Studies in Universal Logic

Amirouche Moktefi
Sun-Joo Shin
Editors

Visual Reasoning with Diagrams

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Studies in Universal Logic

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Visual Reasoning with Diagrams

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Preface

1. Valid Reasoning and Formalization

Deductive logic, all of us would agree, is the study of valid reasoning. Valid reasoning is the process of extracting certain information from given information. Logical systems are invented to make this process almost mechanical so that we may adopt them to save time and effort as well as to carry out valid reasoning in an accurate way. Realizing that mechanical processes go hand in hand with formalization, we should not be surprised to encounter much formalism in the literature on logic. At the same time, one could be intrigued by the following observation: The formalism studied in the logic literature is quite homogeneous—limited to almost only symbolic formal systems. *Why does symbolization almost exclusively dominate the enterprise of formalism in logic?* This is our opening question. We do not even pretend to answer the question decisively, but only aspire to reveal some important aspects about valid reasoning in terms of different forms of representation and to highlight some of the main motivations behind the project of the current volume.

One might be puzzled by the opening question itself, if one assumes that formalization is identical with symbolization. According to this outlook, symbolic systems are the only kind of medium to formalize our valid reasoning processes, and hence, there is no mystery about the homogeneity of logical systems. Let us examine this view by breaking it into two steps: To explore (i) the relation between valid reasoning and symbols/diagrams and (ii) the relation between formalization and symbols/diagrams. After outlining theoretical issues involved in each step, we will illustrate our positions about (i) and (ii) in the second and the third sections, respectively.

First, is valid reasoning (which is the primary mission of logic) itself tied up with a certain type of medium, that is, symbolic representation? Hence, is it the case that information-extraction is carried out only through symbol-manipulation? If so, it would not be surprising that formalization, which aims to mechanize valid reasoning process, should be limited to symbolic systems only. However, if valid reasoning itself does not dictate a certain form of medium, there would be no prima facie reason to equate formalization and symbolization. For now, we would simply like to point to examples in our daily life reasoning process: Maps, pictures, charts, and diagrams, as well as sentences and symbols are all used to carry out reasoning. Of course, pictures may be misleading. But, so may sentences. In the next section, we present extensive and historical examples to illustrate visual reasoning, that is, how valid reasoning is carried out by the use of diagrams. Thus, we would like to conclude that symbols are one of the main methods adopted in carrying out valid reasoning, but are far from being the only method. Then, why have symbols been the almost exclusive medium of formal systems?

What is the essence of formal systems? Not being a case-by-case approach, formalization aims to mechanize processes so that errors may be eliminated and at the same time effectiveness may be achieved. Indeed, accuracy and efficiency are the goals of a formal system. A system that does not assure accuracy would be useless. On the other hand, if we cared only about accuracy, that is we could take as much time as we want and we could rely on processes as elaborate as we want, then there would be no point of having a system, either. We explore further how these two desiderata are obtained so that we may get to the essence of formalization. Then, our opening question about the dominance of symbolic formal systems might find some answer.

Let us start with the accuracy desideratum. An error, we say, takes place in the case of deductive reasoning when we (wrongly) infer a false piece of information from given true information. In order to secure accuracy, we want to have a mechanism to prevent a move from true to false information. An obvious obstacle to this enterprise is that there is an infinite number of cases to get to falsity from truth. How do we come up with a way to predict and prevent these non-denumerably infinite cases? This is a dilemma almost every system has to face, and at the same time it is precisely the reason why we desire to have a system, instead of case-by-case approaches. Another perplexing element is how to deal with semantic values, that is, truth and falsity, in a system. To sum up: How can we manage an infinite number of semantic relation cases at a mechanical level?

A clever solution to this challenge is to stipulate a finite number of permissible syntactic manipulations. Only those permissible inferences are allowed in a system, and since permissible steps are finite, a system can block an error by checking each move against a finite set of rules. It should be noted that these are syntactic transformations, not semantic ones. How do we know that a finite number of syntactic manipulations guarantee the accuracy that a formal system strives for? First, we would like to determine how a finite number of rules conquer an infinite number of valid/non-valid reasoning cases. Second, we need to theorize two-way traffic between syntax and semantics. Valid reasoning is understood in terms of semantic concepts, that is, true or false. At the same time, permissible inference is defined in terms of permissible syntactic alteration.

These questions demand a meta-level of justification: The soundness and completeness proofs of a system (semi-)resolve the tension between infinity and finiteness and at the same time uphold a legitimate exchange between syntax and semantics. If a system is sound, any inference obtained by its rules is valid, and if a system is complete, any valid inference is obtainable in terms of syntactic inference. In spite of a conceptual discrepancy, as far as the extension goes, syntactic and semantic inferences coincide. Here is a triumph of formal systems: A finite number of mechanical syntactic manipulations conquered the infinite cases of semantic territory.

