Nonlinear Optimization with Financial Applications

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Preface

This book has grown out of undergraduate and postgraduate lecture courses given at the University of Hertfordshire and the University of Bergamo. Its primary focus is on numerical methods for nonlinear optimization. Such methods can be applied to many practical problems in science, engineering and management: but, to provide a coherent secondary theme, the applications considered here are all drawn from financial mathematics. (This puts the book in good company since many classical texts in mathematics also dealt with commercial arithmetic.) In particular, the examples and case studies are concerned with portfolio selection and with time-series problems such as fitting trendlines and trend-channels to market data.

The content is intended to be suitable for final-year undergraduate students in mathematics (or other subjects with a high mathematical or computational content) and exercises are provided at the end of most sections. However the book should also be useful for postgraduate students and for other researchers and practitioners who need a foundation for work involving development or application of optimization algorithms. It is assumed that readers have an understanding of the algebra of matrices and vectors and of the Taylor and Mean Value Theorems in several variables. Prior experience of using computational methods for problems such as solving systems of linear equations is also desirable, as is familiarity with iterative algorithms (e.g., Newton's method for nonlinear equations in one variable).

The approach adopted in this book is a blend of the practical and theoretical. A description and derivation is given for most of the currently popular methods for continuous nonlinear optimization. For each method, important convergence results are outlined (and we provide proofs when it seems instructive to do so). This theoretical material is complemented by numerical illustrations which give a flavour of how the methods perform in practice.

It is not always obvious where to draw the line between general descriptions of algorithms and the more subtle and detailed considerations relating to research issues. The particular themes and emphases in this book have grown out of the author's experience at the Numerical Optimization Centre (NOC). This was established in 1968 at the Hatfield College of Technology (predecessor of the University of Hertfordshire) as a centre for research in optimization techniques. Since its foundation, the NOC has been engaged in algorithm development and consultancy work (mostly in the aerospace industry). The NOC staff has included, at various times, Laurence Dixon, Ed Hersom, Joanna Gomulka, Sean McKeown and Zohair Maany who have all made contributions to the state-of-the-art in fields as diverse as quasi-Newton methods, sequential quadratic programming, nonlinear least-squares, global optimization, optimal control and automatic differentiation.

The computational results quoted in this book have been obtained using a Fortran90 module called SAMPO. This is based on the NOC's OPTIMA library – a suite of subroutines for different types of minimization problem. The name SAMPO serves as an acronym for Software And Methods for Portfolio Optimization. (However, it is also worth mentioning that *The Sampo* appears in Finnish mythology as a magical machine which grinds out endless supplies of corn, salt and gold. Its relevance to a book about financial mathematics needs no further comment.) The SAMPO software is not described in detail in this book (which does not deal with any programming issues). Interested readers can obtain it from an `ftp` site as specified in the appendix. Some of the student exercises can be attempted using SAMPO but most of them can also be tackled in other ways. For instance, the SOLVER tool in Microsoft Excel can handle both constrained and unconstrained optimization problems. Alternatively, users of MATLAB, can access a comprehensive toolbox of optimization procedures. The NAG libraries in C and Fortran include a wide range of minimization routines and the NEOS facility at the Argonne National Laboratories allows users to submit optimization problems via email.

I am indebted to a number of people for help in the writing of this book. Besides the NOC colleagues already mentioned, I would like to thank Alan Davies and all the staff in the Mathematics Department at the University of Hertfordshire for their support. I am particularly grateful to Don Busolini and Steve Kane, for introducing me to the financial applications in this book, and to Steve Parkhurst for sharing the lecture courses which underpin it. I have received encouragement and advice from Marida Bertocchi of the Universty of Bergamo, Alistair Forbes of the National Physical Laboratory, Berc Rustem of

Imperial College and Ming Zuo of the University of Alberta. Useful comments and criticisms have also been made by students who were the guinea-pigs for early versions of the text. Surrounded by such clouds of witnesses, any mistakes or omissions that remain in the text are entirely my responsibility.

My thanks are also due to John Martindale and Angela Quilici-Burke at Kluwer Academic Publishing for their encouragement and help with the preparation of the final version of the book.

Most of all, my deepest thanks go to my wife Nancy Mattson who put up with the hours I spent *incommunicado* and crouched over my laptop! Nancy does not share my attachment to computational mathematics: but she and I do have a common commitment to poetry. Therefore, scattered through the book, the reader will find a number of short mathematically-flavoured poems which occupy pages that would otherwise have been mere white space. To those who have asked how mathematical aptitude interacts with poetic composition a reply could be

Two cultures

Poets show, don't tell:
build metaphors from concrete
and specific bricks.

