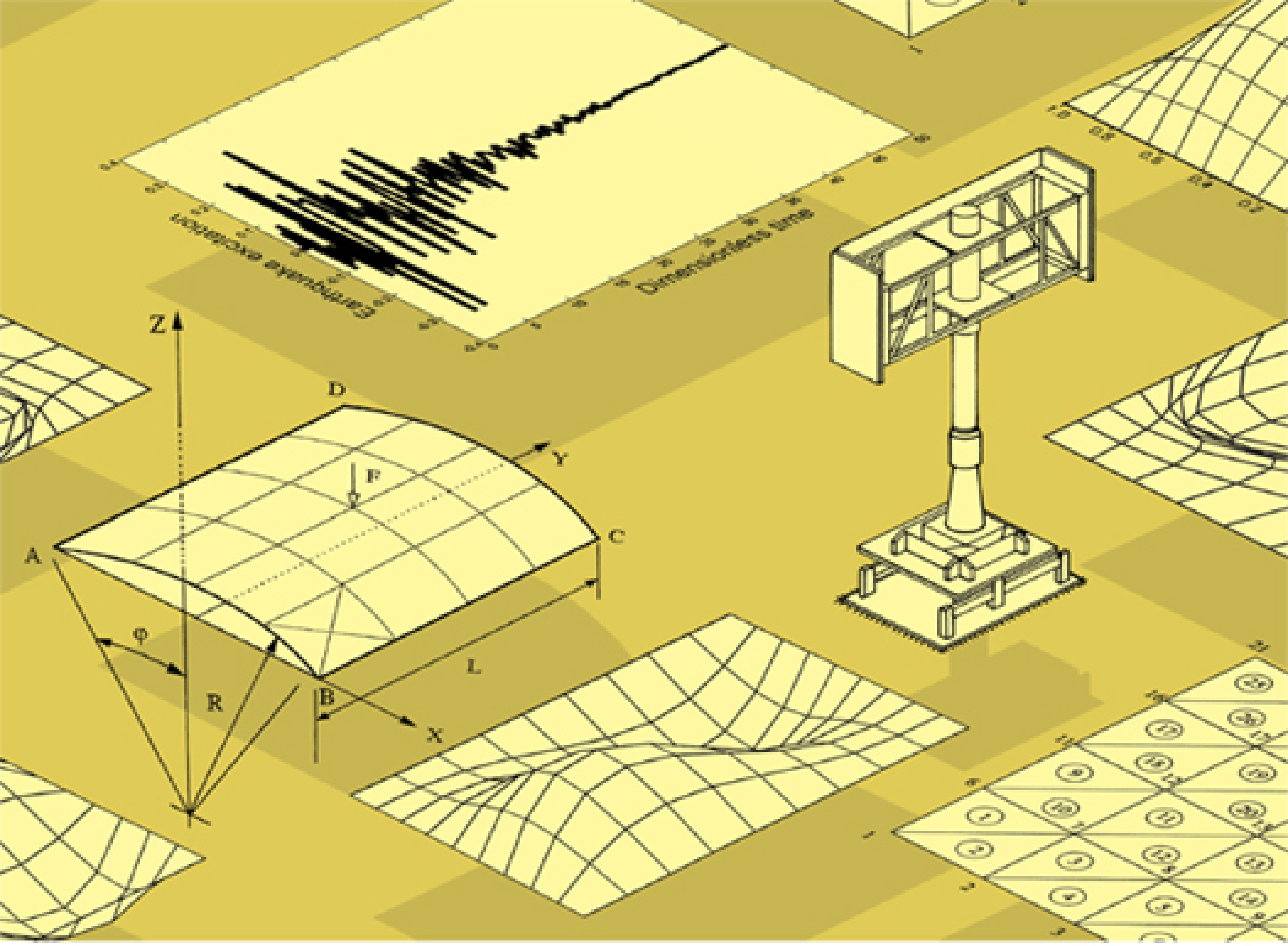

Cho W. S. To

Stochastic Structural Dynamics

Application of Finite Element Methods

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Contents

STOCHASTIC STRUCTURAL DYNAMICS

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APPLICATION OF FINITE ELEMENT METHODS

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WILEY

This edition first published 2014

Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

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Library of Congress Cataloging-in-Publication Data

To, Cho W. S.
Stochastic structural dynamics : application of finite element methods / Cho W. S. To. — First edition.
pages cm
Includes bibliographical references and index.
ISBN 978-1-118-34235-0 (hardback)
1. Structural dynamics–Engineering. 2. Finite element method. 3. Stochastic analysis. I. Title.
TA654.T59 2014
624.1’70151922–dc23
2013023517

A catalogue record for this book is available from the British Library.

