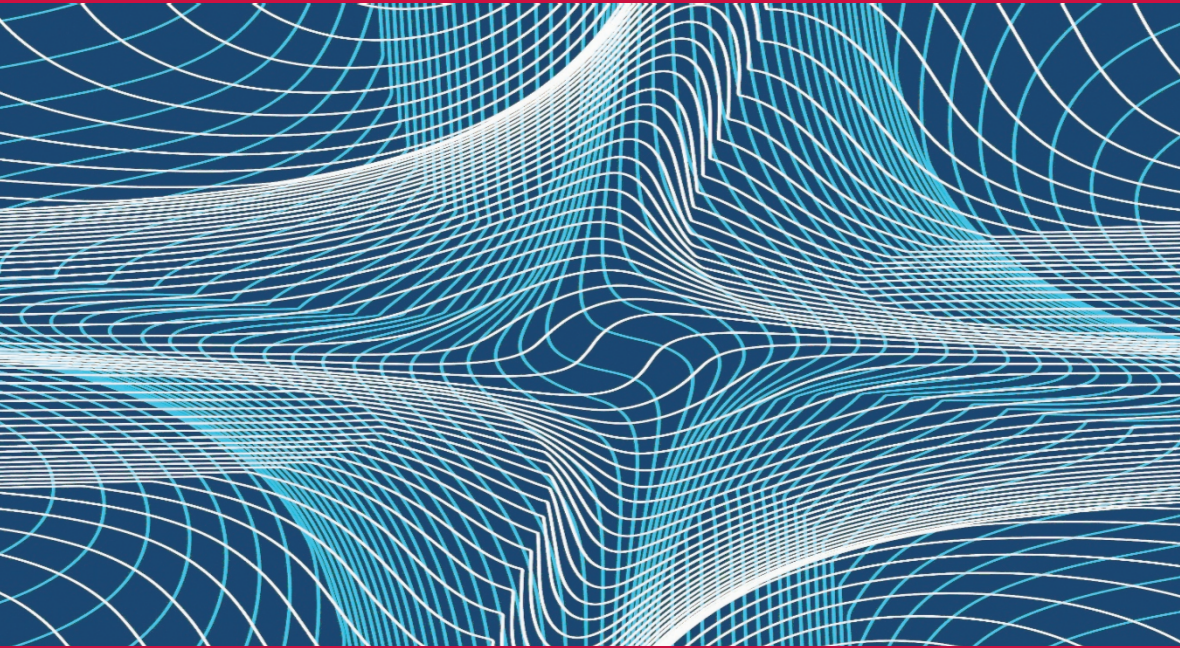


NUMERICAL METHODS IN ENGINEERING SERIES



Variational Methods for Engineers with Matlab®

Eduardo Souza de Cursi

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Variational Methods for Engineers with Matlab®

Series Editor
Piotr Breitkopf

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Introduction

Variational methods may be seen as a new language leading to new formulations and new methods of solution of equations, where the central concept is the physical notion of work. In this language, we do not say that a quantity is null, but we say that its work is null for all the possible values of a connected variable. For instance, we do not say that a displacement is null, but that the associated work is null for any force. However, we do not say that a force is null, but that its work is null for any displacement. In the simplest situation, we do not say that $x = 0$, but we say that “ x is a real number such that its product for any other real number is equal to zero”, e.g. $x \in \mathbb{R}$ and $xy = 0, \forall y \in \mathbb{R}$. Although this modification and the equivalence between both the formulations seem trivial, it implies a deep conceptual change, as it will be seen in the following. Moreover, despite the view expressed by Richard Feynman – from the standpoint of Physics – in [FEY 85], these two formulations may not be equivalent – from the standpoint of Mathematics – in certain situations: it may become necessary to adopt a complex theoretical framework in order to obtain such an equivalence – that will never be complete.