In mathematics
the abstract and general's
our bread and butter.

For more on this subject, the reader may wish to consult [66, 67, 68].

Chapter 1

PORTFOLIO OPTIMIZATION

1. Nonlinear optimization

Optimization involves finding minimum (or maximum) values of functions. Practical applications include calculating a spacecraft launch trajectory to maximize payload in orbit or planning distribution schedules to minimize transportation costs or delivery times. In such cases we seek values for certain *optimization variables* to obtain the *optimum* value for an *objective function*

We now give a simple example of the formulation of an optimization problem. Suppose we have a sum of money M to split between three managed investment funds which claim to offer percentage rates of return r_1, r_2 and r_3. If we invest amounts y_1, y_2 and y_3 we can expect our overall return to be

$$R = \frac{r_1 y_1 + r_2 y_2 + r_3 y_3}{M} \%.$$

If the management charge associated with the i-th fund is calculated as $c_i y_i$, then the total cost of making the investment is

$$C = c_1 y_1 + c_2 y_2 + c_3 y_3.$$

Suppose we are aiming for a return $R_p\%$ and that we want to pay the least charges to achieve this. Then we need to find y_1, y_2 and y_3 to solve the problem

$$\text{Minimize} \quad c_1 y_1 + c_2 y_2 + c_3 y_3 \tag{1.1.1}$$

$$\text{subject to} \quad r_1 y_1 + r_2 y_2 + r_3 y_3 = M R_p; \quad y_1 + y_2 + y_3 = M \tag{1.1.2}$$

$$\text{and} \quad y_1 \geq 0, \; y_2 \geq 0, \; y_3 \geq 0. \tag{1.1.3}$$

This is an optimization problem involving both equality and inequality *constraints* to restrict the values of the variables. The inequalities are included because investments must obviously be positive. If we tried to solve the problem without these restrictions then an optimization algorithm would attempt to reduce costs by making one or more of the y_i large and negative.

In fact, since they only involve linear expressions, (1.1.1) – (1.1.3) represents a *Linear Programming* problem. However, we do not wish to limit ourselves to linear programming (which is a substantial body of study in its own right – see, for instance, [1] – and involves a number of specialised methods not covered in this book). Instead, we shall be concerned with the more general problem in which the function and/or the constraints are nonlinear.

The simple investment decision considered above could be represented as a *nonlinear* programming problem if we imagined that an attempt to make a negative investment would be penalised by a very high management charge! Thus we could consider the problem

$$\text{Minimize} \quad c_1y_1 + c_2y_2 + c_3y_3 + K\sum_{i=1}^{3}\psi(y_i) \tag{1.1.4}$$

$$\text{subject to} \quad r_1y_1 + r_2y_2 + r_3y_3 = MR_p \quad \text{and} \quad y_1 + y_2 + y_3 = M. \tag{1.1.5}$$

where K is a large positive constant and the function ψ is defined by

$$\psi(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$$

The objective function (1.1.4) is now nonlinear (being linear for some values of the y_i and quadratic for others) and it features two linear equality constraints.

Equality constraints can sometimes be used to eliminate variables from an optimization problem. Thus, since $y_3 = M - y_1 - y_2$, we can transform (1.1.4), (1.1.5) into the problem of minimizing the two-variable function

$$c_1y_1 + c_2y_2 + c_3(M - y_1 - y_2) + K\sum_{i=1}^{2}\psi(y_i) + K\psi(M - y_1 - y_2) \tag{1.1.6}$$

subject to the single constraint

$$r_1y_1 + r_2y_2 + r_3(M - y_1 - y_2) = MR_p. \tag{1.1.7}$$

Obviously (see Exercise 1 below) we could go further and use the equality constraint (1.1.7) to express y_1 in terms of y_2. Our problem could then be expressed as an *unconstrained* minimization of a function of a single variable.

In the chapters which follow we shall describe several methods for solving nonlinear optimzation problems, with and without constraints. Applications of these methods will be taken from the field of portfolio optimization, using ideas due to Markowitz [2, 3] which are introduced in the next section.

Exercises

1. Reformulate (1.1.6), (1.1.7) to give an unconstrained minimization problem involving y_1 only.

2. Using the values $M = 1000, R_p = 1.25, K = 10^{-3}$ together with

$$c_1 = 0.01,\ \ c_2 = 0.009,\ \ c_3 = 0.014 \ \text{ and } \ r_1 = 1.2,\ \ r_2 = 1.1,\ \ r_3 = 1.4$$

plot a graph of the function obtained in question 1 in the range $0 \le y_i \le M$. Hence deduce the minimum-cost that produces a return of 1.25%.
How does the solution change in the cases where $c_3 = 0.012$ and $c_3 = 0.011$?

