A First Course in Mathematical Logic and Set Theory

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For my parents

PREFACE

This book is inspired by *The Structure of Proof: With Logic and Set Theory* published by Prentice Hall in 2002. My motivation for that text was to use symbolic logic as a means by which to learn how to write proofs. The purpose of this book is to present mathematical logic and set theory to prepare the reader for more advanced courses that deal with these subjects either directly or indirectly. It does this by starting with propositional logic and first-order logic with sections dedicated to the connection of logic to proof-writing. Building on this, set theory is developed using first-order formulas. Set operations, subsets, equality, and families of sets are covered followed by relations and functions. The axioms of set theory are introduced next, and then sets of numbers are constructed. Finite numbers, such as the natural numbers and the integers, are defined first. All of these numbers are actually sets constructed so that they resemble the numbers that are their namesakes. Then, the infinite ordinal and cardinal numbers appear. The last chapter of the book is an introduction to model theory, which includes applications to abstract algebra and the proofs of the completeness and compactness theorems. The text concludes with a note on Gödel's incompleteness theorems.

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On a personal note, I would like to express my gratitude to my parents for their continued caring and support; to my brother and his wife, who will make sure my niece learns her math; to my dissertation advisor, Paul Eklof, who taught me both set theory and model theory; to Robert Meyer, who introduced me to symbol logic; to David Elfman, who taught me about logic through programming on an Apple II; and to my wife, Barb, whose love and patience supported me as I finished this book.

SYMBOLS

CHAPTER 1
PROPOSITIONAL LOGIC

1.1 SYMBOLIC LOGIC

Let us define **mathematics** as the study of number and space. Although representations can be found in the physical world, the subject of mathematics is not physical. Instead, mathematical objects are abstract, such as equations in algebra or points and lines in geometry. They are found only as ideas in minds. These ideas sometimes lead to the discovery of other ideas that do not manifest themselves in the physical world as when studying various magnitudes of infinity, while others lead to the creation of tangible objects, such as bridges or computers.

Let us define **logic** as the study of arguments. In other words, logic attempts to codify what counts as legitimate means by which to draw conclusions from given information. There are many variations of logic, but they all can be classified into one of two types. There is **inductive logic** in which if the argument is good, the conclusion will probably follow from the hypotheses. This is because inductive logic rests on evidence and observation, so there can never be complete certainty whether the conclusions reached do indeed describe the universe. An example of an inductive argument is:

A red sky in the morning means that a storm is coming.

We see a red sky this morning.

Therefore, there will be a storm today.

Whether this is a trust-worthy argument or not rests on the strength of the predictive abilities of a red sky, and we know about that by past observations. Thus, the argument is inductive. The other type is **deductive logic**. Here the methods yield conclusions with complete certainty, provided, of course, that no errors in reasoning were made. An example of a deductive argument is:

All geometers are mathematicians.

Euclid is a geometer.

Therefore, Euclid is a mathematician.

Whether Euclid refers to the author of the *Elements* or is Mr. Euclid from down the street is irrelevant. The argument works because the third sentence must follow from the first two.

As anyone who has solved an equation or written a proof can attest, deductive logic is the realm of the mathematician. This is not to say that there are not other aspects to the discovery of mathematical results, such as drawing conclusions from diagrams or patterns, using computational software, or simply making a lucky guess, but it is to say that to accept a mathematical statement requires the production of a deductive proof of that statement. For example, in elementary algebra, we know that given

$$2x - 5 = 11,$$

we can conclude

$$2x = 6$$

and then

$$x = 3.$$

As each of the steps is legal, it is certain that the conclusion of $x = 3$ follows. In geometry, we can write a two-column proof that shows that

$$\angle B \cong \angle D$$

is guaranteed to follow from

$$ABCD \text{ is a parallelogram.}$$

The study of these types of arguments, those that are deductive and mathematical in content, is called **mathematical logic**.

