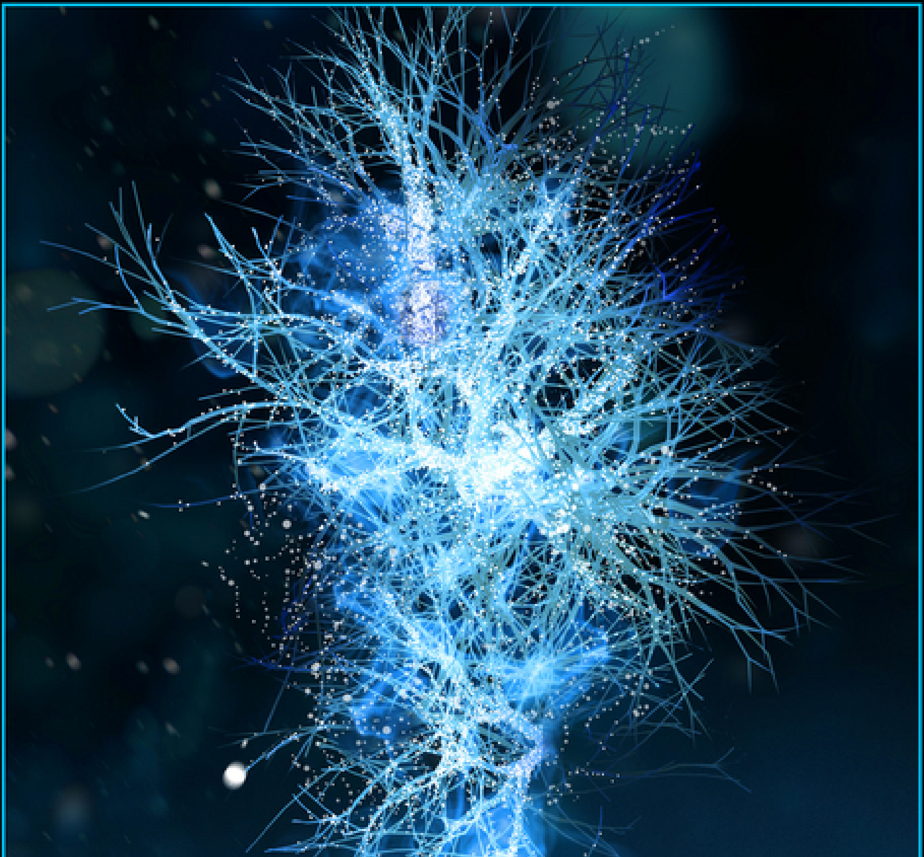
Vladimir Lepetic

# Principles of Mathematics

## *A Primer*



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# PRINCIPLES OF MATHEMATICS

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## A Primer

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*To Ivan and Marija*

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# PREFACE

I suspect that everyone working in academia remembers the days when as a student, being unhappy with the assigned textbook, they promised themselves one day to write the "right" book – one that would be easy and fun to read, have all the necessary material for successfully passing those (pesky) exams, but also be serious enough to incite the reader to dig further. It would, one hoped, open new venues to satisfy readers' curiosity provoked by frequent, but causal, remarks on the themes beyond the scope of an introductory text. Thus, the hope continues, students would be seduced by the intimation of mathematical immensity, sense the "flavor" of mathematical thinking, rather than just learning some "tricks of the trade." Of course, undertaking such an enterprise might easily be considered by many to be too ambitious at best and pure hubris at worst, for many potential readers are likely aware of the plethora of excellent books already available. Well then, why this book and not another one? First, this book is what its title says – a primer. However, unlike many other introductory texts, it is intended for a wide variety of readers; it (or parts of it) can be used as a starter for college undergraduate courses fulfilling "general education" math requirements, but also as an overture to more serious mathematics for students aspiring for careers in math and science. I am sure that rather smart high school students could also use the book to maintain and enhance their enthusiasm for mathematics and science. In any case, I am of the opinion that "introduction" and "rigor" should not exclude each other. Similarly, I don't think that avoiding discussion of weighty issues necessarily makes a text reader-friendly, in particular when mathematics and the sciences are in question. I do think, however, that many profound issues can be introduced accessibly to a beginner and, most importantly, in a way that provokes intellectual curiosity and consequently leads to a better appreciation of the field in question. Another thing, in my opinion equally significant, that any introductory mathematics text should convey is the importance of recognizing the difference and mutual interconnectedness in "knowing," "understanding," and "explaining" (i.e., "being able to explain") something. Admittedly any book, even the most advanced one, is an introduction in such an endeavor; yet one has to start somewhere so why not with a "primer" like this. So, with such philosophy in mind – which by the way can also serve as an apologia, albeit not a very transparent one – all efforts have been made to meticulously follow well-established mathematical formalism and routines. Incidentally, contrary to some educators, I don't think that the standard mathematical routine "definition–lemma–theorem–proof–corollary" is necessarily a deterrent to learning the subject. The only way one can see the "big picture" is to acquire a unique technique that will empower one to do so. The real beauty reveals itself after years of study and practice. If you want to successfully play a musical instrument you need to learn to read the notes, learn some music theory, and then relentlessly practice until you reach a reasonable command of your instrument. After years of doing so, and if you are lucky enough, you become an artist. The payoff, however, is immense.