3. Formulate the problem of finding the maximum return that can be obtained for a fixed management cost C_f. (Note that the problem of maximizing a function $F(x)$ is equivalent to the problem of minimizing $-F(x)$.)

2. Portfolio return and risk

Suppose we have a history of percentage returns, over m time periods, for each of a group of n assets (such as shares, bonds etc.). We can use this information as a guide to future investments. As an example, consider the following data for three assets over six months.

	Return % for					
	January	February	March	April	May	June
Asset 1	1.2	1.3	1.4	1.5	1.1	1.2
Asset 2	1.3	1.0	0.8	0.9	1.4	1.3
Asset 3	0.9	1.1	1.0	1.1	1.1	1.3

Table 1.1. Monthly rates of return on three assets

In general, we can calculate the mean return $\bar{r}_i$ for each asset as

$$\bar{r}_i = \frac{\sum_{j=1}^{m} r_{ij}}{m}.$$

where r_{ij} $(i = 1,..,n,\ \ j = 1,..,m)$ denotes the return on asset i in period j. Hence, for the data in Table 1.1 we get

$$\bar{r}_1 \approx 1.2833;\ \ \bar{r}_2 \approx 1.1167;\ \ \bar{r}_3 \approx 1.0833. \qquad (1.2.1)$$

If we spread an investment across the n assets and if y_i denotes the fraction invested in asset i then the values of the y_i define a *portfolio*. Since *all* investment must be split between the n assets, the invested fractions must satisfy

$$S = \sum_{i=1}^{n} y_i = 1. \tag{1.2.2}$$

The *expected portfolio return* is given by

$$R = \sum_{i=1}^{n} \bar{r}_i y_i. \tag{1.2.3}$$

Thus, for the data in Table 1.1, we might choose to put half our investment in asset 1, one-third in asset 2 and one-sixth in asset 3. This would give an expected return

$$R = 1.2833 \times \frac{1}{2} + 1.1167 \times \frac{1}{3} + 1.0833 \times \frac{1}{6} \approx 1.194\%.$$

The *risk* associated with a particular portfolio is determined from *variances and covariances* that can be calculated from the history of returns r_{ij}. The variance of asset i is

$$\sigma_i^2 = \frac{\sum_{j=1}^{m} (r_{ij} - \bar{r}_i)^2}{m} \tag{1.2.4}$$

while the covariance of assets i and k is

$$\sigma_{ik} = \frac{\sum_{j=1}^{m} (r_{ij} - \bar{r}_i)(r_{kj} - \bar{r}_k)}{m}. \tag{1.2.5}$$

Evaluating (1.2.4), (1.2.5) for the returns in Table 1.1 we get

$$\sigma_1^2 = 0.0181;\ \ \sigma_2^2 = 0.0514;\ \ \sigma_3^2 = 0.0147; \tag{1.2.6}$$

$$\sigma_{12} = -0.0281;\ \ \sigma_{13} = -0.00194;\ \ \sigma_{23} = 0.00528. \tag{1.2.7}$$

The *variance* of the portfolio defined by the investment fractions $y_1, ..., y_n$ is

$$V = \sum_{i=1}^{n} \sigma_i^2 y_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sigma_{ij} y_i y_j \tag{1.2.8}$$

which can be used as a measure of portfolio risk. Thus for a three asset problem

$$V = \sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 + \sigma_3^2 y_3^2 + 2\sigma_{12} y_1 y_2 + 2\sigma_{13} y_1 y_3 + 2\sigma_{23} y_2 y_3. \tag{1.2.9}$$

Using (1.2.6) and (1.2.7), the risk function V for the data in Table 1.1 is

$$0.0181y_1^2 + 0.0514y_2^2 + 0.0147y_3^2 - 0.0562y_1y_2 - 0.00388y_1y_3 + 0.01056y_2y_3.$$

Thus, for a portfolio defined by $y_1 = \frac{1}{2}$, $y_2 = \frac{1}{3}$, $y_3 = \frac{1}{6}$

$$V = 0.0181 \times \frac{1}{4} + 0.0514 \times \frac{1}{9} + 0.0147 \times \frac{1}{36} - 0.0563 \times \frac{1}{6}$$
$$-0.00388 \times \frac{1}{12} + 0.01056 \times \frac{1}{18}$$

which gives a risk $V \approx 0.00153$.