The beginning of the history of variational methods is often brought to the works of the Greek philosopher Aristotle regarding the *lever problem*. Nowadays, lever analysis is an exercise for students, which automatically applies the physical principles of momentum and force equilibrium in order to explain how a lever works. So, it may be difficult to conceive that such knowledge was built up patiently over

more than a millenium and has intrigued mankind for a long time. For more than ten centuries, the reason why a small force was able to move a large load has remained obscure and has been the subject of passionate discussions. For instance, the text *Problemata Mechanica* (*Problems of Mechanics*, attributed to Aristotle, but whose real author is uncertain) starts by saying that [ARI 38]:

Among the problems included in this class are included those concerned with the lever. For, it is strange that a great weight can be moved by a small force, and that, too, when a greater weight is involved. For the very same weight, which a man cannot move without a lever, he quickly moves by applying the weight of the lever.

In this text, the author studies the lever by considering fictitious possible motions – e.g. *virtual* motions – of the system. An explanation is proposed by considering these possible motions and selecting a particular motion among all: the analysis of fictitious movements furnishes the real one. This approach contains the seeds of the modern variational methods: define the virtual motions and find the real motion by analyzing the virtual motions. It is interesting to notice that restrictions are taken into account: it is observed that the extremities of the lever move on a circle – there are *natural* and *unnatural* motions: the natural motion of a load placed on the extremity of the lever corresponds to the tangent direction, while, in fact, the load moves on a circle.

A second ancient author is Heron Alexandrius, who formulated a principle of economy (or of minimum violence of Nature on itself). This principle leads to the principle that light rays follow the shortest path between two points. In the Middle Ages, Jordanus Nemorarius studied the motion of bodies on inclined planes and associated it with the statics of levers having arms of unequal lengths and obtained a result of equality of virtual works [RAD 98, SIM 12].

These ancient texts do not consider the notion of *work*, unknown in ancient times and which needed many centuries to emerge and to be formalized. Tools such as *linear functionals*, *limits* and *differential*

calculus were not available at those times, which limited the development and the formalization of variational methods.

The situation changed radically with the invention of differential and integral calculus. Gottfried Wilhelm Leibniz, Jacques Bernoulli, Jean Bernoulli, Leonhard Euler, Pierre Varignon, Pierre Louis de Maupertuis, Charles-Augustin Coulomb, Jean le Rond D'Alembert and others seized these new tools and used them to develop a new theory of mechanics, which culminated with the publication of Joseph-Louis Lagrange's *Analytical Mechanics* and the formalization of the Principle of Virtual Works.

Lagrange's formalism was extended by William Rowan Hamilton, who introduced a unified point of view connecting optics and dynamics. Hamilton's approach led to new variational formulations in Physics, usually referred to as *Hamiltonian Mechanics*, which is a highly reducing expression – in fact, the works of Hamilton led to formulations involving many fields, such as, for instance, classical mechanics, quantum mechanics, thermodynamics, electromagnetism, etc. It furnished a unified point of view, which may be considered as not completely exploited at this date. Moreover, significant works by Charles Jacobi, Joseph Liouville, Henri Poincaré, Alexandre Lyapounov, Lev Pontryagin, Andrei Kolmogorov and many other researchers extended Hamilton's theory, namely to applications in automatics and control.

The development of information technology has popularized numerical methods based on variational formulations such as, for instance, finite elements, finite volumes, fundamental solutions, smooth particle hydrodynamics and so on. A number of contributions led to these developments, namely the works of Boris Galerkin, Paul Dirac, Maurice Fréchet, Norbert Wiener, Sergei Sobolev, Solomon Bochner, Laurent Schwarz among a large set of researchers.

The revolution of variational methods is still under development, since new connections between variational formulations and probabilities have been highlighted by recent developments. The exploration of these relationships opens up new perspectives and gives us a glimpse of exciting prospects for the future.

Integrals

Variational methods are closely connected to integrals. So, before starting our variational adventures, let us recall some elements about integrals and their numerical evaluation.