Let us stop here to relate the above discussions about accuracy to our question—the relation between formalization and symbols. In order to achieve the accuracy desideratum all we need is to prove that each permissible syntactic inference is a semantically valid step. There is no constraint on a medium of a formal system, as long as we can establish syntax and semantics in a non-ambiguous way. Are symbols the only kind of medium for which we can set up syntax? Many might think so, since we are very much used to thinking about syntax only in terms of sentences. On the other hand, since we have been using many different forms of representation, e.g. sentences, pictures, and sounds, to carry out reasoning, obviously semantics is not limited to particular kinds of media. However, some might argue that while ordinary piecemeal reasoning can be carried out in multi-representational forms, only symbolic systems allow systematic formal semantics.

We claim syntax and semantics are not bound to a certain form of representation at all. Pictures, just like sentences, can have their own syntax, and as long as they represent something, semantics can be defined as well. Syntax should tell us what are the vocabulary, the well-formed units, and the transformation rules. It is often the case that we judge whether a given representation is symbolic or diagrammatic depending on what its vocabulary is. In the case of Euler representation, we see circles as its vocabulary, and we therefore call it diagrammatic. Likewise, almost every logical system in logic textbooks adopts symbols, e.g., A, B, x, y, z, ∀, ∃, as vocabulary, and such systems are therefore considered symbolic systems. Well-formedness of a unit is defined so that certain kinds of arrangement, either spatial or linear, are acceptable in a system. Transformation rules can be stipulated as to which manipulations between two well-formed units are allowed in a system. Semantics, again, does not have to be tied to one form or another to carry out the job. Hence, there is nothing intrinsic about symbols or diagrams in terms of the feasibility of setting up syntax and semantics. Now, our opening question—the exclusiveness of symbolic formalization—has become even more mysterious. The third section revisits this mystery and at the same time presents a relatively recent movement for visual formalization. But first, more historical visual reasoning considerations are in order.

2. Valid Reasoning and Diagrams

A look at historical literature shows that diagrams seem to have always been used by logicians. However, uses vary. Of course, visual devices have long and often been used in educational contexts as heuristic and mnemonic tools. For instance, logic students have for centuries been familiar with squares of opposition and logic trees. These and similar structures offer in a glance a survey of the relations between propositions (Fig. 1) or illustrate the working of a logic process such as dichotomy division (Fig. 2). As such, they do usually accompany logical arguments developed in words or with the appeal to symbolic notation. However, the widespread use of such devices should not make us forget that other schemes have also been designed to carry out logical reasoning independently. Such diagrams, known to John Venn as analytical diagrams, were particularly appreciated in the 18th and 19th centuries, a period which could fairly be considered as the golden age of logic diagrams. Interestingly, this period also witnessed a growing interest in symbolic notation.

Both diagrammatic and symbolic methods were widely used during that period to solve logical problems. Broadly speaking, a syllogistic problem was understood as checking the validity of an inference where premises and conclusions were given. In the mid-nineteenth century, symbolic logicians working under the influence of mathematical practices, tended rather to offer a set of premises and look for what conclusion(s) is/are to be drawn. In both situations, logicians invented tools and methods to solve those problems with the appeal of symbolic, diagrammatic or even sometimes mechanical devices. Leonhard Euler used diagrams alone, while George Boole made use of symbols merely. Charles S. Peirce devised

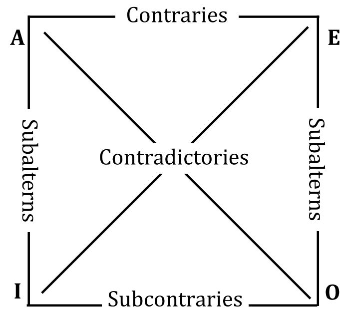

Fig. 1 Square of opposition

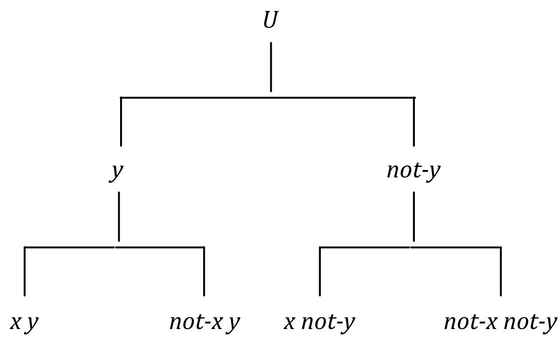

Fig. 2 Logic tree

both symbolic and diagrammatic methods. For the same purpose, Lewis Carroll designed a board and colored counters that were sold with his books. Some other logicians like William Stanley Jevons and Allan Marquand invented logic machines too. One appeals to either method depending on the problem being faced and what sounds convenient. Sometimes, the combination of several methods is also highly appreciated when dealing with complex problems. Finally, an appeal to more than one method, say both diagrammatic and symbolic, has proven useful in ascertaining individual results, each method being carried out independently in order to compare those results with each other.