The return and risk functions (1.2.3) and (1.2.8) can be written more conveniently using matrix-vector notation. Expression (1.2.3) is equivalent to

$$R = \bar{r}^T y \tag{1.2.10}$$

where $\bar{r}$ denotes the column vector $(\bar{r}_1, \bar{r}_2, ..., \bar{r}_n)^T$ and y is the column vector of invested fractions $(y_1, y_2, ..., y_n)^T$. Moreover, we can express (1.2.8) as

$$V = y^T Q y \tag{1.2.11}$$

where the *variance-covariance matrix* Q is defined by

$$Q = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{in} \\ \sigma_{12} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \sigma_{ij} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \sigma_{1n} & \sigma_{2n} & \dots & \sigma_{nn} \end{pmatrix}. \tag{1.2.12}$$

In (1.2.12) we have used the equivalent notation

$$\sigma_{ii} = \sigma_i^2. \tag{1.2.13}$$

Thus, for data in Table 1.1,

$$V = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} 0.0181 & -0.0281 & -0.00194 \\ -0.0281 & 0.0514 & 0.00528 \\ -0.00194 & 0.00528 & 0.0147 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

We can also write (1.2.2) in vector form as

$$S = e^T y = 1 \tag{1.2.14}$$

where e denotes the n-vector $(1, 1, .., 1)^T$.

Exercises

1. Returns on assets such as shares can be obtained from day-to-day stock market prices. If the closing prices of a share over five days trading are

$$P_1 = 4.62,\ P_2 = 4.39,\ P_3 = 3.64,\ P_4 = 3.89,\ P_5 = 4.23$$

calculate the corresponding returns using the formula

$$r_i = 100\frac{P_i - P_{i-1}}{P_{i-1}}.$$

What returns are given by the alternative formula

$$r_i = 100\log_e(\frac{P_i}{P_{i-1}})?$$

Show that the two formulae for r_i give similar results when $|P_i - P_{i-1}|$ is small.

2. Calculate the mean returns and the variance-covariance matrix for the asset data in Table 1.2.

	Return % for				
	Day 1	Day 2	Day 3	Day 4	Day 5
Asset 1	0.15	0.26	0.18	0.04	0.06
Asset 2	0.04	-0.07	-0.05	0.07	0.03
Asset 3	0.11	0.21	0.06	-0.06	0.12

Table 1.2. Daily rates of return on three assets

3. Optimizing two-asset portfolios

We begin by considering simple portfolios involving only two assets. There are several ways to define an optimal choice for the invested fractions y_1 and y_2 and each of them leads to a one-variable minimization problem. The ideas introduced in this section can be extended to the more realistic case when there are n (> 2) assets.

The basic minimum risk problem

A major concern in portfolio selection is the minimization of risk. In its simplest form, this means finding invested fractions $y_1, ..., y_n$ to solve the problem

Minrisk0

$$\text{Minimize} \quad V = y^T Q y \quad \text{subject to} \quad \sum_{i=1}^{n} y_i = 1. \qquad (1.3.1)$$

Like the earlier example (1.1.4), (1.1.5), this is an equality constrained minimization problem for which a number of solution methods will be given in later chapters. Our first approach involves using the constraint to eliminate y_n and yield an unconstrained minimization problem involving only $y_1,...,y_{n-1}$.

In the particular case of a two-asset problem, we wish to minimize $V = y^TQy$ with the constraint $y_1 + y_2 = 1$. If we write x in place of the unknown fraction y_1 then $y_2 = 1 - y_1 = 1 - x$. We can now define

$$\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.3.2}$$

and then it follows that $y = \alpha + \beta x$ so that problem **Minrisk0** becomes

$$\text{Minimize} \quad V = (\alpha + \beta x)^T Q(\alpha + \beta x). \tag{1.3.3}$$

Expanding the matrix-vector product we get

$$V = \alpha^T Q\alpha + (2\beta^T Q\alpha)x + (\beta^T Q\beta)x^2.$$

At a minimum, the first derivative of V is zero. Now

$$\frac{dV}{dx} = 2\beta^T Q\alpha + 2\beta^T Q\beta x \tag{1.3.4}$$

and so there is a stationary point at

$$x = -\frac{\beta^T Q\alpha}{\beta^T Q\beta}. \tag{1.3.5}$$

This stationary point will be a minimum (see chapter 2) if the second derivative of V is positive. In fact

$$\frac{d^2V}{dx^2} = 2\beta^T Q\beta$$

and we shall be able to show later on that this cannot be negative.