The history of infinite sums is such as integrals may be brought to Zeno of Elea's paradox about the grain of millet. As described by Simplicius [FAI 98]:

Tell me, Protagoras, said he, does one grain of millet make a noise when it falls, or does the ten-thousandth part of a grain? On receiving the answer that it does not, he went on: Does a measure of millet grains make a noise when it falls, or not? He answered, it does make a noise. Well, said Zeno, does not the statement about the measure of millet apply to the one grain and the ten-thousandth part of a grain? He assented, and Zeno continued, Are not the statements as to the noise the same in regard to each? For as are the things that make a noise, so are the noises. Since this is the case, if the measure of millet makes a noise, the one grain and the ten-thousandth part of a grain make a noise.

Here, Zeno considers the problem of the sum of small negligible components, what is the principle of the integration of infinitesimal

contributions. In these ancient times, the concept of the limit was not known, so the notion of infinite sums and the evaluation of areas remained unsolved. In fact, many philosophers considered the question of limits, such as Antiphon the sophist, who argued that continuously doubling the number of sides of a polygon inscribed in a circle gives as result a polygon whose sides coincide with the circumference and having the same area as the circle [PEN 02]. Eudoxus of Cnidus proposed a generalization which led to the exhaustion *method*, which was extensively used for the evaluation of areas for about 2 thousand years. Archimedes applied this method to the evaluation of surfaces and volumes by considering sums of “lines” [BAU 09, BLO 11]. Archimedes’ method was improved by Thabit ibn Qurra and inspired Bonaventura Cavalieri and Gilles de Roberval, namely the principle of *indivisibles*. Many researchers enriched the works of Cavalieri. For instance, Evangelista Torricelli introduced indivisibles having a thickness and Blaise Pascal considered triangular and pyramidal sums of indivisibles leading to the evaluation of double and triple integrals.

The differential and integral calculus introduced by Gottfried Leibniz and Isaac Newton opened up a new era. Agustin-Louis Cauchy introduced the concept of the definite integral and chose the notation proposed by Jean-Baptiste Joseph Fourier, which is used today. A complete formalization was proposed by Georg Riemann and led to the Riemann sums and integrals.

At this moment, the horizon seemed clear, but clouds appeared with the works of Karl Weierstrass and Richard Dedekind, which drew attention to fundamental inconsistencies in the theory. Georg Cantor, Camille Jordan and the proposition of a new set and measure theory furnished a new impulse. Emile Borel, René Baire and Henri Lebesgue formed the trinity that constructed a complete formalization of the integrals in the new framework and that gave the impression of taking away the difficulties raised by their predecessors.

But the adventure continues, since one of the most important foundations of the actual theory is the axiom of choice, which

generates a paradox which has not yet been solved: the Banach-Tarski paradox, which establishes that a volume cannot be defined for a sphere of radius one. Research regarding this paradox will probably lead to new developments and evolutions in our understanding of these old concepts of areas and volumes, issues that have concerned humanity for millennia and whose complete solution eludes our sagacity every time we think we are near to their complete understanding.

1.1. Riemann integrals

The first formal theory concerning integrals was proposed by Riemann and was provoked by his interest in *Fourier series*. In 1807, a paper by Siméon-Denis Poisson [POI 08] mentioned that Joseph Fourier had proposed the representation of some functions by trigonometric series. The ideas of Fourier immediately aroused controversy. Much of the criticism was related to formal aspects. In his book *Théorie analytique de la chaleur* [FOU 22], Fourier made the following comment about trigonometric series:

One could doubt that there existed such a function, but this issue will be clarified later.

In fact, as often in the history of science, the explanation furnished by Fourier was not complete and has raised fundamental questions, namely what are the functions that may be represented by a trigonometric series and the evaluation of integrals for the determination of the coefficients? Integrals were introduced by Isaac Newton and Gottfried Leibniz in the 17th Century, but their theory was the subject of discussion years before acceptance by the scientific community. About one century after, integrals were being taught in university courses, such as, for instance, those that Augustin-Louis Cauchy presented [CAU 23, CAU 29], but the formal theory waited for the works of Georg Friedrich Bernhard Riemann about the questions raised by Fourier, namely his habilitation thesis [RIE 67]. In this thesis, Riemann introduced the fundamental elements for the definition of an integral. Basically, the evaluation of

