinite) time horizon.
• models with several conflicting objectives.
• boundary (*transversality*) conditions.

These extended results can be usefully applied to various questions in economics and finance, including social decision making and policy analysis, forecasting, market simulation, sensitivity analysis, comparative static and dynamic analysis, planning, mechanism design, and empirical investigations. If an economic system behaves so as to optimize some objective, then a computed optimum of a model may be used for forecasting some way into the future. However, the book is focussed on optimal social decision making (social choice).

1.2. Welfare economics and social choice: Modelling and Applications

A central issue in economics and finance, concerning welfare economics, is to find a normative framework and methodology for social decision-making, so as to choose the socially desirable (multi-agent or even aggregate) state of the economy, a task popularly known as "social choice". Optimisation methods based on welfare economics can aid such social decision making ((Islam 2001a).

The optimisation models of economics and finance can, therefore, be interpreted as *models for normative social choice* which specify optimal social welfare in the economy and financial sector satisfying the static and dynamic constraints of the economy since these models can generate a set of aggregative and disaggregstive optimal decisions. choices or allocation of resources for the society. This approach is in the line of arguments advanced in the paradigm of new^3 welfare economics (Islam 2001b; Clarke and Islam, 2004). It has the following main elements: 1) the possibility perspective of social choice theory; 2) measurability of social welfare based on subjective or objective measures; 3) the extended welfare criteria; 4) operationalisation of welfare economics and social choice (which was the original motivation of classical economists for developing the discipline of welfare economics), and 5) a multi-disciplinary system approach incorporating welfaristic and non-welfaristic elements of social welfare.

Any welfare economic analysis of issues in economics and financial policies involves the application of the following multidisciplinary criteria of moral philosophy and welfare economics: efficiency, rationality, equity, liberty, freedom, capabilities and functioning (see Hausman and McPherson, 1996) for a survey of these criteria). This framework of new^3 welfare economies provides the scope for evaluating economic outcomes in terms of social welfare (and efficiency, utility) as well as other criteria of welfare economics and moral philosophy such as rights, liberty, morality, etc. (see Hausman and McPherson, 1996 for a survey of the concepts and issues and their economic implications).

The incorporation of this approach in optimisation modelling is possible through the choice of the social discount rate, the objective function (extended welfare criteria incorporating welfaristic and non-welfaristic elements of social welfare), terminal conditions, time horizon, and the modelling structure.

In making such an application of optimisation models, several conceptual and methodological issues in social choice theory and welfare economics (which have dominated the controversy about the possibility of social choice) needs to be resolved including the following (Islam 2001b):

- The nature value judgment about the nature of individual well-being or welfare (such as in utilitarianism or welfarism, capability,) etc.
- Possibilities for measurability of utility and welfare (cardinality or ordinality).
- Interpersonal comparability of utility and welfare.
- The nature of marginal utility of income (constancy or variability).
- The role of distributional concerns in welfare judgment (the intensity of preferences).
- The choice of a measurement and accounting method (nature of preference indexing, numerical calculations, etc.).
- The extent of informational requirements for decision making.

These issues can be considered from the impossibility (Arrow, 1951) or possibility perspectives (Sen, 1970). The possibility perspective approach requires a set of axioms including cardinality, intertemporal comparability, and the relevance of the intensity of preferences. In this possibility approach (see Sen 1999), there is an urge for the need for, amongst others, finding a suitable method and information broadening for developing an optimistic social choice theory for useful social welfare analysis and judgment. This can be accomplished by developing an operational approach to social choice. This is an especially immediate task in applied welfare economics, although work in this area has not progressed far. In Islam (2001a, 2001b) and Clarke and Islam (2004), a paradigm has been developed for new^3 welfare economics, for normative operational social choices based on the possibility perspective.

The choice of the elements for a social norm is controversial, since each specification relates to some form of value judgment in a welfare economics choice model, and a choice significantly affects the pattern and level of social welfare. A specification of the the elements of a social choice should be based within the framework of some paradigm of welfare economics. The new^3 welfare economics paradigm adopts the following set of assumptions and elements of an operational approach to social choice aqnd decision making:

- Definition of well-being and welfare: the social welfaristic approach (Islam 2001a and 2001b) - economic activities, which improve net social welfare, are justified.
- The possibility of the specification of aggregate social welfare criteria and index: the possibility theorem perspective.
- Time preference: different discounting approaches for intertemporal

equity - depending on the preference of the society.