We now consider a numerical example based on the following Table 1.3.

	Return % for					
	January	February	March	April	May	June
Asset 1	1.2	1.3	1.4	1.5	1.1	1.2
Asset 2	1.3	1.0	0.8	0.9	1.4	1.3

Table 1.3. Monthly rates of return on two assets

These are, in fact, simply the returns for the first two assets in Table 1.1 and so $\bar{r}$ is given by (1.2.1) and the elements of Q come from (1.2.6) and (1.2.7). Specifically we have

$$\bar{r} = \begin{pmatrix} 1.2833 \\ 1.1167 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0.0181 & -0.0281 \\ -0.0281 & 0.0514 \end{pmatrix}. \tag{1.3.6}$$

If we use Q from (1.3.6) then (1.3.3) becomes

$$V = 0.0514 - 0.1590x + 0.1256x^2.$$

Hence, by (1.3.4) and (1.3.5),

$$\frac{dV}{dx} = -0.159 + 0.2512x \quad \text{and so} \quad x = \frac{0.159}{0.2512} \approx 0.633.$$

Since V has a positive second derivative,

$$\frac{d^2V}{dx^2} = 0.2512,$$

we know that a minimum of V has been found. Hence the minimum-risk portfolio for the assets in Table 1.3 has $y_1 = x \approx 0.633$. Obviously the invested fraction $y_2 = 1 - y_1 \approx 0.367$ and so the "least risky" strategy is to invest about 63% of the capital in asset 1 and 37% in asset 2. The portfolio risk is then $V \approx 0.00112$ and, using the $\bar{r}$ values in (1.3.6), the expected portfolio return is

$$R \approx 0.633 \times \bar{r}_1 + 0.367 \times \bar{r}_2 \approx 1.22\%.$$

Exercise
Find the minimum-risk portfolio involving the first two assets in Table 1.2.

Optimizing return and risk

The solution to problem **Minrisk0** can sometimes be useful; but in practice we will normally be interested in both risk and return rather than risk on its own.

In a rather general way we can say that an optimal portfolio is one which gives "biggest return at lowest risk". One way of trying to determine such a portfolio is to consider a composite function such as

$$F = -R + \rho V = -\bar{r}^T y + \rho y^T Q y. \tag{1.3.7}$$

The first term is the negative of the expected return and the second term is a multiple of the risk. If we choose invested fractions y_i to minimize F then we can expect to obtain a *large* value for return coupled with a *small* value for risk. The positive constant ρ controls the balance between return and risk.

Based on the preceding discussion, we now consider the problem

Risk-Ret1

$$\text{Minimize} \quad F = -\bar{r}^T y + \rho y^T Q y \quad \text{subject to} \quad \sum_{i=1}^{n} y_i = 1. \qquad (1.3.8)$$

As in the previous section, we can use the equality constraint to eliminate y_n and then find the unconstrained minimum of F, considered as a function of $y_1, \dots, y_{n-1}$. In particular, for the two-asset case, we can we write x in place of the unknown y_1 and define α and β by (1.3.2) so that $y = \alpha + \beta x$. Problem **Risk-Ret1** then becomes

$$\text{Minimize } F = -\bar{r}^T \alpha - (\bar{r}^T \beta)x + \rho[\alpha^T Q \alpha + 2(\beta^T Q \alpha)x + (\beta^T Q \beta)x^2]. \quad (1.3.9)$$

Differentiating, we get

$$\frac{dF}{dx} = -\bar{r}^T \beta + 2\rho\beta^T Q\alpha + 2\rho(\beta^T Q\beta)x$$

and so a stationary point of F occurs at

$$x = \frac{\bar{r}^T \beta - 2\rho\beta^T Q\alpha}{2\rho(\beta^T Q\beta)}. \qquad (1.3.10)$$

We now consider the data in Table 1.3. As before, values for $\bar{r}$ and Q are given by (1.3.6) and so F becomes

$$F = 0.1256\rho x^2 - (0.1667 + 0.1589\rho)x - 1.1167 + 0.05139\rho.$$

In order to minimize F we solve

$$\frac{dF}{dx} = 0.2512\rho x - (0.1667 + 0.1589\rho) = 0.$$

This gives

$$x = \frac{0.6636}{\rho} + 0.6326 \qquad (1.3.11)$$

and this stationary point is a minimum because the second derivative of F is 0.2512ρ which is positive whenever $\rho > 0$.