The book contains six chapters and literally hundreds of solved problems. In addition, readers will find that every chapter ends with a number of supplementary exercises. It is my experience that in a one-semester course with class meeting twice a week, an instructor can leisurely cover two chapters of his/her choice, and they would still have ample time to pick and choose additional topics from other chapters they deem interesting/relevant. The first chapter contains a fairly detailed introduction to Set Theory, and it may be also considered as a conceptual/"philosophical" introduction to everything that follows. Chapters 2 and 3 on Logic and Proofs follow naturally, although I would not have many arguments against those who would prefer to start with Chapters 2 and 3 and subsequently discuss Set Theory. Readers interested only in Functions and/or Group Theory can, after Chapter 1, immediately jump to Chapters 4 and/or 5. Similarly, the last chapter on Linear Algebra can be approached independently provided, of course, the reader has been at least briefly introduced to the necessary preliminaries from Chapters 1, 4, and 5. Finally, the case could be made that a text of this kind, in order to justify an implicitly hinted philosophy, should necessarily have a chapter on Topology and Category Theory. I wholeheartedly agree. However, that would require adding at least another 300 pages to this primer, and the sheer volume of such a book would likely be a deterrent rather than an enticement for a beginner and thus defeat its very purpose. Postponing topology and category theory for some later time hopefully will be just a temporary weakness. It is not unreasonable to expect that after carefully going through Sets and Functions, for instance, the reader will anticipate further subtleties in need for clarification and reach for a book on topology. Similarly, those wondering about a possible theory that would subsume all others might find Category Theory an appropriate venue to reach that goal. In any case, provoking intellectual curiosity and imagination is the main purpose of this text, and the author wishes that the blame for any failure in this endeavor could be put, or at least partially placed, on the shoulders of the course instructor. Alas, the shortcomings are all mine.

I am aware that it is impossible, and no effort would be adequate, to express my gratitude to all of my teachers, colleagues, and students who have over decades influenced my thinking about mathematics. This impossibility notwithstanding, I need to mention Ivan Supek who, at my early age, introduced me to the unique thinking about mathematics, physics, and philosophy, coming directly from Heisenberg, whose assistant and personal friend he had been for years. Vladimir Devidé who discovered for me the world of Gödel and, many years later, my PhD adviser, Louis Kauffman, who put the final touch on those long fermenting ideas.

Lastly, I want to thank Ivan Lepetic, who painstakingly read the whole manuscript, made many corrections and improvements, and drew the illustrations.

VLADIMIR LEPETIC

# 1

# SET THEORY

> "The question for the ultimate foundations and the ultimate meaning of mathematics remains open; we do not know in which direction it will find its final solution nor even whether a final objective answer can be expected at all.
>
> "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization."
>
> H. Weyl[1]

## 1.1 INTRODUCTION

The fact that you chose to read this book makes it likely that you might have heard of Kurt Gödel,[2] the greatest logician since Aristotle,[3] whose arguably revolutionary discoveries influenced our views on mathematics, physics, and philosophy,

[1]Hermann Klaus Hugo Weyl (1885–1955), German mathematician, Yearbook of the American Philosophical Society, 1943 (copyright 1944).
[2]Kurt Gödel (1906–1978), Austrian–American logician, mathematician, and philosopher.
[3]J.A. Wheeler said that "if you called him the greatest logician since Aristotle you'd be downgrading him" (quoted in Bernstein, J., *Quantum Profiles*, Princeton University Press, 1991. Also in Wang, H. *A Logical Journey,* MIT Press, 1996).

*Principles of Mathematics: A Primer*, First Edition. Vladimir Lepetic.

comparable only to the discoveries of quantum mechanics. Well, even if you have not heard of him I want to start by rephrasing his famous theorem:

*Mathematics is inexhaustible!*

Notwithstanding the lack of a definition of what mathematics is, that still sounds wonderful, doesn't it? At this point, you may not fully understand the meaning of this "theorem" or appreciate its significance for mathematics and philosophy. You may even disagree with it, but I suppose you would agree with me that mathematics is the study of abstract structures with enormous applications to the "real world." Also, wouldn't you agree that the most impressive features of mathematics are its certainty, abstractness, and precision? That has always been the case, and mathematics continues to be a vibrant, constantly growing, and definitely different discipline from what it used to be. I hope you would also agree (at least after reading this book) that it possesses a unique beauty and elegance recognized from ancient times, and yet revealing its beauty more and more with/to every new generation of mathematicians. Where does it come from? Even if you accept the premise that it is a construct of our mind, you need to wonder how come it represents/reflects reality so faithfully, and in such a precise and elegant way. How come its formalism matches our intuition so neatly? Is that why we "trust" mathematics (not mathematicians) more than any other science; indeed, very often we define truth as a "mathematical truth" without asking for experimental verification of its claims? So, it is definitely reasonable to ask at the very beginning of our journey (and we will ask this question frequently as we go along): Does the world of mathematics exist outside of, and independently of, the physical world and the actions of the human mind? Gödel thought so. In any case, keep this question in mind as you go along – it has been in the minds of mathematicians and philosophers for centuries.