This account should not be understood as the story of a happy and continuous development. Not only were different methods also in rivalry, but within diagrammatic reasoning itself various schemes were in competition. Of course, diagrams needed to be accurate in order to enter the contest. However, other criteria were in order as to the efficiency of the diagrams, the type of information they could represent, their naturalness and the visual aid they would provide. If we consider the method of representation, one can broadly distinguish two main types of diagrams. The first method, usually attributed to Euler though it was known prior to him, aims at representing given information in strictness. For instance, if we were asked to represent the proposition "All x are y", we need only to draw a circle x within a circle y (Fig. 3). This way, we do represent what is actually known: class x is included in class y. However, it might be observed that our diagram represents x as being strictly included in y, while our proposition holds also for the case where x and y are identical. In order to represent this potential information, we would do well to appeal to another method of representation attributed to Venn. The idea is to first represent all possible relations between terms, then to mark the cells to indicate their state. In the above proposition, two terms were involved (x and y), which means that there are four

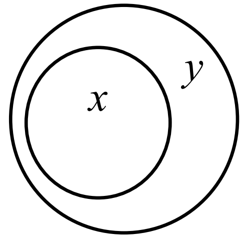

Fig. 3 Euler diagram

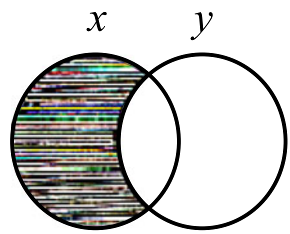

Fig. 4 Venn diagram

possible combinations: xy, x *not*-y, *not*-x y, *not*-x *not*-y, as shown in Fig. 2. These can be represented with two intersecting circles that divide the space in the desired manner. Then, the representation of proposition "All x are y" is obtained simply by shading the compartment x *not*-y to indicate its emptiness (Fig. 4).

Both diagrams do represent the same proposition "All x are y". Figure 3 looks more intuitive, but Fig. 4 is more accurate as it leaves room for the possibility of having classes x and y be identical. So far we have only represented information with these diagrams. Solving logic problems requires manipulating information in order to extract a conclusion from a set of premises. We will not give a detailed account here as such examples will be found in the chapters of this volume. However, we will explain the general idea for working such problems. First, we have to represent diagrammatically the propositions given as premises. Then, we remove the figures (here circles) that correspond to the terms that we do not want to have in the conclusion. In the case of a syllogism, that would be the middle term. Consequently, we get a diagram that represents new information as to the relation between the saved terms. Now, all we have to do is to express that relation in concrete form in order to tell what the conclusion of the argument is.

In the above example, we appealed to circles in order to represent classes. It must be noted that other shapes could have been used as well. Actually, linear diagrams have also been used by logicians and it can be easily shown that their method of representation might also be recognized as being Euler or Venn types. The same can be said about tabular diagrams that were praised in the late decades of the 19th century, thanks to their representation of a closed universe and to their advantageous use when the number of terms increases. Of course, when it comes to solving logic problems involving a high number of terms, say above six terms, diagrams become complex and lose substantially the visual aid one would expect from such devices. However, this is a practical inconvenience that should not dispute the theoretical feasibility of solving such problems diagrammatically. It must also be said here that the problem arises similarly with symbolic methods.

The variety and rivalry of diagrammatic methods that were used throughout the 19th century should not be considered as a symptom of their ambiguity. Actually, the very same situation can be observed with symbolic notations, many being aimed by their inventors to supersede rival symbolisms. Early notations used by Boole, and his followers Venn and Jevons, were equational. Later, several inclusional systems were developed by Peirce, Hugh MacColl and Ernst Schröder. Other logicians such as Carroll and Oscar H. Mitchell rather favored notations with subscripts. It is interesting to note that even for symbolic notation, visual properties certainly played a crucial role in their conception, acceptation or rejection. For instance, Peirce and Schröder's symbols for inclusion ("$-\!\!\!<$" and "$\mathrel{=\!\!\!\!<}$" respectively) look like a combination of mathematical symbols "$=$" and "$<$". As such, they suggest that a class is strictly included or is identical to another class the same way a number is strictly inferior or is equal to another number. In addition, those symbols were asymmetrical the same way modern notations for implication are. As such, they were more convenient to represent inclusion than MacColl's symbol (":") which was symmetrical. Christine Ladd-Franklin adopted Peirce's symbol for inclusion, but unlike him, she preferred to invent new symbols that would be symmetrical for intersection ("$\vee$") and exclusion ("$\overline{\vee}$").