For $\rho = 5$, (1.3.11) gives $y_1 = x \approx 0.765$. This implies $y_2 \approx 0.235$ and the portfolio expected return is about 1.244% with risk $V \approx 0.00333$. If we choose $\rho = 10$ (thus placing more emphasis on reducing risk) the optimal portfolio is

$$y_1 \approx 0.7, \; y_2 \approx 0.3 \quad \text{giving} \quad R \approx 1.233\% \;\text{ and }\; V \approx 0.00167.$$

Exercises

1. Solve (1.3.9) using data for assets 2 and 3 in Table 1.2 and taking $\rho = 100$.

2. Based on the data in Table 1.3, determine the range of values of ρ for which the solution of (1.3.9) gives x lying between 0 and 1. What, *in general*, is the range for ρ which ensures that (1.3.10) gives $0 \le x \le 1$?

Minimum risk for specified return

Problem **Risk-Ret1** allows us to balance risk and return according to the choice of the parameter ρ. Another approach could be to fix a target value for return, say R_p%, and to consider the problem

Minrisk1

$$\text{Minimize} \quad V = y^T Q y \tag{1.3.12}$$

$$\text{subject to} \quad \sum_{i=1}^{n} \bar{r}_i y_i = R_p \quad \text{and} \quad \sum_{i=1}^{n} y_i = 1. \tag{1.3.13}$$

One way to tackle **Minrisk1** is to consider the *modified* problem

Minrisk1m

$$\text{Minimize} \quad F = y^T Q y + \frac{\rho}{R_p^2}(\sum_{i=1}^{n} \bar{r}_i y_i - R_p)^2 \quad \text{subject to} \quad \sum_{i=1}^{n} y_i = 1. \tag{1.3.14}$$

At a minimum of F we can expect the risk $y^T Q y$ to be small and also that the return $\bar{r}^T y$ will be close to the target figure R_p. As in **Risk-Ret1**, the value of the parameter ρ will determine the balance between return and risk.

For the two-asset case we can solve **Minrisk1m** by eliminating y_2 and using the transformation $y = \alpha + \beta x$ where $x = y_1$. We then get the problem

$$\text{Minimize} \quad F = (\alpha + \beta x)^T Q (\alpha + \beta x) + \frac{\rho}{R_p^2}(\bar{r}^T \alpha - R_p + \bar{r}^T \beta x)^2. \tag{1.3.15}$$

After some simplification, and writing $\bar{\rho}$ in place of ρ / R_p^2, F becomes

$$\alpha^T Q \alpha + \bar{\rho}(\bar{r}^T \alpha - R_p)^2 + 2(\beta^T Q \alpha + \bar{\rho}(\bar{r}^T \alpha - R_p)\bar{r}^T \beta)x + (\beta^T Q \beta + \bar{\rho}(\bar{r}^T \beta)^2)x^2.$$

A minimum of F occurs when its first derivative is zero, i.e. at

$$x = -\frac{\beta^T Q \alpha + \bar{\rho}(\bar{r}^T \alpha - R_p)\bar{r}^T \beta}{\beta^T Q \beta + \bar{\rho}(\bar{r}^T \beta)^2} \tag{1.3.16}$$

For the assets in Table 1.3, values of $\bar{r}$ and Q are given in (1.3.6) and so

$$\bar{r}^T \alpha \approx 1.1167; \quad \bar{r}^T \beta \approx 0.1667; \quad \beta^T Q \alpha \approx -0.0794; \quad \beta^T Q \beta \approx 0.1256.$$

Hence (1.3.16) gives

$$x \approx -\frac{-0.0794 + 0.1667\bar{\rho}(1.1167 - R_p)}{0.1256 + 0.1667^2\bar{\rho}}.$$

If the target return R_p is 1.25% then (1.3.16) becomes

$$x \approx -\frac{-0.0794 - 0.0222\bar{\rho}}{0.1256 + 0.0278\bar{\rho}} \approx -\frac{-0.0794 - 0.0142\rho}{0.1256 + 0.0178\rho}.$$

Setting $\rho = 10$ gives $x \approx 0.2214/0.3036 \approx 0.73$. Hence the corresponding invested fractions are

$$y_1 \approx 0.73, \quad y_2 \approx 0.27$$

giving portfolio risk and return $V \approx 0.00233$ and $R \approx 1.24\%$.

Solutions with $\bar{r}^T y$ closer to its target value can be obtained by increasing the parameter ρ. Thus, when $\rho = 100$, we get the solution

$$y_1 \approx 0.79, \quad y_2 \approx 0.21 \quad \text{with} \quad y^T Q y \approx 0.0042 \;\text{ and }\; \bar{r}^T y \approx 1.25\%.$$

Exercises

1. For the data in Table 1.3, estimate the minimum-risk portfolio for a target return of 1.2% using values of $\rho = 10, 100, 1000$ in the function (1.3.15).