The set theory that we start with comes as a culmination of 2000 years of mathematics, with the work of the German mathematician George Cantor[4] in the 1890s. As much as the inception of set theory might have had (apparently) modest beginnings, there is virtually no mathematical field in which set theory doesn't enter as the very foundation of it. And it does it so flawlessly, so naturally, and in such a "how-could-it-be-otherwise" way, that one wonders why it took us so long to discover it. And arguably, there is no concept more fundamental than the concept of the set. (Indeed, try to answer the question: What is a real number without referring to set theory?) Be it as it may, now we have it. We start our journey through the "Principles," with the basic formalism of set theory.

No one shall be able to drive us from the paradise that Cantor created for us.[5]

D. Hilbert

[4]Georg Ferdinand Ludwig Philipp Cantor (1845–1918), German mathematician, the "father" of Set Theory.

[5]David Hilbert (1862–1943), German mathematician.

## 1.2 SET THEORY – DEFINITIONS, NOTATION, AND TERMINOLOGY – WHAT IS A SET?

You are probably familiar with the notion in mathematics of a set as a collection, an aggregate or a "group"[6] of certain "(some)things," or a collection of certain "objects"[7] that form a whole. We assume the existence of some domain of those "objects," out of which our mind will build a "whole." Cantor suggested that one should *imagine a set as a collection into a whole A of definite and separate objects of our intuition or our thought.* These objects are called members or elements of a set. For example, we can consider the set of all planets in the solar system,[8] or the set of all living people on Earth. Or, we can consider the set of all living females on this planet. Those would be well-defined sets, and by the very "definition," that is, the description of the set, our mind effortlessly constructs the concept of a "whole." On the other hand, calling for a set of all tall men, or a set of all big planets, triggers a similar concern. What is "a tall man" or "a big planet?" Obviously, describing a set of real objects by means of their characteristics can be problematic due to the imprecision of everyday language. So, it is fair to say that once the nature of objects defining a set is unambiguously stated, the whole entity, and not the individual elements, becomes the object of our study. Consequently, what we care about is the relationship between different sets as well as the very consistency of the "set" concept.

As you can see, at the very beginning of our discussion, we are introducing a concept that looks, to say the least, pretty vague, especially since we are doing mathematics, which we expect to be the epitome of precision. So, at this point in the process of devising our theory – *The Naïve Set Theory* – we will use the words "*set*" and "*is an element of*" without properly defining them. We will simply assume that we know exactly what they mean and hope that we won't run into any inconsistencies and paradoxes. In addition, we need the basic logical vocabulary consisting of "*not,*" "*and,*" "*or,*" and "*if … then …* ." That's it! With so little, how can one satisfy the credo of modern mathematics – a "philosophy" by the name of Cantorism – that *everything (mathematical) is a set*? This idea is not as outlandish as you may think, so I suggest you wait for a while before deciding whether to accept this doctrine or not. Remember the Pythagoreans[9] who thought that everything is a natural number. You can imagine their dismay upon

[6]To be precise, we want to make sure that here by the "group" we do not mean the mathematical term "group" as in Group Theory, but simply a group of certain objects or elements.
[7]The term "object" could be misleading too, for sometimes by the "object" people instinctively think of "(some)thing" that is, a "thing" that can be touched, seen, and so on. Since objects of a set theory can be ordinary things, like pencils, chairs, people, or animals, and they can also be very abstract in nature, like numbers, functions, and ideas, maybe the term "entity" instead of the "object" would be more appropriate.
[8]Of course, "all" in this case, by mathematical standards, might be a somewhat imprecise quantifier, but let's assume at this point that there will be no surprises of striping off a "planetary status" of an object in our solar system, as we have recently witnessed in the case of Pluto.
[9]Religious sect founded by Pythagoras of Samos (ca. 570–495) Ionian–Greek philosopher.