ISBN 9781118342350

Lidong Leighton and Lizhen Jane

Preface

Stochastic structural dynamics is concerned with the studies of dynamics of structures and structural systems that are subjected to complex excitations treated as random processes. In engineering practice, many structures and structural systems cannot be dealt with analytically and therefore the versatile numerical analysis techniques, the finite element methods (FEM) are employed.

The parallel developments of the FEM in the 1950's and the engineering applications of stochastic processes in the 1940's provided a combined numerical analysis tool for the studies of dynamics of structures and structural systems under random loadings. In the open literature, there are books on statistical dynamics of structures and books on structural dynamics with chapter(s) dealing with random response analysis. However, a systematic treatment of stochastic structural dynamics applying the FEM seems to be lacking. The present book is believed to be the first relatively in-depth and systematic treatment on the subject. It is aimed at advanced and specialist level. It is suitable for classes taken by master's degree level post-graduate students and specialists.

The present book has seven chapters and ten appendices. Chapter 1 introduces the displacement-based FEM, element equations of motion for temporally and spatially stochastic systems, hybrid stress-based element equations of motion, incremental variational principle and mixed formulation-based nonlinear element matrices, constitutive relations and updating of configurations and stresses.

Chapter 2 is concerned with the spectral analysis and response statistics of linear structural systems. It includes evolutionary spectral analysis, evolutionary spectra of engineering structures, modal analysis and time-dependent response statistics, and response statistics of engineering structures.

Direct integration methods for linear structural systems are presented in Chapter 3. The stochastic central difference method with time co-ordinate transformation and its application, extended stochastic central difference method for narrow-band excitations, stochastic Newmark family of algorithms, and their applications to plate structures are presented in this chapter.

Modal analysis and response statistics of quasi-linear structural systems are covered in Chapter 4. Modal analysis of temporally stochastic quasi-linear systems and the bi-modal approach are included. Response analysis of plate structures by the Melosh-Zienkiewicz-Cheung bending plate element, and the high precision triangular plate element are presented.

Chapter 5 is concerned with the application of the direct integration methods for response statistics of quasi-linear structural systems. Recursive covariance matrices of displacements of cantilever pipes containing turbulent fluids and subjected to modulated white noise as well as narrow-band random excitations are derived in this chapter.

Direct integration methods for temporally stochastic nonlinear structural systems subjected to stationary and nonstationary random excitations are presented in Chapter 6. A brief introduction to the statistical linearization techniques is included. Symplectic members of the deterministic and stochastic versions of the Newmark family of algorithms are identified. The stochastic central difference method with time co-ordinate transformation and adaptive time schemes are introduced and applied to the computation of large responses of plate and shell structures.

Chapter 7 is concerned with the presentation of the direct integration methods for temporally and spatially stochastic nonlinear structural systems. The stochastic FEM or probabilistic FEM is introduced. The stochastic central difference method for temporally and spatially stochastic structural systems subjected to stationary and nonstationary random excitations are developed. Application of the method to spatially homogeneous and non-homogeneous shell structures is made.

Finally, a word about symbols is in order. Mathematically, random variables and random processes are different. But without ambiguity the same symbols for random variables and processes are applied in the present book, unless it is stated otherwise.

Acknowledgments

Thanks are due to the author's several former graduate students, Gregory Zidong Chen, Sherwin Xingling Dai, Derick Hung, Meilan Liu, Irewole Raphael Orisamolu, and Bin Wang who provided various drawings in this book.

The author would like to express his sincere thanks to Paul Petralia, Senior Editor and his project team members, Tom Carter, Sandra Grayson, Anna Smart, and Liz Wingett.