Propositions

To study arguments, one must first study sentences because they are the main parts of arguments. However, not just any type of sentence will do. Consider

all squares are rectangles.

The purpose of this sentence is to affirm that things called squares also belong to the category of things called rectangles. In this case, the assertion made by the sentence is correct. Also, consider,

circles are not round.

This sentence denies that things called circles have the property of being round. This denial is incorrect. If a sentence asserts or denies accurately, the sentence is **true**, but if it asserts or denies inaccurately, the sentence is **false**. These are the only **truth values** that a sentence can have, and if a sentence has one, it does not have the other. As arguments intend to draw true conclusions from presumably true given sentences, we limit the sentences that we study to only those with a truth value. This leads us to our first definition.

DEFINITION 1.1.1

A sentence that is either true or false is called a **proposition**.

Not all sentences are propositions, however. Questions, exclamations, commands, or self-contradictory sentences like the following examples can neither be asserted nor be denied.

- *Is mathematics logic?*
- *Hey there!*
- *Do not panic.*
- *This sentence is false.*

Sometimes it is unclear whether a sentence identifies a proposition. This can be due to factors such as imprecision or poor sentence structure. Another example is the sentence

it is a triangle.

Is this true or false? It is impossible to know because, unlike the other words of the sentence, the meaning of the word *it* is not determined. In this sentence, the word *it* is acting like a variable as in $x + 2 = 5$. As the value of x is undetermined, the sentence $x+2 = 5$ is neither true nor false. However, if x represents a particular value, we could make a determination. For example, if $x = 3$, the sentence is true, and if $x = 10$, the sentence is false. Likewise, if *it* refers to a particular object, then *it is a triangle* would identify a proposition.

There are two types of propositions. An **atom** is a proposition that is not comprised of other propositions.

Examples include

the angle sum of a triangle equals two right angles

and

some quadratic equations have real solutions.

A proposition that is not an atom but is constructed using other propositions is called a **compound proposition**. There are five types.

- A **negation** of a given proposition is a proposition that denies the truth of the given proposition. For example, the negation of $3 + 8 = 5$ is $3 + 8 \neq 5$. In this case, we say that $3 + 8 = 5$ has been **negated**. Negating the proposition *the sine function is periodic* yields *the sine function is not periodic*.
- A **conjunction** is a proposition formed by combining two propositions (called **conjuncts**) with the word *and*. For example,

the base angles of an isosceles triangle are congruent,
and a square has no right angles

is a conjunction with *the base angles of an isosceles triangle are congruent* and *a square has no right angles* as conjuncts.

- A **disjunction** is a proposition formed by combining two propositions (called **disjuncts**) with the word *or*. The sentence

the base angles of an isosceles triangle are congruent,
or a square has no right angles

is a disjunction.

- An **implication** is a proposition that claims a given proposition (called the **antecedent**) entails another proposition (called the **consequent**). Implications are also known as **conditional propositions**. For example,

if rectangles have four sides, then squares have for sides **1.1**

is a conditional proposition. Its antecedent is *rectangles have four sides*, and its consequent is *squares have four sides*. This implication can also be written as

rectangles have four sides implies that squares have four sides,

squares have four sides if rectangles have four sides,

rectangles have four sides only if squares have four sides,

and

if rectangles have four sides, squares have four sides.

A conditional proposition can also be written using the words *sufficient* and *necessary*. The word *sufficient* means “adequate” or “enough,” and *necessary* means “needed” or “required.” Thus, the sentence

rectangles having four sides is sufficient for squares to have four sides

translates (1.1). In other words, the fact that rectangles have four sides is enough for us to know that squares have four sides. Likewise,

squares having four sides is necessary for rectangles to have four sides

is another translation of the implication because it means that squares must have four sides because rectangle have four sides. Summing up, the antecedent is sufficient for the consequent, and the consequent is necessary for the antecedent.

- A **biconditional proposition** is the conjunction of two implications formed by exchanging their antecedents and consequents. For example,

if rectangles have four sides, then squares have four sides,
and if squares have four sides, then rectangles have four sides.