learning of the incommensurability of the side and the diagonal of a square. The discovery of $\sqrt{2}$ must have been a catastrophe for this secluded sect, let alone the pain of disclosing the findings to the uninitiated. Legend has it that for his unfortunate discovery Hippasus[10] was drowned by the members of this mystic brotherhood. Later, we learned about certain other sets of numbers – the set of real numbers, for instance – which is fundamentally more "infinite" than anything we knew before. To understand those we definitely need sets.[11] We may continue on this rather vague path and also say that a set is a "thing" that is a collection of other things (which themselves could be sets) called the elements of the set. These hazy definitions by synonym suffice for most purposes, for our mind is able to grasp (the essence of) the concept regardless of the abstractness of the definition. Indeed, we want these concepts to be sufficiently abstract in order to avoid contradictions, especially when dealing with the foundation of mathematics. At the same time, very few so "simple" ideas in mathematics proved to be so fecund with the repercussions to almost all fields of mathematics. Not surprisingly, Mathematical Logic and Philosophy of Mathematics in particular became exceptionally interesting and rich fields notwithstanding the paradoxes spurred by much ingenious work on the foundations of mathematics and set theory.

So, before we start with the formalism of set theory, I want to tell you something rather funny and interesting, something that will keep showing up over and over again in the foundation of mathematics. This will certainly provoke some curiosity in you and at the same time show you the richness of ideas that set theory contains, and how our mind detects paradoxes in apparently simple concepts – concepts that this very mind came up with. The following is known as the Russell[12] Paradox. (Remember, the notion of "*elementhood*" or "*membership*" does not prevent us from thinking of sets as being elements of (i.e., belonging to) other sets.) So, let's follow Cantor and imagine all the *definite distinguishable concepts of your/our intellect.* One of them could be the idea of unicorns – it doesn't matter that you/we know they don't "exist." (They do exist in your mind, right?) Well, let's think about the collection of definite concepts of our intellect that doesn't contain itself. Let me explain. It is easy to think of, say, a set of all horses (or unicorns if you wish) on Earth. This set obviously represents a set that does not contain itself as a member. A set of horses is not a horse, of course. Now, can you think of a set that would be a member of itself? How about a set of all ideas? It is an idea, right? So is it a member of itself? Or, how about a set of all sets? It is a reasonable idea too. But, it is again also a set, hence a member of itself. Well, let's think about it. Let's call any set that doesn't contain itself

[10]Hippasus of Metapontum (ca. fifth century BC), Pythagorean philosopher.

[11]Could it be that even sets are not "everything"? Well, yes! It is possible that we may need an even more fundamental structure to address, among other things, the even "greater," Absolute Infinities. The discussion of those we leave for some other time.

[12]Bertrand Arthur William Russell (1872–1970), British philosopher, logician, and mathematician.

as one of its elements an ordinary set, say, $\mathcal{O}$ and the one that does – an extraordinary, $\mathcal{E}$. Now, here is what Russell said: Consider a set of all ordinary sets $\mathcal{O}$. It exists – Cantor said so – since it is a distinguishable concept of one's intuition or one's thought. So we could safely claim:

1. $\mathcal{O}$ is an ordinary set!
   Suppose not. Suppose it is extraordinary and thus contains itself as one of its elements. But every set in $\mathcal{O}$ is ordinary. Thus $\mathcal{O}$ is ordinary. But this is a contradiction! Therefore, our assumption was wrong; $\mathcal{O}$ is definitely ordinary. Well, is it? No!? What if we say:
2. $\mathcal{O}$ is an extraordinary set!
   Suppose not. Suppose $\mathcal{O}$ is ordinary. Since $\mathcal{O}$ contains all ordinary sets, it has to contain itself as one of its members. But that makes it extraordinary. This is a contradiction. Our assumption that $\mathcal{O}$ is ordinary was wrong. Therefore, $\mathcal{O}$ is extraordinary.

Obviously (1) and (2) are contradictory.
Here is another well-known example of a finite set, which we cannot properly make out[13]:

Consider two sets of adjectives: set $\mathcal{A}$ of self-descriptive adjectives we call *autologous* (*autological)* and set $\mathcal{H}$ of nonself-descriptive adjectives, called *heterologous* (*heterological*), that is, the set of all adjectives not belonging to $\mathcal{A}$. For example, set $\mathcal{A}$ contains adjectives such as *English*, *finite*, *derived*, and *pentasyllabic*. That is, they do have the properties they describe. On the other hand, set $\mathcal{H}$ contains adjectives such as *German*, *French*, *black, white*, and *monosyllabic*, that is, obviously none of them belongs to $\mathcal{A}$. Now, what about "*heterologous*"? Which set does it belong to? What I am asking is this: Is "*heterologous*" heterologous?

If this sounds confusing to you, and it's perfectly all right if it does, for it is confusing indeed. Here is Russell again with an analogous "story" (and I assure you this is not some silly game of words) to help us out:

> *There is a small town with only one (strange) barber. The strange thing about him is that he shaves all men in town that do not shave themselves. Now, does he shave himself or not?*

So, what are we to make of it? At the very beginning, we are dealing only with two concepts, "*set*" and "*an element of*," and we are faced with a fundamental problem of definitions that seems irresolvable. We cannot allow a seed of contradictions sitting at the very concept we want as our foundation. How do we start?