Finally, the author would also like to thank Elsevier Science for permission to reproduce the following figures. Figures 1.1 and 1.2 are from To, C. W. S. (1979): Higher order tapered beam finite elements for vibration analysis, *Journal of Sound and Vibration*, **63(1)**, 33–50. Figures 1.3 and 1.4 are from To, C. W. S. and Liu, M. L. (1994): Hybrid strain based three-node flat triangular shell elements, *Finite Elements in Analysis and Design*, **17**, 169–203. Figures 2.1 through 2.3 are from To, C. W. S. (1982): Nonstationary random responses of a multi-degree-of-freedom system by the theory of evolutionary spectra, *Journal of Sound and Vibration*, **83(2)**, 273–291. Figures 2.12, 2.13, and 2B.1 are from To, C. W. S. (1984): Time-dependent variance and covariance of responses of structures to non-stationary random excitations, *Journal of Sound and Vibration*, **93(1)**, 135–156. Figures 2.14 through 2.19 are from To, C. W. S. and Wang, B. (1993): Time-dependent response statistics of axisymmetrical shell structures, *Journal of Sound and Vibration*, **164(3)**, 554–564. Figures 2.20 through 2.26 are from To, C. W. S. and Wang, B. (1996): Nonstationary random response of laminated composite structures by a hybrid strain-based laminated flat triangular shell finite element, *Finite Elements in Analysis and Design*, **23**, 23–35. Figures 3.2 through 3.10 are from To, C. W. S. and Liu, M. L.

(1994): Random responses of discretized beams and plates by the stochastic central difference method with time co-ordinate transformation, *Computers and Structures*, **53(3)**, 727–738. Figures 3.11 through 3.15, and Figures 5.9 through 5.14 are from Chen, Z. and To, C. W. S. (2005): Responses of discretized systems under narrow band nonstationary random excitations, *Journal of Sound and Vibration*, **287**, 433–458. Figures 4.1 through 4.7 are from To, C. W. S. and Orisamolu, I. R. (1987): Response of discretized plates to transversal and in-plane non-stationary random excitations, *Journal of Sound and Vibration*, **114(3)**, 481–494. Figures 6.1 through 6.7, and 6.10 through 6.13 are from To, C. W. S. and Liu, M. L. (2000): Large nonstationary random responses of shell structures with geometrical and material nonlinearities, *Finite Elements in Analysis and Design*, **35**, 59–77.

Chapter 1

Introduction

The parallel developments of the finite element methods (FEM) in the 1950's [1,2] and the engineering applications of the stochastic processes in the 1940's [3, 4] provided a combined numerical analysis tool for the studies of dynamics of structures and structural systems under random loadings. There are books on statistical dynamics of structures [5, 6] and books on structural dynamics with chapter(s) dealing with random response analysis [7, 8]. In addition, there are various monographs and lecture notes on the subject. However, a systematic treatment of the stochastic structural dynamics applying the FEM seems to be lacking. The present book is believed to be the first relatively in-depth and systematic treatment of the subject that applies the FEM to the field of stochastic structural dynamics.

Before the introduction to the concept and theory of stochastic quantities and their applications with the FEM in subsequent chapters, the two FEM employed in the investigations presented in the present book are outlined in this chapter. Specifically, Section 1.1 is concerned with the derivation of the temporally stochastic element equation of motion applying the displacement formulation. The consistent element stiffness and mass matrices of two beam elements, each having two nodes are derived. One beam element is uniform and the other is tapered. The corresponding temporally and spatially stochastic element equation of motion is derived in Section 1.2. The element equations of motion based on the mixed formulation are introduced in Section 1.3. Consistent element matrices for a beam of uniform cross-sectional area are obtained. This beam element has two nodes, each of which has two degrees-of-freedom (dof). This beam element is applied to show that stiffness matrices derived from the displacement and mixed formulations are identical. The incremental variational principle and element matrices based on the mixed formulation for nonlinear structures are presented in Section 1.4. Section 1.5 deals with constitutive relations and updating of configurations and stresses. Closing remarks for this chapter are provided in Section 1.6.

1.1 Displacement Formulation-Based Finite Element Method

Without loss of generality and as an illustration, the displacement formulation based element equations of motion for temporally stochastic linear systems are presented in this section. These equations are similar in form to those under deterministic excitations. It is included in Sub-section 1.1.1 while application of the technique for the derivation of element matrices of a two-node beam element of uniform cross-section is given in Sub-section 1.1.2. The tapered beam element is presented in Sub-section 1.1.3.