$$I = \int_{\Omega} f(x)dx$$

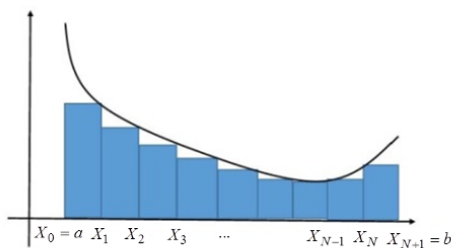
requires a subdivision of Ω into a finite number of non-recovering subsets, i.e. a partition $\wp = \{\Omega_i; 1 \leq i \leq N\}$ of Ω :

$$\bigcup_{i=1}^N \Omega_i = \Omega, \quad \Omega_i \cap \Omega_j = \emptyset, \text{ if } i \neq j.$$

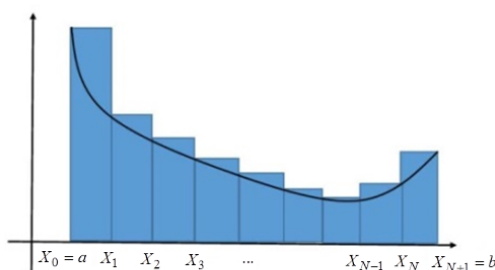
For instance, when $x \in \mathbb{R}$ and $\Omega = (a, b) \subset \mathbb{R}$, we may consider a family of N subintervals $\Omega_i = (x_i, x_{i+1})$ ($i = 1, \dots, N$) such that $a = x_1 < x_2 < \dots < x_N < x_{N+1} = b$. For each subinterval Ω_i , the function f has a maximum \overline{f}_i and a minimum \underline{f}_i . Thus, we may consider the Riemann sums:

$$\overline{R}(f, \wp) = \sum_{i=1}^N \overline{f}_i (x_{i+1} - x_i), \quad \underline{R}(f, \wp) = \sum_{i=1}^N \underline{f}_i (x_{i+1} - x_i),$$

which verify $\underline{R}(f, \wp) \leq \overline{R}(f, \wp)$ (see Figure 1.1).



(a) $\underline{R}(f, \wp)$



(b) $\overline{R}(f, \wp)$

Figure 1.1. Riemann sums: partitions of the horizontal axis

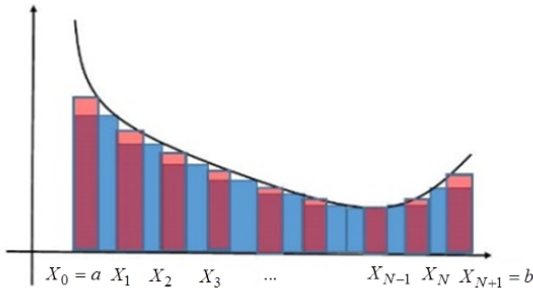
Adding supplementary points to the partition increases \underline{R} and decreases \overline{R} (Figure 1.1), so that, \underline{R} has as cluster point an upper bound \underline{I} and \overline{R} has as cluster point a lower bound \overline{I} , $\underline{R}(f, \varphi) \leq \underline{I} = \overline{I} \leq \overline{R}(f, \varphi)$. When both these values coincide, we say that their common value is $I = \underline{I} = \overline{I}$.

A practical estimation of the value of I is:

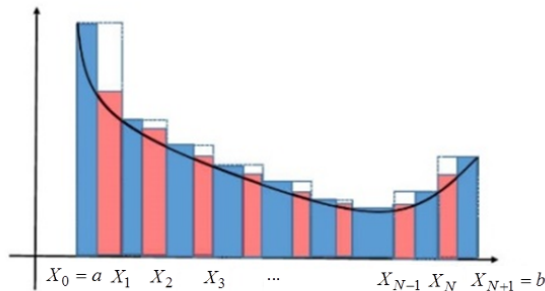
$$I \approx \frac{1}{2} \left(\underline{R}(f, \varphi) + \overline{R}(f, \varphi) \right),$$

i.e.

$$I \approx \sum_{i=1}^N \left(\frac{\overline{f}_i + \underline{f}_i}{2} \right) (x_{i+1} - x_i)$$