[13]Due to Kurt Grelling (1886–1942) and Leonard Nelson (1882–1927), German mathematicians and philosophers.

How do we build a fundamental structure of mathematics, a structure precise enough and rich enough, to encapsulate "all of mathematics" and all the rules of inference, without contradictions and without any ambiguities? Mathematicians and philosophers have been thinking about these questions for thousands of years, going back to Euclid's[14] axiomatic treatment of geometry, to Leibnitz's[15] ideas of mathematical logic, to Hilbert's dream of unifying all of mathematics under the umbrella of a formal axiomatic system, to the works of Cantor, Russell, and Whitehead,[16] and many others. In any case, the theory that Cantor developed, indeed a mathematical theory unlike any before, proved to be the best candidate to fulfill that. Mathematics arose on a system of axioms and precise formalism, which we want to be

1. consistent;
2. complete; and
3. decidable.

That a formal system is "consistent" means that we should not be able to prove, in finitely many steps, an assertion and its negation at the same time. *A* and *not-A* cannot (should not) be true at the same time. By "complete" we mean a system that is rich enough to allow us to determine whether *A* or *not-A* is a theorem, that is, a true statement. And finally, "decidable" refers to what is known as "the decision problem" (the famous "Entscheidungsproblem" in German), that is, a procedure, an algorithm by which we can (always) determine, in a finite number of steps, whether something is a theorem or not. That's what we want. Not much to ask for, wouldn't you say? After all, consistent and complete should imply that a decision procedure is at hand. Well, it's not. It can't be done! Mr Gödel said so.[17]

Here is how Hilary Putnam[18] "illustrates" Gödel's theorem:

(i) *That, even if some arithmetical (or set-theoretical) statements have no truth value, still, to say of any arithmetical (or set-theoretical) statement that it has (or lacks) a truth value is itself always either true or false (i.e. the statement either has a truth value or it doesn't).*

[14] Euclid (of Alexandria) (ca. 325–270 BC), Greek mathematician/geometer.
[15] Gottfried Wilhelm Leibniz (1646–1716), German mathematician and philosopher.
[16] Alfred North Whitehead (1861–1947), British mathematician, logician, and philosopher.
[17] "The human mind is incapable of formulating all its mathematical intuitions, that is, if it has succeeded in formulating some of them, this very fact yields new intuitive knowledge, for example, the consistency of this formalism. This may be called the 'incompletability' of mathematics." Kurt Gödel, *Collected Works,* Oxford University Press, 2001.
[18] Putnam, H., *Mathematics Without Foundation,* in Philosophy of Mathematics, 2nd ed., Cambridge University Press, 1983.

(ii) *All and only decidable statements have a truth value.*

*For a statement that a mathematical statement $S$ is decidable may itself be undecidable. Then, by (ii), it has no truth value to say "$S$ has a truth value" (in fact falsity; since if $S$ has a truth value, then $S$ is decidable, by (ii), and if $S$ is decidable, then "$S$ is decidable" is also decidable). Since it is false (by the previous parenthetical remark) to say "$S$ has a truth value" and since we accept the equivalence of "$S$ has a truth value" and "$S$ is decidable", then it must also be false to say "$S$ is decidable". But it has no truth value to say "$S$ is decidable". Contradiction.*

Did you get it? Think about it. It literally grows on you. The whole point of all of "this" is that you start getting a "feel" for what mathematics really is and where we are actually "going." Anyway, after this "warm-up," let's start slowly and from the beginning.

First, we assume that there is a *domain,* or *universe* $\mathcal{U}$, of objects, some of which are sets.

Next, we need the formalism in which all our statements about sets can be precisely written – let's call it ***the language of set theory*** $\mathcal{L}$. This formal language contains a specific *alphabet,* that is, a list of symbols that we judiciously use and a number of specific statements that are called *axioms.* What are they? Well, in order to start somewhere and in order to avoid an infinite regress, we choose (there has to be (?)) a set of propositions that are not proved (not provable) but can be used in sound construction of our formalism. In addition, we create a basis for (all?) mathematics, which is inherently beautiful, and thus we can use it as an aesthetical criterion that all other sciences can measure up to. Similarly, there exists a collection of (mathematical) words or symbols that we do not define in terms of others – undefined does not mean meaningless – but simply take as given. Those we call primitives. This idea is as old as mathematics itself. Remember Euclid? The first lines of his *Elements* read as follows:

1. A point is that which has no parts.
2. A curve is length without width.
3. The extremity of a curve is a point.
4. A surface is that which has only a length and a width.
5. The extremity of a surface is a curve, and so on.