2. If, for a two-asset problem, R_p is taken as $0.5(\bar{r}_1 + \bar{r}_2)$, is it necessarily true that the solution to **Minrisk1m** has y_1 and y_2 both approximately equal to 0.5? What can you say about the possible ranges for y_1 and y_2 when ρ is large?

4. Minimimum risk for three-asset portfolios

In the case of a three-asset portfolio we can reduce problem **Minrisk1** to a one-variable minimization by using *both* constraints to eliminate the invested fractions y_2 and y_3. Equations (1.3.13) imply

$$\bar{r}_2 y_2 + \bar{r}_3 y_3 = R_p - \bar{r}_1 y_1 \tag{1.4.1}$$

$$y_2 + y_3 = 1 - y_1 \tag{1.4.2}$$

Multiplying (1.4.2) by $\bar{r}_3$ and subtracting it from (1.4.1) gives

$$(\bar{r}_2 - \bar{r}_3) y_2 = R_p - \bar{r}_1 y_1 - \bar{r}_3 + \bar{r}_3 y_1 \tag{1.4.3}$$

and hence

$$y_2 = \tilde{\alpha}_2 + \tilde{\beta}_2 y_1 \quad \text{where} \quad \tilde{\alpha}_2 = \frac{R_p - \bar{r}_3}{\bar{r}_2 - \bar{r}_3} \quad \text{and} \quad \tilde{\beta}_2 = \frac{\bar{r}_3 - \bar{r}_1}{\bar{r}_2 - \bar{r}_3}. \tag{1.4.4}$$

Moreover, (1.4.2) gives $y_3 = 1 - y_1 - y_2$ which simplifies to

$$y_3 = \tilde{\alpha}_3 + \tilde{\beta}_3 y_1 \quad \text{where} \quad \tilde{\alpha}_3 = 1 - \tilde{\alpha}_2 \quad \text{and} \quad \tilde{\beta}_3 = -(1 + \tilde{\beta}_2). \qquad (1.4.5)$$

If we write x in place of the unknown y_1 and also define

$$\tilde{\alpha} = \begin{pmatrix} 0 \\ \tilde{\alpha}_2 \\ \tilde{\alpha}_3 \end{pmatrix} \quad \text{and} \quad \tilde{\beta} = \begin{pmatrix} 1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \end{pmatrix}$$

then $y = \tilde{\alpha} + \tilde{\beta}x$. The risk V can now be expressed as a function of x, i.e.,

$$V = (\tilde{\alpha} + \tilde{\beta}x)^T Q(\tilde{\alpha} + \tilde{\beta}x) = \tilde{\alpha}^T Q\tilde{\alpha} + (2\tilde{\beta}^T Q\tilde{\alpha})x + (\tilde{\beta}^T Q\tilde{\beta})x^2. \qquad (1.4.6)$$

Hence

$$\frac{dV}{dx} = 2\tilde{\beta}^T Q\tilde{\alpha} + 2(\tilde{\beta}^T Q\tilde{\beta})x$$

and a stationary point of V occurs at

$$x = -\frac{\tilde{\beta}^T Q\tilde{\alpha}}{\tilde{\beta}^T Q\tilde{\beta}}. \qquad (1.4.7)$$

Exercises

1. Show that, for the asset data in Table 1.1, the mean returns $\bar{r}$ in (1.2.1) imply $\tilde{\alpha} \approx (0,\ 3.5,\ -2.5)^T$ and $\tilde{\beta} \approx (1,\ -6,\ 5)^T$ when the target return is $R_p = 1.2\%$. Hence show that the minimum risk strategy for an expected return of 1.2% is to spread the investment in the ratio 0.53 : 0.32 : 0.15 over assets 1, 2 and 3.

2. Find the minimum-risk portfolio that will give an expected return of 1.1% for the assets in Table 1.1.

3. For the data in Table 1.2, find the minimum-risk portfolio to achieve an expected return $R_p = 0.5\%$.

4. Suppose $y_1, ...y_4$ are invested fractions for a four-asset portfolio. Use (1.2.3) and (1.2.2) to obtain expressions – similar to (1.4.4) and (1.4.5) – for y_3 and y_4 in terms of y_1 and y_2.