1.1.1 Derivation of element equations of motion

The Rayleigh-Ritz (RR) method approximates the displacement by a linear set of admissible functions that satisfy the geometric boundary conditions and are *p* times differentiable over the domain, where *p* is the number of boundary conditions that the displacement must satisfy at every point of the boundary of the domain. The admissible functions required by the RR method are constructed employing the finite element displacement method with the following steps:

(*a*) idealization of the structure by choosing a set of imaginary reference or node points such that on joining these node points by means of imaginary lines a series of finite elements is formed;

(*b*) assigning a given number of dof, such as displacement, slope, curvature, and so on, to every node point; and

(*c*) constructing a set of functions such that every one corresponds to a unit value of one dof, with the others being set to zero.

Having constructed the admissible functions, the element matrices are then determined. For simplicity, the damping matrix of the element will be disregarded. Thus, in the following the definition of consistent element mass and stiffness matrices in terms of deformation patterns usually referred to as shape functions is given.

Assuming the displacement *u*(*x, t*) or simply *u* at the point *x* (for example, in the three-dimensional case it represents the local co-ordinates *r, s* and *t* at the point) within the *e*'th element is expressed in matrix form as

(1.1) $\boldsymbol{u}(x,t) = \boldsymbol{N}(x)\boldsymbol{q}(t)$,

where *N*(*x*) or simply *N* is a matrix of element shape functions, and ***q***(*t*) or ***q*** a matrix of nodal dof with reference to the local axes, also known as the vector of nodal displacements or generalized displacements.

The matrix of strain components ε thus takes the form

(1.2) $\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{q}$,

where ***B*** is a differential of the shape function matrix *N*.

The matrix of stress components σ is given by

(1.3) $\sigma = D\varepsilon$,

where D is the elastic matrix.

Substituting Eq. (1.2) into (1.3) gives

(1.4) $\sigma = DBq$.

In order to derive the element equations of motion for a conservative system, the Hamilton's principle can be applied

(1.5) $L = T - (U + W)$,

where T and $(U + W)$ are the kinetic and potential energies, respectively.

It may be appropriate to note that for a non-conservative system or system with non-holonomic boundary conditions, the modified Hamilton's principle [9] or the virtual power principle [10, 11] may be applied. Non-holonomic systems are those with constraint equations containing velocities which cannot be integrated into relations in co-ordinates or displacements only. An example of a non-holonomic system is the bicycle moving down an inclined plane in which enforcing no slipping at the contact point gives rise to non-holonomic constraint equations. Another example is a disk rolling on a horizontal plane. In this case enforcing no slipping at the contact point also give rise to non-holonomic constraint equations.

The kinetic energy density of the element is defined as

(1.6) $dT = ½\, \rho\, \dot{u}^T \dot{u}\, dV$

where ρ is the density of the material, dV is the incremental volume, and the over-dot denotes the differentiation with respect to time t.

By making use of Eq. (1.6), the kinetic energy of the element becomes

(1.7) $$T = \frac{1}{2} \iiint_V \rho\, \dot{u}^T \dot{u}\, dV .$$

The strain energy density for a linear elastic body is defined as

$$dU = \frac{1}{2}\varepsilon^T \sigma \, dV = \frac{1}{2}\varepsilon^T D \varepsilon \, dV \, . \qquad (1.8)$$

The potential energy for a linearly elastic body can be expressed as the sum of internal work, the strain energy due to internal stress, and work done by the body forces and surface tractions. Thus,

$$U + W = \iiint_V dU - \iiint_V u^T \overline{Q} \, dV - \iint_S u^T \overline{Y} \, dS \, , \qquad (1.9)$$

where S now is the surface of the body on which surface tractions $\overline{Y}$ are prescribed. The last two integrals on the right-hand side (rhs) of Eq. (1.9) represent the work done by the external random forces, the body forces $\overline{Q}$ and surface tractions $\overline{Y}$. In the last equation the over-bar of a letter designates the quantity is specified.