- Units of measurement; market and shadow prices of goods and services
- Methods for modelling: efficient allocation or optimisation modelling.
- Institutions: various alternative institutions can be assumed such as competitive market economy, mixed economy, or planning - depending on the underlying social organization.

The main argument of this book is that mathematical models can be developed, incorporating the above elements of new[3] welfare economics; they can provide useful information to understand social choice in relevant economic, social, environmental and financial issues, and formulating appropriate policies.

The general structure of an optimisation model in economics and finance, containing the above elements, and suitable for normative social choice or decision making (see Craven 1995; Islam 2001a; Laffont, 1988) is as follows:

$$W = f(y) \text{ subject to } g(y) \in S,$$

where: $I = [a, b]$;

W is an indicator of social welfare;

V is the space of functions;

$f(y)$ is a scalar or vector valued social welfare functional of society.

y is a vector of variables or functions of economic and financial sub-systems;

$g(y)$ is a constraint function (including economic and financial factors);

S is a convex cone, describing a feasible set of the economy; and

$\mathbf{R}^n$ is Euclidian space of n dimensions.

In the above social welfare model, a social welfare function of the Bergson-Samuelson form is specified to embed social welfare judgments about alternative states of resource allocation in the economy. (For further details, see Islam, 2001a.) Social welfare, and factors affecting it, are assumed to be measurable and quantifiable. The problem of normative social choice in decision making is represented by the optimization model, based on the possibility perspective of social choice. It is operational, since it may be applied to real life conditions, for finding optimal decisions in society. The model can represent the economic organization of a competitive market or planning system (the selection of a system of social organization depends on the social preferences assumed in the model). A model, containing an objective function, constraints and boundary conditions, can represent the socio-economic factors relevant for decision making. *These general assumptions are made for the various models in this book*; specific assumptions for each model are discussed in the relevant cases.

The optimal solution to the welfare optimisation social choice problem exists (i.e., an optimal decision, choice, or policy exits) if the problem satisfies the Weierstrass theorem; and if the objective function is convex, x* is a global solution.

The set S represents the static or dynamic economic and financial systems. The objective function f(x) is the social welfare functional embedding social choice criteria. Different value judgements and different theories of welfare economics and social choice, and various sub-systems of the economy can be incorporated in this social choice program by making different assumptions about different functions, parameters and the structure of the above model. The above control model can embed and address the issues of welfare economics and social choice discussed above if it is based on a proper specification of the method for aggregation of individual welfare, welfare criteria, cost benefit consideration, and institutional mechanisms assumed for society. *The results of the model can specify the optimal choices* regarding optimal dynamic welfare and resource allocation and price structure, the optimal rate and valuation of consumption, capital accumulation, and other economic activities, and optimal institutional and mechanism design. Further discussion on construction of welfare economic modelling is given in Islam (2001a), Heal (1973), Chakravarty (1969), and Fox, Sengupta and Thorbecke (1973).

The above social choice model is a finite horizon free terminal time continuous optimisation problem and it is deterministic, and open loop with social welfare maximization criteria. Other possible forms of social choice models include dynamic game models and with other types of end points and transversality conditions; overtaking, catching up and Rawlsian optimality criteria; with different types of constraints, discontinuities and jumps; and with uncertainty. These social choice models may also represent equilibrium and disequilibrium economic systems, adaptive dynamics, social learning, chaotic behaviour, artificial intelligence and genetic algorithm.

In such an optimisation model of social choice, the following set of elements should be specified:

- an economic model (including social, financial, and environmental constraints;
- the length of the planning horizon;
- the choice of an optimality criterion or an intertemporal utility function;
- the discount rate, representing the rate of time preference; and
- the terminal or transversality conditions.

The specification of the elements is a political economic exercise involving substantial value judgment on the part of the modeller. Depending on the value judgment of the modeller, a particular form of each element can be specified.

1.3. The objectives

The objective of this book is to provide extensions to the existing methods for optimisation in economics and finance which can be appropriately used for normative social choice, based on the possibility perspective of social choice and the other elements of new[3] welfare economics discussed above, as well as for other exercises such as sensitivity analysis, simulation of market behaviour, forecasting, and comparative static and dynamic analysis. The focus of the book is on the methods for optimisation, not on the social choice issues in optimisation models, in economics and finance. This book has not taken any particular perspective in social value judgments, and therefore the details of the choice of the elements are not provided here. We have left the specification of various elements of welfare economics and optimisation modelling in a possible general form. A modeller can choose a set of specific elements according to his/her value judgment (see also Islam, 2001a), to develop a model for a particular economy.