Surely, you feel some uneasiness about these statements. Still, the whole gigantic structure of Euclidean geometry, unquestioned for 2000 years, is based on these axioms. Putting aside the controversy among mathematicians on how fundamental these axioms are in general, as well as the question of their effectiveness, these axioms are needed and they are here to stay.

We also need the *formal rules of inference* so that the *language* we use is precise enough to derive all the theorems of our theory.

In addition to the aforementioned four basic symbols, we will soon need some more. So, we list the somewhat extensive alphabet of the language we are going to use throughout the book:

$\in$ : element; a member; $x \in A$ : $x$ is an element of set $A$

$\notin$ : not an element; not a member; $x \notin A$ : $x$ is not an element of set $A$

$\ni$ : such that; sometimes "s.t."

$^{c}$ : complement; $A^{c}$: complement of set $A$

$\setminus$ : difference; $A \setminus B$: $A$ difference $B$; sometimes just: $A$ "minus" $B$

$\Delta$ : symmetric difference: $A \Delta B$: symmetric difference of $A$ and $B$

$\subseteq$ : subset; $A \subseteq B$: $A$ is a subset of $B$

$\subset$ : proper subset: $A \subset B$

$\cap$ : intersection: $A \cap B$

$\cup$ : union: $A \cup B$

$\varnothing$ : the empty set

$\times$ : Cartesian product; $A \times B$: Cartesian product of sets $A$ and $B$

$\mathbf{N}$ : the natural numbers

$\mathbf{Z}$ : the integers

$\mathbf{Q}$ : the rational numbers

$\mathbf{R}$ : the real numbers

$\mathbf{Z}^{+}$ : the nonnegative integers

$\mathbf{Q}^{+}$ : the nonnegative rational numbers

$\mathbf{R}^{+}$ : the nonnegative real numbers

$|A|$ : the cardinal number (cardinality) of $A$

$\forall$ : for all; for every; for any; $\forall x \in A$: for every $x$ from $A$

$\exists$ : there exists

$\exists!$: there exists a unique …

$\nexists$ : (same as $\sim \exists$) does not exist

$\wedge$ : and; sometimes also "&"

$\vee$ : or

$\rightarrow$ : "conditional"; "implication"; $a \rightarrow b$ if $a$ then $b$. Sometimes same as "$\Rightarrow$"

$\leftrightarrow$ : "biconditional"; $a \leftrightarrow b$ $a$ if and only if $b$; "iff"; Sometimes same as "$\Leftrightarrow$"

$\sim$ : "negation"; "it is not the case that"; "opposite of"

$=$ : equal

$\equiv$ : equivalent

iff: "if and only if"; "$\Leftrightarrow$"; "$\leftrightarrow$"

**Definition 1.1** A set is said to be a well-defined set iff there is a method of determining whether a particular object is an element of that set.

The precise "description" of a set and its elements is based on the following axioms.

**Axiom 0 (Set Existence)**[19] There exists a *set*, that is, $\exists A\ (A = A)$. In other words, we postulate that there exists something, a "thing," an entity, we call a *set.*

Once a set $A$ is given, we say that "*a is an element of A*" or that "*a is a member of A,*" and we write $a \in A$. Similarly, if $a$ is not a member of $A$, we simply write $a \notin A$.

It is worth mentioning again that the expression "*an element of,*" that is, an elementhood relation, is also the elemental concept for which it is difficult to find a suitable alternative, so we also take it as an undefined predicate.

**Example 1.1**

$$A = \{a, b, c, d, e, f\}$$

is a set whose elements are $a, b, c, d, e, f,$ that is, $a \in A, b \in A, c \in A,$ and so on. This is nicely illustrated by the Venn diagram (Figure 1.1).

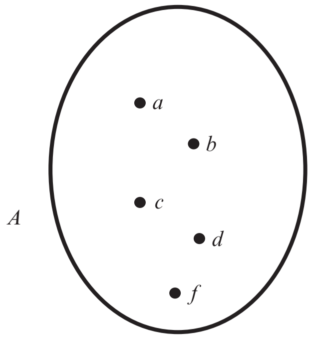


**Figure 1.1** Venn diagram

Often it is convenient, especially when it is impossible to list all the elements of a set, to introduce a set using the so-called set-builder notation. We write

$$A = \{x | P(x)\}$$

and we read: $A$ is a set of all $x$, such that $P(x)$, where $P(x)$ designates some property that all $x$'*s* possess, or $P$ is a condition that specifies some property of all objects $x$.

[19]We will have more to say about these axioms later.