Applying Eq. (1.8), the total potential of the element from Eq. (1.9) becomes

$$U + W = \frac{1}{2}\iiint_V \varepsilon^T D \varepsilon \, dV - \iiint_V u^T \overline{Q} \, dV - \iint_S u^T \overline{Y} \, dS \, . \qquad (1.10)$$

Substituting Eqs. (1.7) and (1.10) into (1.5), the functional of a linearly elastic element,

$$L = \frac{1}{2}\iiint_V \left(\rho \, \dot{u}^T \dot{u} - \varepsilon^T D \varepsilon + 2u^T \overline{Q}\right) dV + \iint_S u^T \overline{Y} \, dS \, . \qquad (1.11)$$

On substituting Eqs. (1.1) through (1.3) into the last equation and using the matrix relation $(XY)^T = Y^T X^T$, the Lagrangian becomes

$$L = \frac{1}{2}\iiint_V \left(\rho \dot{q}^T N^T N \dot{q} - q^T B^T D B q + 2 q^T N^T \overline{Q}\right) dV + \iint_S q^T N^T \overline{Y} dS \,. \quad (1.12)$$

Applying Hamilton's principle, it leads to

$$\int_{t_1}^{t_2} \left(\delta\dot{q}^T \iiint_V \rho N^T N dV \dot{q} - \delta q^T \iiint_V B^T D B \, dV q + \delta q^T \iiint_V N^T \overline{Q} dV + \delta q^T \iint_S N^T \overline{Y} dS \right) dt = 0 \,. \quad (1.13)$$

Integrating the first term inside the brackets on the left-hand side (lhs) of Eq. (1.13) by parts with respect to time *t* results

$$\int_{t_1}^{t_2} \delta\dot{q}^T \iiint_V \rho N^T N dV \dot{q} \, dt = \left[\delta q^T \iiint_V \rho N^T N dV \dot{q} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q^T \iiint_V \rho N^T N dV \ddot{q} \, dt \,. \quad (1.14)$$

According to Hamilton's principle, the tentative displacement configuration must satisfy given conditions at times t_1 and t_2, that is,

$$\delta q(t_1) = 0 \,, \qquad \delta q(t_2) = 0 \,.$$

Hence, the first term on the rhs of Eq. (1.14) vanishes.

Substituting Eq. (1.14) into (1.13) and rearranging, it becomes

$$\int_{t_1}^{t_2} \delta q^T \left(\iiint_V \rho N^T N dV \ddot{q} + \iiint_V B^T D B dV q - \iiint_V N^T \overline{Q} dV - \iint_S N^T \overline{Y} dS \right) dt = 0 . \quad (1.15)$$

As the variations of the nodal displacements δ*q* are arbitrary, the expressions inside the parentheses must be equal to zero in order that Eq. (1.15) is satisfied. Therefore, the equation of motion for the *e*'th element in matrix form is

$$m\ddot{q} + kq = f , \quad (1.16)$$

where the element mass and stiffness matrices are defined, respectively as

$$m = \iiint_V \rho N^T N dV , \quad k = \iiint_V B^T D B dV ,$$

and the element random load matrix

$$f = \iiint_V N^T \overline{Q} dV + \iint_S N^T \overline{Y} dS .$$

Applying the generalized co-ordinate form of displacement model the displacement can be expressed as

$$u = \Phi \zeta , \quad (1.17)$$

where Φ is a matrix of function of variables *x* and ζ is the vector of generalized co-ordinates, also known as generalized displacement amplitudes. The coefficient matrix may be determined by introducing the nodal co-ordinates successively into Eq. (1.17) such that the vector *u* and matrix Φ become the nodal displacement vector *q* and coefficient matrix *C*, respectively. That is,

$$q = C\zeta . \quad (1.18)$$

Hence, the generalized displacement amplitude vector

$$\zeta = C^{-1} q , \quad (1.19)$$

where C^{-1} is the inverse of the coefficient matrix also known as the transformation matrix and is independent of the variables ***x***.