To remove the redundancy in this sentence, notice that the first conditional can be written as

rectangles have four sides only if squares have four sides

and the second conditional can be written as

rectangles have four sides if squares have four sides,

resulting in the biconditional being written as

rectangles have four sides if and only if squares have four sides

or the equivalent

rectangles having four sides is necessary and sufficient
for squares to have four sides.

Propositional Forms

As a typical human language has many ways to express the same thought, it is beneficial to study propositions by translating them into a notation that has a very limited collection of symbols yet is still able to express the basic logic of the propositions. Once this is done, rules that determine the truth values of propositions using the new notation can be developed. Any such system designed to concisely study human reasoning is called a **symbolic logic**. Mathematical logic is an example of symbolic logic.

Let p be a finite sequence of characters from a given collection of symbols. Call the collection an **alphabet**. Call p a **string** over the alphabet. The alphabet chosen so that p can represent a mathematical proposition is called the **proposition alphabet** and consists of the following symbols.

- **Propositional variables**: Uppercase English letters, $P, Q, R, \cdots$, or uppercase English letters with subscripts, $P_n, Q_n, R_n, \cdots$, where $n = 0, 1, 2, \cdots$
- **Connectives**: $\nvdash$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- **Grouping symbols**: (,), [,].

The sequences $P \vee Q$ and $P_1 Q_1 \wedge \leftrightarrow ((($ and the **empty string**, a string with no characters, are examples of strings over this alphabet, but only certain strings will be chosen for our study. A string is selected because it is able to represent a proposition. These strings will be determined by a method called a **grammar**. The grammar chosen for our present purposes is given in the next definition. It is given **recursively**. That is, the definition is first given for at least one special case, and then the definition is given for other cases in terms of itself.

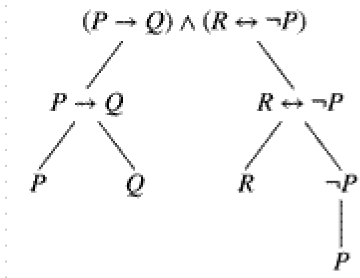
DEFINITION 1.1.2

A **propositional form** is a nonempty string over the proposition alphabet such that

- every propositional variable is a propositional form.
- $\neg p$ is a propositional form if p is a propositional form.
- $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$, and $(p \leftrightarrow q)$ are propositional forms if p and q are propositional forms.

We follow the convention that parentheses can be replaced with brackets and outermost parenthesis or brackets can be omitted. As with propositions, a propositional form that consists only of a propositional variable is an **atom**. Otherwise, it is **compound**.

The strings P, Q_1, $\neg P$, $(P_1 \vee P_2) \wedge P_3$, and $(P \rightarrow Q) \wedge (R \leftrightarrow \neg P)$ are examples of propositional forms. To prove that the last string is a propositional form, proceed using Definition 1.1.2 by noting that $(P \rightarrow Q) \wedge (R \leftrightarrow \neg P)$ is the result of combining $P \rightarrow Q$ and $R \leftrightarrow \neg P$ with $\wedge$. The propositional form $P \rightarrow Q$ is from P and Q combined with $\rightarrow$, and $R \leftrightarrow \neg P$ is from R and $\neg P$ combined with $\leftrightarrow$. These and $\neg P$ are propositional forms because P, Q, and R are propositional variables. This derivation yields the following **parsing tree**:



The parsing tree yields the **formation sequence** of the propositional form:

$$P, Q, R, \neg P, P \to Q, R \leftrightarrow \neg P, (P \to Q) \wedge (R \leftrightarrow \neg P).$$

The sequence is formed by listing each distinct term of the tree starting at the bottom row and moving upwards.