(a) $\underline{R}(f, \varphi)$



(b) $\overline{R}(f, \varphi)$

Figure 1.2. Adding supplementary points increases \underline{R} and decreases \overline{R}

In practice, the values of $\overline{f_i}$ and $\underline{f_i}$ are not determined exactly, but are approximated by using the values $f_i = f(x_i)$ and $f_{i+1} = f(x_{i+1})$:

$$I \approx \sum_{i=1}^N \left(\frac{f_i + f_{i+1}}{2} \right) (x_{i+1} - x_i) \quad [1.1]$$

This formula is known as the *trapezoidal rule*. A simple estimative is furnished by the *mean value approximation*:

$$I \approx \frac{b-a}{N} \sum_{i=1}^{N+1} f_i .$$

For $\Omega = (a_1, b_1) \times (a_2, b_2)$ or $\Omega = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$, we may consider families of intervals associated with each component and the Cartesian product of these families. Analogous Riemann sums $\underline{R}(f, \wp) \leq \overline{R}(f, \wp)$ may be defined in this case.

1.2. Lebesgue integrals

Riemann's theory was quickly challenged by the works of Richard Dedekind and Karl Weierstrass. Their work on the foundations of function theory led, on the one hand, to the development of set theory by Georg Cantor and, on the other hand, to measure theory initiated by Giuseppe Peano and Camille Jordan, then formalized by Emile Borel and René Baire. Henri Lebesgue adopted the point of view of measure theory in order to redefine integrals. The theory initiated by Lebesgue is not yet the end of integration theory, since measure theory produces sets that are not measurable (for instance, Vitali sets, presented in the pamphlet [VIT 05]) and, as a consequence, functions that are not measurable. These elements result straight from the axiom of choice (see, for instance, [SOU 10]) and lead to fundamental difficulties, from those presented Felix Hausdorff [HAU 14], and Stefan Banach and Alfred Tarski [BAN 24]. These difficulties have not been solved to date and promise interesting developments in the future.

The numerical evaluation of the Lebesgue integral

$$I = \int_{\Omega} f$$

requires more information than the evaluation of a Riemann integral; we need a partition $\wp = \{\Omega_i: 1 \leq i \leq N\}$ of Ω and a partition $\mathcal{F} = \{\mathcal{F}_i: 1 \leq i \leq M\}$ of the image $f(\Omega)$.

For instance, let us consider the situation where both Ω and $f(\Omega)$ are intervals. We define \mathcal{F} by taking points $y_{min} = y_1 < y_2 < \dots < y_N < y_{M+1} = y_{max}$, where $y_{min} = \min\{f(x): x \in \Omega\}$ and $y_{max} = \max\{f(x): x \in \Omega\}$. In practice, y_{min} and y_{max} may be approximations of these values satisfying where $y_{min} \leq \min\{f(x): x \in \Omega\}$ and $y_{max} \geq \max\{f(x): x \in \Omega\}$.

Let us denote by ℓ the Lebesgue measure, given by:

$$\ell((\alpha, \beta)) = \beta - \alpha, \quad \ell((\alpha_1, \beta_1) \times (\alpha_2, \beta_2)) = (\beta_1 - \alpha_1)(\beta_2 - \alpha_2), \dots$$

and

$$\ell_i = \ell(f^{-1}([y_i, y_{i+1}))) = \ell(\{x \in \Omega: y_i \leq f(x) < y_{i+1}\})$$

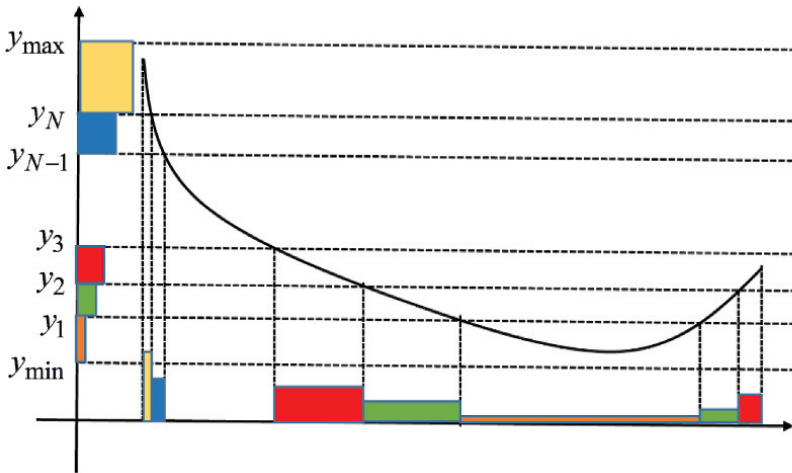


Figure 1.3. Lebesgue's approach to integration:
partition of the vertical axis

We have:

$$I_{inf} = \sum_{i=1}^M y_i \ell_i \leq I \leq I_{sup} = \sum_{i=1}^M y_{i+1} \ell_i$$

Let us introduce:

$$\mu_{inf}(y) = \ell \left(S_{inf}(y) \right), \quad S_{inf}(y) = \{x \in \Omega : f(x) < y\}.$$

Then

$$\ell_i = \mu_{inf}(y_{i+1}) - \mu_{inf}(y_i)$$

and

$$\begin{aligned} I_{inf} &= \sum_{i=1}^M y_i \ell_i = \sum_{i=1}^M y_i \left(\mu_{inf}(y_{i+1}) - \mu_{inf}(y_i) \right) \\ &= \sum_{i=2}^{M+1} y_{i-1} \mu_{inf}(y_i) - \sum_{i=1}^M y_i \mu_{inf}(y_i) \\ &= y_{M+1} \mu_{inf}(y_{M+1}) - y_1 \mu_{inf}(y_1) - \sum_{i=2}^{M+1} (y_i - y_{i-1}) \mu_{inf}(y_i). \end{aligned}$$

Thus,

$$I_{inf} \rightarrow y_{M+1} \mu_{inf}(y_{M+1}) - y_1 \mu_{inf}(y_1) - \int_{y_1}^{y_{M+1}} \mu_{inf}(y) dy,$$

where the integral in this formula is a Riemann integral. So,

$$I \geq y_{M+1} \mu_{inf}(y_{M+1}) - y_1 \mu_{inf}(y_1) - \int_{y_1}^{y_{M+1}} \mu_{inf}(y) dy. \quad [1.2]$$

Analogously,

$$\begin{aligned} I_{sup} &= \sum_{i=1}^M y_{i+1} \ell_i = \sum_{i=1}^M y_{i+1} \left(\mu_{inf}(y_{i+1}) - \mu_{inf}(y_i) \right) = \sum_{i=2}^{M+1} y_i \mu_{inf}(y_i) - \\ &\sum_{i=1}^M y_{i+1} \mu_{inf}(y_i) = y_{M+1} \mu_{inf}(y_{M+1}) - y_1 \mu_{inf}(y_1) - \sum_{i=1}^M (y_{i+1} - y_i) \mu_{inf}(y_i). \end{aligned}$$

Thus, we have

$$I_{sup} \rightarrow y_{M+1} \mu_{inf}(y_{M+1}) - y_1 \mu_{inf}(y_1) - \int_{y_1}^{y_{M+1}} \mu_{inf}(y) dy,$$

and

$$I \leq y_{M+1} \mu_{inf}(y_{M+1}) - y_1 \mu_{inf}(y_1) - \int_{y_1}^{y_{M+1}} \mu_{inf}(y) dy. \quad [1.3]$$

Equations [1.2] and [1.3] show that:

$$I = y_{M+1}\mu_{\inf}(y_{M+1}) - y_1\mu_{\inf}(y_1) - \int_{y_1}^{y_{M+1}} \mu_{\inf}(y) dy. \quad [1.4]$$

Notice that, if $y_{\min} \leq \min\{f(x): x \in \Omega\}$ and $y_{\max} \geq \max\{f(x): x \in \Omega\}$, then $\mu_{\inf}(y_1) = 0$, while $\mu_{\inf}(y_{M+1}) = b - a$ and we have:

$$I = (b - a)y_{M+1} - \int_{y_1}^{y_{M+1}} \mu_{\inf}(y) dy.$$

Let

$$\mu_{\sup}(y) = \ell(S_{\sup}(y)), \quad S_{\sup}(y) = \{x \in \Omega: f(x) \geq y\}.$$