Substituting Eq. (1.19) into (1.17) one has

(1.20) $$u = \Phi C^{-1} q \ .$$

Comparing Eqs. (1.1) and (1.20), one has the shape function matrix

(1.21) $$N = \Phi C^{-1} \ .$$

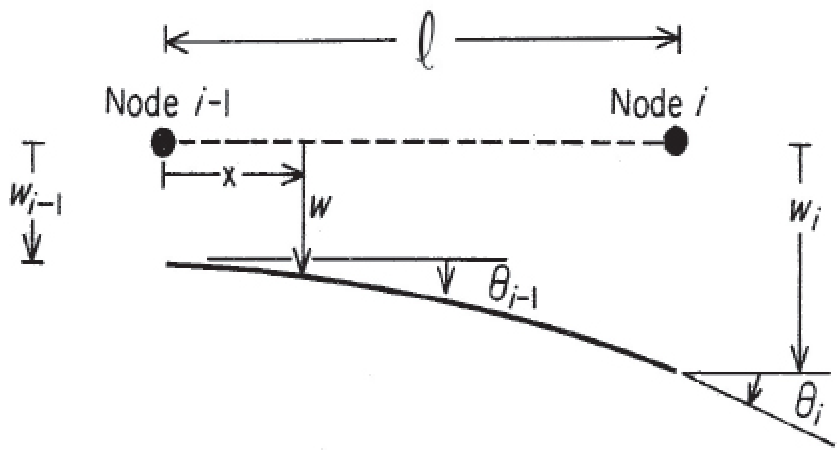
On application of Eqs. (1.16) and (1.21), the element mass, stiffness and load matrices can be evaluated.

To provide a more concrete illustration of the shape function matrix and a better understanding of the steps in the derivation of element mass and stiffness matrices, a uniform beam element is considered in the next sub-section.

1.1.2 Mass and stiffness matrices of uniform beam element

The uniform beam element considered in this sub-section has two nodes, each of which has two dof. The latter include nodal transverse displacement, and rotation or angular displacement about an axis perpendicular to the plane containing the beam and the transverse displacement. For simplicity, the theory of the Euler beam is assumed. The cross-sectional area A and second moment of area I are constant. Let ρ and E be the density and modulus of elasticity of the beam. The bending beam element is shown in Figure 1.1 where the edge displacements and angular displacements are included. The convention adopted in the figure is sagging being positive.

Figure 1.1 Uniform beam element with edge displacements.



Applying Eq. (1.17) so that the transverse displacement at a point inside the beam element can be written as

(1.22a, b) $$u = w = \Phi\zeta, \qquad \Phi = [1 \quad x \quad x^2 \quad x^3]$$

(1.22c) $$\zeta^T = [\zeta_1 \quad \zeta_2 \quad \zeta_3 \quad \zeta_4].$$

Consider the nodal values. At $x = 0$, $w = w_{i-1}$ and $\theta = \partial w/\partial x = \theta_{i-1}$ so that upon application of Eq. (1.22a) one has

(1.23a, b) $$w_{i-1} = [1 \quad 0 \quad 0 \quad 0]\zeta, \qquad \theta_{i-1} = [0 \quad 1 \quad 0 \quad 0]\zeta.$$

Similarly, at $x = l$, $w = w_i$ and $\theta = \theta_i$ so that upon application of Eq. (1.22a) it leads to

(1.23c, d) $$w_i = [1 \quad \ell \quad \ell^2 \quad \ell^3]\zeta, \qquad \theta_i = [0 \quad 1 \quad 2\ell \quad 3\ell^2]\zeta.$$

Re-writing Eq. (1.23) in matrix form as in Eq. (1.18), one has

(1.24) $$q = \begin{pmatrix} w_{i-1} \\ \theta_{i-1} \\ w_i \\ \theta_i \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix}.$$

Thus, the inverse of matrix C becomes

(1.25) $$C^{-1} = \frac{1}{\ell^3} \begin{bmatrix} \ell^3 & 0 & 0 & 0 \\ 0 & \ell^3 & 0 & 0 \\ -3\ell & -2\ell^2 & 3\ell & -\ell^2 \\ 2 & \ell & -2 & \ell \end{bmatrix}.$$

Making use of Eqs. (1.22b) and (1.25), the shape function matrix by Eq. (1.21) is obtained as

(1.26) $$N = \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \end{bmatrix},$$

in which

$$N_{11} = 1 - 3\xi^2 + 2\xi^3, \qquad N_{12} = x\left(1 - 2\xi + \xi^2\right), \qquad \xi = \frac{x}{\ell},$$

$$N_{13} = 3\xi^2 - 2\xi^3, \qquad N_{14} = -x\left(\xi - \xi^2\right).$$

Substituting Eq. (1.26) into the equation for element mass matrix defined in Eq. (1.16), one can show that

$$m = \rho A \int_0^\ell N^T N dx = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ . & 4\ell^2 & 13\ell & -3\ell^2 \\ symmetric & . & 156 & -22\ell \\ . & . & . & 4\ell^2 \end{bmatrix}.$$