For instance, if we want $A$ to be a set of all natural numbers greater or equal to 5 we write:

$$A = \{x|\ x \geq 5, \quad x \in \mathbf{N}\ \}$$

Certainly nothing prevents us from considering a set whose elements are also sets. In other words, we can have a set $X = \{x,\ y,\ w,\ z\}$, where $x,\ y,\ w,$ and $z$ are sets themselves. ■

**Example 1.2** Suppose we consider

$$X = \{\text{Alice}, \text{Bob}\}$$

as a set whose two elements are persons Alice and Bob. *Set X* is definitely different from set, say,

$$Y = \{\text{Alice}, \{\text{Bob}\}\}$$

which also has two elements, but this time the elements are: Alice and $\{Bob\}$, that is, the element $\{Bob\}$ is itself a set containing one element – *Bob.*

Formally, we write:

$$\text{Alice} \in Y, \text{Bob} \notin Y, \text{but}\{\text{Bob}\} \in Y$$

Of course, we could have constructed a set

$$Z = \{\{\text{Alice}, \{\text{Bob}\}\}\}$$

which has only one element, namely, $Y$. Do you see why? It may help if we represent sets by Venn diagrams, where $X, Y,$ and $Z$ (Figure 1.2) look as follows:

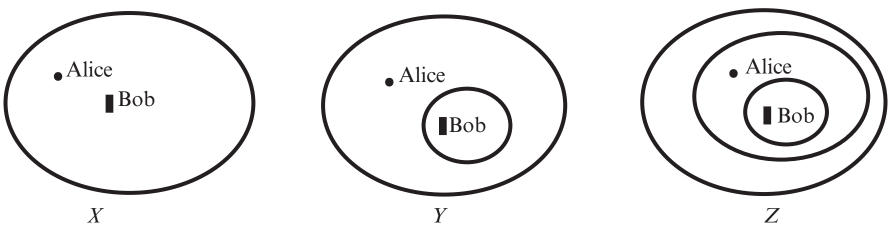


**Figure 1.2** Sets *X, Y,* and *Z*

■

**Axiom 1 (Axiom of extensionality)** A set is uniquely determined by the elements it contains, that is, two sets are considered equal if they have the same elements. Less clearly but often said: a set is determined by its extension.

**Example 1.3** Sets $A = \{a, b, c, d\}$ and $B = \{d, a, a, a, b, c, c, d\}$ are considered the same, that is, we say that $A = B$. ■

So, we have

**Definition 1.2** Given sets $A$ and $B$, we say that $A$ equals $B$, and we write $A = B$ if and only if every element of $A$ is an element of $B$ and every element of $B$ is an element of $A$. For the sake of completeness and more precision (at this point maybe prematurely[20]), using formal logic notation, we express this as follows:

$$A = B \leftrightarrow (\forall x)(x \in A \leftrightarrow x \in B)$$

**Definition 1.3** Given two sets $A$ and $B$, we say that $A$ is a subset of $B$, and we write $A \subseteq B$ if and only if every element of $A$ is also an element of $B$ (Figure 1.3), that is,

$$A \subseteq B \leftrightarrow (\forall \;\; x \in A, x \in B)$$

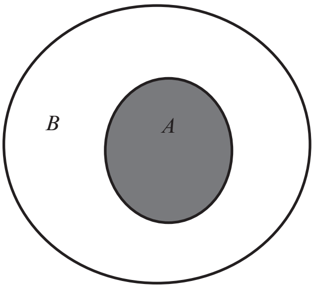


**Figure 1.3** Subset $A \subseteq B$

Note that $B$ could be "larger" than $A$, that is, that all elements of $A$ are elements of $B$, but not all elements of $B$ are elements of $A$. To distinguish between these subtleties, we state the following

**Definition 1.4** Given two sets $A$ and $B$, we say that $A$ is a proper subset of $B$, $A \subseteq B,$ if and only if every element of $A$ is an element of $B$, but not all elements of $B$ are elements of $A$.

Equality of sets can now be restated as

$$A = B \leftrightarrow A \subseteq B \& B \subseteq A$$

[20]This formalism will become more clear after you have studied Chapter 2.

What we are saying here is that two sets are considered equal solely on the basis of their elements (i.e., what's in the sets and how many) and not on the "arrangement" or a repeat of some of the elements in the respective sets.

**Example 1.4** Show that, if a set $A$ is a set of all integers $n$, where every $n$ is expressible as $n = 2p$, with $p \in \mathbf{Z}$, that is,

$$A = \{n \in \mathbf{Z} | n = 2p, \quad p \in \mathbf{Z}\}$$

and $B$ analogously described as

$$B = \{m \in \mathbf{Z} \,| m = 2q - 2, \quad q \in \mathbf{Z}\}$$

then $A = B$.

**Solution** Set $A$ is the set of all even integers. We would like to see whether any integer of the form $2p$, for some $p \in \mathbf{Z}$, can also be written in the form $2q$–$2$, for some $q \in \mathbf{Z}$. Suppose there is an $n \in \mathbf{Z}$, such that $n = 2p$, for some integer $p$ we want to find an integer $q$, so that $n = 2q - 2$. Thus,

$$\begin{aligned} 2q - 2 &= 2p \\ 2q &= 2p + 2 = 2(p + 1) \\ q &= p + 1 \end{aligned}$$

Therefore, for $n = 2p$, and $p \in \mathbf{Z}$, $q = p + 1$. It follows that

$$2q - 2 = 2(p + 1) - 2 = 2p + 2 - 2 = 2p$$

Hence, $A \subseteq B$.