5. Two- and three-asset minimum-risk solutions

It is useful to have computer implementations of the solution methods described in sections 3 and 4. The results given in this section have been obtained using a suite of software called `SAMPO`, which is written in Fortran90. It is *not* essential for the reader to understand or use this software (although it can be downloaded from an `ftp` site as described in the appendix). Solutions to **Minrisk0**, **Risk-Ret1** and **Minrisk1** can easily be coded in other languages or expressed as spreadsheet calculations. Results obtained with the `SAMPO` software will be quoted extensively this book in order to compare the performance of different optimization methods. However, we shall defer discussion of this until later chapters. For the moment we simply note that the results below are from a program **`sample1`** which is designed to read asset data (like that appearing in Tables 1.1 – 1.3) and then calculate solutions to **Minrisk0**, **Risk-Ret1** or **Minrisk1**.

The first problems we consider are based on data for the first two assets in Table 1.2. The expected returns turn out to be $\bar{r}_1 \approx 0.138$, $\bar{r}_2 \approx 0.004$ and the variance-covariance matrix is

$$Q \approx \begin{pmatrix} 0.0065 & -0.0039 \\ -0.0039 & 0.0029 \end{pmatrix}.$$

The solution to problem **Minrisk0** appears in Table 1.4.

y_1	y_2	R	V_{min}
0.397	0.603	0.057%	2.4×10^{-4}

Table 1.4. Solution of **Minrisk0** for first two assets in Table 1.2

Table 1.5 shows solutions to **Risk-Ret1** for the same data. These are obtained by minimizing (1.3.9) for various values of the weighting parameter ρ.

ρ	y_1	y_2	R	V
10	0.787	0.213	0.109%	2.85×10^{-4}
100	0.436	0.564	0.062%	2.7×10^{-4}
1000	0.401	0.599	0.058%	2.4×10^{-4}

Table 1.5. Solutions of **Risk-Ret1** for first two assets in Table 1.2

The values of y_1 and y_2 vary appreciably with ρ and tend towards the solution of **Minrisk0** as ρ gets larger. It can be shown (Exercise 3 below) that this will be the case for *any* asset data and is not just a feature of the present problem.

We now turn to problem **Minrisk1m**, still considering the first two assets in Table 1.2. Table 1.6 shows the solutions obtained by minimizing (1.3.15) for various values of ρ when $R_p = 0.1\%$. We observe that the choice of ρ is quite important. When ρ is small, the invested fractions which minimize (1.3.15) do not yield the target value for expected return; and it is only as ρ increases

ρ	y_1	y_2	R	V^*
0.01	0.56	0.44	0.079%	7.0×10^{-3}
0.1	0.688	0.312	0.096%	1.7×10^{-3}
1	0.713	0.287	0.1%	1.96×10^{-3}
10	0.716	0.284	0.1%	1.99×10^{-3}

Table 1.6. Solutions of **Minrisk1m** for first two assets in Table 1.2

that R tends to the required value $R_p = 0.1\%$. If we take $\rho > 10$ then – for this particular problem – there will not be any significant change in y_1 and y_2. (However, for different asset data, values of ρ very much greater than 10 might be needed before an acceptable solution to **Minrisk1** was obtained.)

We next consider a three-asset problem involving the data in Table 1.1. We can use the program `sample1` to solve **Minrisk1** by minimizing (1.4.6). Table 1.7 shows the results obtained for various values of target return R_p.

R_p	y_1	y_2	y_3	V^*
1.15%	0.295	0.229	0.475	4.4×10^{-3}
1.2%	0.530	0.321	0.149	1.3×10^{-3}
1.25%	0.764	0.413	-0.177	1.8×10^{-3}

Table 1.7. Solutions of **Minrisk1** for assets in Table 1.1

The first two rows of Table 1.7 show that a reduction in the value of R_p does not *necessarily* imply a reduced risk. If we decrease the target expected return from 1.2% to 1.15% we bias investment *away from* the better performing asset 1. This means that the positive co-variance σ_{23} in (1.2.7) will contribute more to V than the negative co-variances σ_{12}, σ_{13} and so the optimum value of risk is not so small. By comparing rows two and three of Table 1.7, however, we see that an increase in R_p from 1.2% to 1.25% does produce a corresponding increase in risk.

The negative value for y_3 in the third row of Table 1.7 is not an error. It indicates that, for a target return $R_p = 1.25\%$, the optimal investment strategy involves *short selling* – i.e., selling shares in asset 3 *even though the investor does not own them.* This can be done by borrowing the shares from a broker with the intention of returning them later. The strategy is only effective if the price of the shares falls because then the cost of buying replacement shares at a later date is less than the receipts from the sale of the borrowed ones.