Then,

$$\ell_i = \mu_{\sup}(y_i) - \mu_{\sup}(y_{i+1})$$

and

$$I_{\inf} = \sum_{i=1}^M y_i \ell_i = \sum_{i=1}^M y_i (\mu_{\sup}(y_i) - \mu_{\sup}(y_{i+1})) = \sum_{i=1}^M y_i \mu_{\sup}(y_i) - \sum_{i=2}^{M+1} y_{i-1} \mu_{\sup}(y_i) = y_1 \mu_{\sup}(y_1) - y_{M+1} \mu_{\sup}(y_{M+1}) + \sum_{i=2}^{M+1} (y_i - y_{i-1}) \mu_{\sup}(y_i).$$

Thus,

$$I_{\inf} \rightarrow y_1 \mu_{\sup}(y_1) - y_{M+1} \mu_{\sup}(y_{M+1}) + \int_{y_1}^{y_{M+1}} \mu_{\sup}(y) dy,$$

where the integral in this formula is a Riemann one. So,

$$I \geq y_1 \mu_{\sup}(y_1) - y_{M+1} \mu_{\sup}(y_{M+1}) + \int_{y_1}^{y_{M+1}} \mu_{\sup}(y) dy. \quad [1.5]$$

Analogously,

$$I_{\sup} = \sum_{i=1}^M y_{i+1} \ell_i = \sum_{i=1}^M y_{i+1} (\mu_{\sup}(y_i) - \mu_{\sup}(y_{i+1})),$$

so that

$$I_{sup} = y_1 \mu_{sup}(y_1) - y_{M+1} \mu_{sup}(y_{M+1}) - \sum_{i=1}^M (y_{i+1} - y_i) \mu_{sup}(y_i) .$$

Thus, we have:

$$I_{sup} \rightarrow y_1 \mu_{sup}(y_1) - y_{M+1} \mu_{sup}(y_{M+1}) + \int_{y_1}^{y_{M+1}} \mu_{sup}(y) dy ,$$

and

$$I \leq y_1 \mu_{sup}(y_1) - y_{M+1} \mu_{sup}(y_{M+1}) + \int_{y_1}^{y_{M+1}} \mu_{sup}(y) dy . \quad [1.6]$$

Equations [1.5] and [1.6] show that:

$$I = y_1 \mu_{sup}(y_1) - y_{M+1} \mu_{sup}(y_{M+1}) + \int_{y_1}^{y_{M+1}} \mu_{sup}(y) dy . \quad [1.7]$$

Notice that, if $y_{min} \leq \min\{f(x): x \in \Omega\}$ and $y_{max} \geq \max\{f(x): x \in \Omega\}$, then $\mu_{sup}(y_1) = b - a$, while $\mu_{sup}(y_{M+1}) = 0$ and we have:

$$I = (b - a) y_1 + \int_{y_1}^{y_{M+1}} \mu_{sup}(y) dy .$$

1.3. Matlab® classes for a Riemann integral by trapezoidal integration

The one-dimensional (1D) trapezoidal rule is implemented in Matlab® in the intrinsic function `trapz` and may be extended to multidimensional situations. Assume that the structure `p` has fields `p.x`, `p.y`, `p.z` corresponding to the coordinates of the points x_i, y_j, z_k of the partitions of the intervals and `p.dim` corresponding to the dimension, while table `F` contains the values of $f - F_i = f(x_i)$, or $F_{ij} = f(x_i, y_j)$, or $F_{ijk} = f(x_i, y_j, z_k)$, according to the dimension of the integral. Let us summarize the data in a structure `data` such