EXAMPLE 1.1.3

Make the following **assignments**:

$$p := R \leftrightarrow (P \wedge Q),$$
$$q := (R \leftrightarrow P) \wedge Q.$$

The symbol := indicates that an assignment has been made. It means that the propositional form on the right has been assigned to the lowercase letter on the left. Using these designations, we can write new propositional forms using p and q. The propositional form $p \wedge q$ is

$$[R \leftrightarrow (P \wedge Q)] \wedge [(R \leftrightarrow P) \wedge Q]$$

with the formation sequence,

$$P, Q, R, P \wedge Q, R \leftrightarrow P,$$
$$R \leftrightarrow (P \wedge Q), (R \leftrightarrow P) \wedge Q, [R \leftrightarrow (P \wedge Q)] \wedge [(R \leftrightarrow P) \wedge Q],$$

and $\nvdash q \to p$ is

$$\neg[(R \leftrightarrow P) \wedge Q] \to [R \leftrightarrow (P \wedge Q)]$$

with the formation sequence

$$P, Q, R, R \leftrightarrow P, P \wedge Q, (R \leftrightarrow P) \wedge Q, R \leftrightarrow (P \wedge Q),$$
$$\neg[(R \leftrightarrow P) \wedge Q], \neg[(R \leftrightarrow P) \wedge Q] \to [R \leftrightarrow (P \wedge Q)].$$

Interpreting Propositional Forms

Notice that determining whether a string is a propositional form is independent of the meaning that we give the symbols. However, as we do want these symbols to convey meaning, we assume that the propositional variables represent atoms and set this interpretation on the connectives:

$\neg$	not
$\wedge$	and
$\vee$	or
$\rightarrow$	implies
$\leftrightarrow$	if and only if

Because of this interpretation, name the compound propositional forms as follows:

$\neg p$	negation
$p \wedge q$	conjunction
$p \vee q$	disjunction
$p \rightarrow q$	implication
$p \leftrightarrow q$	biconditional

EXAMPLE 1.1.4

To see how this works, assign some propositions to some propositional variables:

P := *The sine function is not one-to-one.*

Q := *The square root function is one-to-one.*

R := *The absolute value function is not onto.*

The following symbols represent the indicated propositions:

- $\nvdash R$

  The absolute value function is onto.

- $\nvdash P \vee \nvdash Q$

  The sine function is one-to-one, or the square root function is not one-to-one.

- $Q \rightarrow R$

  If the square root function is one-to-one, the absolute function is not onto.

- $R \leftrightarrow P$

  The absolute value function is not onto if and only if the sine function is not one-to-one.

- $P \wedge Q$

  The sine function is not one-to-one, and the square root function is one-to-one.

- $\nvdash P \wedge Q$

The sine function is one-to-one, and the square root function is one-to-one.

- $\nvdash (P \wedge Q)$

It is not the case that the sine function is not one-to-one and the square root function is one-to-one.

The proposition

the absolute value function is not onto if and only if both the sine function is not one-to-one and the square root function is one-to-one

is translated as $R \leftrightarrow (P \wedge Q)$ and

the absolute value function is not onto if and only if the sine function is not one-to-one, and the square root function is one-to-one

is translated as $(R \leftrightarrow P) \wedge Q$. If the parenthesis are removed, the resulting string is $R \leftrightarrow P \wedge Q$. It is simpler, but it is not clear how it should be interpreted. To eliminate its ambiguity, we introduce an **order of connectives** as in algebra. In this way, certain strings without parentheses can be read as propositional forms.

DEFINITION 1.1.5 [Order of Connectives]

To interpret a propositional form, read from left to right and use the following precedence:

- propositional forms within parentheses or brackets (innermost first),
- negations,
- conjunctions,
- disjunctions,
- conditionals,
- biconditionals.

EXAMPLE 1.1.6

To write the propositional form $\nvdash P \vee Q \wedge R$ with parentheses, we begin by interpreting $\nvdash P$. According to the order of operations, the conjunction is next, so we evaluate $Q \wedge R$. This is followed by the disjunction, and we have the propositional form $\nvdash P \vee (Q \wedge R)$.