(1.27)

Similarly, the element stiffness matrix is obtained as

$$k = \frac{EI}{\ell^6} \int_0^\ell B^T B dx = \frac{2EI}{\ell^3} \begin{bmatrix} 6 & 3\ell & -6 & 3\ell \\ . & 2\ell^2 & -3\ell & \ell^2 \\ symmetric & . & 6 & -3\ell \\ . & . & . & 2\ell^2 \end{bmatrix},$$

(1.28)

in which

$$B = \frac{\partial^2 N}{\partial x^2} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \end{bmatrix}, \qquad B_{11} = 12x - 6\ell,$$

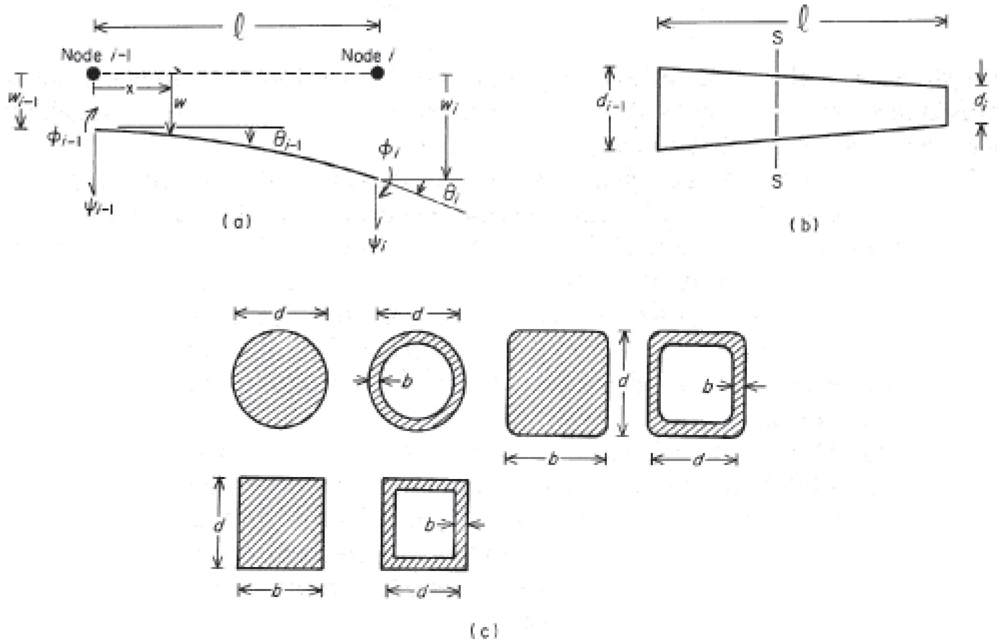
$$B_{12} = 6x\ell - 4\ell^2, \qquad B_{13} = -12x + 6\ell, \qquad B_{14} = 6x\ell - 2\ell^2.$$

1.1.3 Mass and stiffness matrices of higher order taper beam element

The tapered beam element considered in this sub-section has two nodes, each of which has four dof. The latter include nodal displacement, rotation or angular displacement, curvature, and shear dof. This is the higher order tapered beam element first developed and presented by the author [12].

The tapered beam element of length ℓ, shown in Figure 1.2, is assumed to be of homogeneous and isotropic material. Its cross-sectional area and second moment of area are, respectively given by

Figure 1.2 Linearly tapered beam element: (a) beam element with edge forces; (b) tapered beam element; (c) cross-section at section S-S in (b).



(1.29) $$A(x) = c_1 b(s) d(x) , \qquad I(x) = c_2 b(s) d^3(x) ,$$

where c_1 and c_2 depend on the shape of the beam cross-section. For an elliptic-type closed curve cross-section, they are given by [13]

(1.30a, b) $$c_1 = \frac{\Gamma\left(\frac{1}{\mu_1}+1\right)\Gamma\left(\frac{1}{\mu_2}+1\right)}{\Gamma\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+1\right)} , \quad c_2 = \frac{\Gamma\left(\frac{1}{\mu_1}+1\right)\Gamma\left(\frac{3}{\mu_2}+1\right)}{12\Gamma\left(\frac{1}{\mu_1}+\frac{3}{\mu_2}+1\right)} ,$$

in which Γ(.) is the gamma function, and μ_1 and μ_2 are real positive numbers which need not be integers. When $\mu_1 = \mu_2 = 1$, the cross-section is a triangle and in this case the factor 1/12 in c_2 should be replaced by 1/9. When $\mu_1 = \mu_2 = 2$, the cross-section is an ellipse. As μ_1 and μ_2 each approaches infinity, it is a rectangle.