Let's now assume that an integer can be expressed as $m = 2q - 2$, for some $q \in \mathbf{Z}$. Suppose, furthermore, that

$$2p = 2q - 2 = 2(q - 1)$$

that is,

$$p = q - 1$$

So, if $m = 2q - 2$, with $q \in \mathbf{Z}$, we write

$$2p = 2(q - 1) = 2q - 2$$

We conclude that $B \subseteq A$. Since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$ by definition of set equality. ■

**Example 1.5** Let $A$ be a set of all solutions of the equation $x^2 = 2x$, and let $B$ be a set of all solutions of the equation $(x - 1)^2 = 1$. Then, it is easy to see that $A = B$. ■

**Axiom 2 (Comprehension axiom)**[21]

(i) For any *reasonable*[22] property $P$, there exists a set containing exactly those elements that are defined by that property; In particular, mathematical entities that have a certain property in common constitute a set.

Certainly nothing prevents us from considering a set whose elements are also sets. In other words, we can have a set $X = \{x, y, w, z\}$, where $x, y, w$, and $z$ are sets themselves. So we postulate:

(ii) Sets are mathematical entities, and, hence, they may in turn appear as elements of a set.

This is one of the reasons why one should not restrict oneself on a style of letters that represent sets. Thus, although we will most frequently use capitals to designate sets, occasionally it will be more convenient to use lowercase letters.

**Example 1.6** Let $x_1, x_2, \dots, x_n$ be a collection of $n$ sets, then

$$X = \{x_1, x_2, \dots, x_n\}$$

is also a set (Figure 1.4).

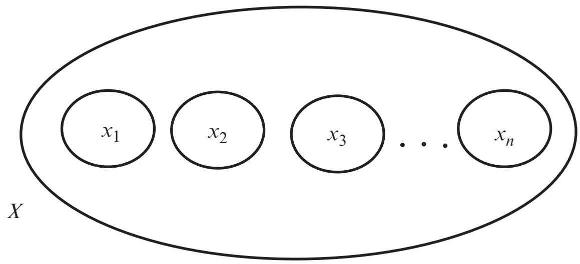


**Figure 1.4**

[21] This is sometimes called the Comprehensive principle.
[22] What is "*reasonable*" is debatable, and in any case a rather vague concept. We won't be discussing these subtleties here.

Having elements of a set being sets themselves gives us more flexibility in dealing with only one kind of object. Thus, we don't need to postulate the existence of every possible element of the various structures we intend to study. It follows, let's emphasize this again, that every set $x$ is a unique element of another set, namely, $\{x\}$.

After accepting the fact that the elements of a set are sets, let's take a closer look at Axiom 2: Let $X$ be a set, and let $Y$ be a set whose elements are exactly those elements $x \in X$ with a property $P$, that is,

$$Y = \{x \in X | P(x)\}$$

So, let the particular property be $x \notin x$. (Remember, $x$ is a set.) In other words, whatever set $X$ may be, if

$$Y = \{x \in X | x \notin x\}$$

then for every $y$,

$$y \in Y \quad \text{iff} \quad y \in X \quad \text{and} \quad y \notin y \tag{*}$$

Is it possible that $Y \in X$? Let's see. If $Y \in X$, we have two possibilities: either $Y \in Y$ or $Y \notin Y$. Suppose $Y \in Y$. Then, from $Y \in X$ and (*) it follows that $Y \notin Y$ – obviously a contradiction. Suppose that $Y \notin Y$. Then, again, from $Y \in X$ and (*) it follows that $Y \in Y$ – a contraction again. We conclude that it is impossible that $Y \in X$. (You may remember this argument from before.)

Now, let me digress a bit and say something about two very important concepts that will be discussed in much more detail in Chapter 4. Many readers are familiar with the concepts of *relations* and *function*: For the time being, let's just say that:

A *relation* $R$ is uniquely determined by pairs of elements $x$ and $y$ that are somehow related.

A *function* $f : X \to Y$ is uniquely determined by the pairs of two objects, an argument $x \in X$ and a functional value $f(x) \in Y$.

Now let's look at these via Axioms 1 and 2: For instance, the usual relation $\leq$ on the set of natural numbers describes a particular property, so we can construct a set $R$ consisting of pairs of natural numbers $(a, b)$ where $a \leq b$, that is

$$R = \{(a, b) | a \leq b, a, b \in \mathbf{N}\}$$

Similarly, we think of a function $f$ as the following set of pairs:

$$f = \{(x, f(x)) | x \in X, f(x) \in Y\}$$

■