that `data.points = p` and `data.values = F`. Then we may use the class below (Program 1.1).

```

classdef riemann
    methods (Static, Access = private)
        function v = trapz2(x1,x2,F)
            sy = zeros(size(x1));
            for i = 1: length(sy)
                sy(i) = trapz(x2,F(i,:));
            end;
            v = trapz(x1,sy);
        end
        function v = trapz3(x1,x2,x3,F)
            sz = zeros(length(x1),length(x2));
            for i = 1: length(x1)
                for j = 1: length(x2)
                    sz(i,j) = trapz(x3,F(i,j,:));
                end;
            end;
            sy = zeros(size(x1));
            for i = 1: length(x1)
                sy(i) = trapz(x2,sz(i,:));
            end;
            v = trapz(x1,sy);
        end
    end
    methods(Static)
        function v = trpzd(data)
            p = data.points;
            F = data.values;
            n = p.dim;
            switch n
                case 1
                    v = trapz(p.x,F);
                case 2
                    v = riemann.trapz2(p.x,p.y,F);
                case 3
                    v = riemann.trapz3(p.x,p.y,p.
z,F);
            end
        end
    end
end

```

```

        otherwise
            v = [];
            disp('error in arguments for
trpzd');
        end
    end
    function v = mean_value(data)
        p = data.points;
        F = data.values;
        n = p.dim;
        switch n
            case 1
                v = mean(F)*p.measure;
            case 2
                v = mean(mean(F))*p.measure;
            case 3
                v =
mean(mean(mean(F))*p.measure;
            otherwise
                v = [];
                disp('error in arguments for
mean value');
            end
        end
    end
end
end

```

Program 1.1. *A class for the evaluation of Riemann integrals*

This class contains only trapezoidal and mean value integration methods, but it can be enriched by the user with other methods of numerical integration.

EXAMPLE 1.1.– Let us evaluate:

$$I = \int_0^1 dx \int_0^2 (x^2 + y^2) dy$$

by using the points `x=0:0.01:1;` and `y=0:0.01:2;`. Assuming that `F(i, j)=x(i)^2+y(j)^2;`, the code:

```

p.x = x;
p.y = y;
p.dim = 2;
p.measure = 2;
data.points = p;
data.values = F;
v1 = riemann.trpzd(data)
v2 = riemann.mean_value(data)

```

produces $v1 = 3.3334$, $v2 = 3.3433$. The exact value is $\frac{10}{3} \approx 3.3333$. ■

EXAMPLE 1.2.– Let us evaluate:

$$I = \int_0^1 dx \int_0^2 dy \int_0^3 (x^2 + y^2 + z^2) dz$$

by using the points $x=0:0.01:1$; $y=0:0.01:2$; $z=0:0.01:3$. Assuming that $F(i, j, k) = x(i)^2 + y(j)^2 + z(k)^2$, the code:

```

p.x = x;
p.y = y;
p.z = z;
p.dim = 3;
p.measure = 6;
data.points = p;
data.values = F3;
v1 = riemann.trpzd(F3,p)
v2 = riemann.mean_value(F3,p)

```

produces $v1 = 28.0003$, $v2 = 28.0600$. The exact value is 28. ■

The creation of the tables F from a subprogram f evaluating a function f is made by the following class:

```
classdef spam
    properties
    end
    methods (Static, Access = private)
        function v = vspam1(x,f)
            v = zeros(size(x));
            for i = 1: length(x)
                v(i) = f(x(i));
            end;
        end
        function v = vspam2(x,y,f)
            v = zeros(length(x),length(y));
            for i = 1: length(x)
                for j = 1: length(y)
                    v(i,j) = f(x(i),y(j));
                end;
            end;
        end
        function v = vspam3(x,y,z,f)
            v = zeros(length(x),length(y),length(z));
            for i = 1: length(x)
                for j = 1: length(y)
                    for k = 1: length(z)
                        v(i,j,k) = f(x(i),y(j),z(k));
                    end;
                end;
            end;
        end
        function v = tspam1(x,f)
            v = zeros(size(x));
            for i = 1: length(x)
                v(i) = f(x(i));
            end;
        end
        function v = tspam2(x,y,f)
            v = zeros(size(x));
            for i = 1: size(x,1)
                for j = 1: size(x,2)
                    v(i,j) = f(x(i,j),y(i,j));
                end;
            end;
        end;
    end;
end;
```