The cross-sectional dimensions, $b(x)$ and $d(x)$, vary linearly along the length of the element so that

$$b(x) = b_{i-1}\left[1 + (\alpha - 1)\frac{x}{\ell}\right], \quad d(x) = d_{i-1}\left[1 + (\beta - 1)\frac{x}{\ell}\right], \tag{1.31a, b}$$

where $\alpha = b_i/b_{i-1}$ and $\beta = d_i/d_{i-1}$ are the taper ratios for the beam element.

Substituting Eq. (1.31) into (1.29) leads to

$$A(x) = A_{i-1}\left(1 + \gamma_1\xi + \gamma_2\xi^2\right), \tag{1.32a}$$

$$I(x) = I_{i-1}\left(1 + \delta_1\xi + \delta_2\xi^2 + \delta_3\xi^3 + \delta_4\xi^4\right), \tag{1.32b}$$

$$\xi = \frac{x}{\ell}, \quad \gamma_1 = (\alpha - 1) + (\beta - 1), \quad \gamma_2 = (\alpha - 1)(\beta - 1),$$

$$\delta_1 = (\alpha - 1) + 3(\beta - 1), \quad \delta_2 = 3(\alpha - 1)(\beta - 1) + 3(\beta - 1)^2,$$

$$\delta_3 = 3(\alpha - 1)(\beta - 1)^2 + (\beta - 1)^3, \quad \delta_4 = (\alpha - 1)(\beta - 1)^3,$$

A_{i-1} and I_{i-1} are respectively the cross-sectional area and second moment of area associated with Node $i - 1$.

It should be noted that in applying Eq. (1.32) to hollow beams, of square or circular cross-section, for instance, either the ratio b/d must be small or the ratio b/d must be constant because in Eq. (1.29) for a square hollow cross-section $c_1 = 4$ and $c_2 = (2/3)[1 + (b/d)^2]$, and for a circular hollow cross-section $c_1 = \pi$ and $c_2 = (\pi/8)[1 + (b/d)^2]$.

With the cross-sectional area and second moment of area defined, the element mass and stiffness matrices can be derived accordingly. To this end let the transverse displacement of the beam element be

$$w = \sum_{j=1}^{8} \zeta_j x^{j-1}, \quad \text{or} \quad w = \Phi \zeta, \tag{1.33}$$

where the row and column vectors are respectively

$$\Phi = \left[1 \; x \; x^2 \; x^3 \; x^4 \; x^5 \; x^6 \; x^7\right], \qquad \zeta = \left[\zeta_1 \; \zeta_2 \; \zeta_3 \; \zeta_4 \; \zeta_5 \; \zeta_6 \; \zeta_7 \; \zeta_8\right]^T.$$

Equation (1.33) can be identified as Eq. (1.17) in which the displacement function *u* is replaced by *w*. Thus, the nodal displacement vector in Eq. (1.18) for the present tapered beam element becomes

$$q = \left[w_{i-1} \; \theta_{i-1} \; \phi_{i-1} \; \psi_{i-1} \; w_i \; \theta_i \; \phi_i \; \psi_i\right]^T, \quad \phi_i = \frac{\partial^2 w_i}{\partial x^2}, \quad \psi_i = \frac{\partial^3 w_i}{\partial x^3}.$$

The corresponding coefficient matrix in Eq. (1.18) is obtained as [12]

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 & \ell^4 & \ell^5 & \ell^6 & \ell^7 \\ 0 & 1 & 2\ell & 3\ell^2 & 4\ell^3 & 5\ell^4 & 6\ell^5 & 7\ell^6 \\ 0 & 0 & 2 & 6\ell & 12\ell^2 & 20\ell^3 & 30\ell^4 & 42\ell^5 \\ 0 & 0 & 0 & 6 & 24\ell & 60\ell^2 & 120\ell^3 & 210\ell^4 \end{bmatrix}. \tag{1.34}$$

The inverse of matrix *C* can be found to be [12]