Although a variety of models and computation approaches are developed and implemented in this book, *they may all describe social choices,* concerning the maximization of social welfare, intertemporal allocation, and utilization of resources, in relation to the social value judgement expressed in the models.

1.4. An example of an optimal control model

A large part of this book is concerned with optimal control models for economic questions. Such models are generally of the form:

$$\mathrm{MAX}_{x(.),u(.)}\ F^0(x,u) := \int_0^1 f(x(t),u(t),t)dt + \Phi(x(1))$$

$$\text{subject to } x(0)=a,\ \dot{x}(r)=m(x(t),u(t),t),\ q(t)\leq u(t)\leq r(t)\ \ 0\leq t\leq 1).$$

Here the *state function* $x(t)$ could describe capital, the *control function* $u(t)$ could describe consumption; an objective (an integral over a time period, plus an endpoint term) describes a utility to be maximized, subject to a *dynamic equation*, a differential equation determining the state.

A special case is a model for economic growth and development, of which the following is an example. The well known Kendrick-Taylor model for economic growth (Kendrick and Taylor, 1971) describes the change of capital stock $k(t)$ and consumption $c(t)$ with time t by a dynamic differential equation for the time derivative $\dot{k}(t)$, and seeks to maximize a discounted utility function of consumption, integrated over a time period $[0,T]$. The model is expressed as:

$$\mathrm{MAX} \int_0^T e^{-\rho t}c(t)^{\tau}dt \quad \text{subject to } k(0)=k_0,$$

$$\dot{k}(t)=\zeta e^{qt}k(t)^{\beta}-\sigma k(t)-c(t),\quad k(T)=k_T.$$

No explicit bounds are stated for $k(t)$ and $c(t)$. However, both the formulas and their interpretation requires that both $k(t)$ and $c(t)$ remain positive. However, with some values of $u(t)$, the differential equation for $k(t)$ can bring $k(t)$ down to zero. The capital is the *state function* of this optimal control formulation, and the consumption is the *control function.* In general the control function is to be varied, subject to any stated bounds, in order to achieve the maximum. This model includes the standard features, namely an optimality criterion contained in an objective function which consists of the discounted sums of the utilities provided by consumption at every period, a finite planning horizon T, a positive discount rate, boundary conditions, namely initial values of the variables, and parameters and the terminal conditions on the state.

1.5. The structure of the book

Chapter 2 presents the relevant mathematics of optimization, and especially optimal control, including the formulation of dynamic economic and finance models as optimal control problems. Questions discussed include the following:

- When is an optimum reached, and when is it unique?
- Relaxing of convex assumptions, and of maximum to quasimax.
- Multiobjective optimal control, and the Pontryagin conditions for optimality for single-objective and multiobjective problems.

Some qualitatively different effects may occur with nonconvex models, such as non-unique optima, and jumps in the consumption function, which have economic significance.

In Chapter 3, algorithms for computing optimal control are discussed, with reasons for preferring a direct optimization approach, and step- function approximations. A computer package *SCOM* is described, developed by the present authors, for solving a class of optimal control problems in continuous time, using the MATLAB system, but in a different way from the *RIOTS_95* package (Schwartz, 1996), which also uses MATLAB. As in the MISER (Jennings et al., 1998) and OCIM (Craven et al., 1998) packages, the control is parametrised as a step-function, and MATLAB's *constr* package for constrained optimization is used as a subroutine. End-point conditions are simply handled using penalty terms. Much programming is made unnecessary by the matrix features built into MATLAB. Some economic models present computational difficulties because of implicit constraints, and there is some advantage using finite difference approximations for gradients. The Kendrick-Taylor model of economic growth is computed as an example.

Chapter 4 discusses the use of optimal control methods for computing some non-linear continuous optimal welfare, development, and growth models. Results are reported for computing the Kendrick-Taylor optimal-growth model using RIOTS_95 and SCOM programs based on the discretisation approach. Comparisons are made to the computational experiments with OCIM, and MISER. The results are used to compare and evaluate mathematical and economic properties, and computing criteria. While several computer packages are available for optimal control problems, they are not always suitable for particular classes of control problems, including some economic growth models.

Chapter 5 presents some proposed extensions for dynamic optimization modelling in finance, for characterizing optimal intertemporal allocation of financial and physical resources, adapted from developments in other areas of economics and mathematics. The extensions discussed concern (a) the elements of a dynamic optimization model, (b) an improved model including physical capital, (c) some computational experiments. It is sought to model,