So far we have argued that both diagrammatic and symbolic methods were known to logicians and were widely used, together or separately, to solve logic problems. We also pointed out similarities in their conception, development, use and status in the 19th century. Consequently, one legitimately wonders why modern logic turned out to be almost exclusively symbolic in the 20th century. In the next section, we will briefly provide some explanations connected to the logical path explored by Gottlob Frege, and subsequently investigated by Giuseppe Peano and Bertrand Russell. However we would like to end the present section by recalling that Frege himself appealed to a system of graphs in a way that does not differ that much from Peirce's use of existential graphs. These two graph systems show that diagrammatic representations can be used effectively for advanced logic systems, beyond the class logics that we discussed throughout this section.

3. Logical Systems and Diagrams

At the end of the first section, we concluded that a formal system itself does not require any specific kind of representation as long as we can set up its syntax and semantics. In principle, symbols and diagrams have the same status, it seems. Do we, then, actually have visual formal systems in a strict sense? The answer is "yes." Peirce, who was a founder of modern logic, invented both symbolic and graphic logical systems. John Sowa's semantic network is a prime example of how Peirce's Existential Graphs could be adopted as a formal knowledge representation system. Late in the 20th century, Barwise and Etchemendy initiated the project of diagrammatic formalization, and much work has been produced under their leadership. Shin presented a modified version of Venn diagrams as a formal system equipped with its own syntax and semantics. Just as with symbolic systems, the soundness and completeness of a diagrammatic system were proved. Subsequently more diagrams have been formalized and utilized in computer science. Hence, our theoretical discussions of formalization in the first section have materialized.

This is, however, far from being a satisfactory response to our opening question—*Why does symbolization almost exclusively dominate the enterprise of formalism in logic?* On the contrary, one could be more puzzled. We have shown a persistent practice of visual reasoning throughout history in the second section, have argued for a theoretical ground for non-symbolic formal systems in the first section, and have presented diagrammatic logical systems in the above. Then, how could we explain the dominance of symbolic systems over visual systems throughout the 20th century, since modern formal logic was born? Many have pointed out that there has been a prejudice against diagrams. However, the existence of a bias does not explain the dominance of symbolic systems but reiterates it. It would be almost like answering "It is just because I like them" to the question "Why do you like apples?"

One classic complaint against diagrams is that they are misleading. We can easily find examples of misleading diagrams in geometric proofs. For example, in the case of proving a property of a triangle in general the user happens to draw an isosceles triangle and mistakenly uses in the proof the property that the triangle has equal sides. Hence, many adopt diagrams only as a heuristic tool, but not as part of a formal proof. This line of thought, however, misses entirely the main spirit of formalization. As seen in the first section, a formal system is needed precisely to prevent ambiguity and misleading of our reasoning steps, by stipulating permissible inference rules. In a sound formal diagrammatic system there would be no room for misusing diagrams and, hence, diagrams would not be able to mislead us. Then, why have we seen much more symbolic formalization than diagrammatic formalization? Is there any obstacle to formalizing diagrams? We have paved the way to clearing up any theoretical obstacles to formalization of any medium: Formal syntax and semantics do not have to belong to symbols only. Then, is it a practical choice of symbolic formalization over diagrammatic formalization? Acknowledging that this is an extremely important inquiry, we would like to encourage researchers to work on specific diagrammatic systems so that we may find more fine-grained differences among various kinds of representation systems. Fortunately, much work has been under way, and the papers in the current volume are excellent illustrations of the work in this direction. We applaud each author's valuable and creative project presented here in the volume.

Witnessing the surge of interest in visual formalization, one cannot help raising the following question: *What is the source of the new movement for visual formalization?* We suspect that this question is the flip side of our opening puzzle about the long-standing preferred practice of symbolic systems. The dominance of symbols and the new revival of diagrams are directly related to the two main goals of a formal system mentioned in the first section—accuracy and efficiency. In the world of mathematics and logic, the turn of the 20th century brought many surprises, some of which were alarming and quite devastating. Discovering inconsistencies, contradictions and limits of mathematical systems, scholars made accuracy the top priority of formalization and did everything possible to avoid any error. In this context, diagrams with their reputation of leading inference astray could not attract any serious attention, and diagrams were not considered to be something we could formalize. That is, hyper attention to accuracy along with a slightly misunderstood concept of formalization produced a wrong equation between formalization and symbolization.

If we can explain why accuracy became almost the only desideratum of our formal disciplines when facing Russell's paradox, the inconsistency of set theory, different geometries, incompleteness theorems, etc., we can also explain why the other goal of a formal system, that is, efficiency, has recently attracted new attention in our efforts at formalization, in terms of what has been recently pursued in formal systems. Not surprisingly, the age of the computer can easily tell us why efficiency has been highlighted in the research on formalization. Given more than one logically equivalent system, it is natural for us to compare their efficiencies and to choose the most efficient system. Unlike with accuracy, it is not easy to define what efficiency is and how it might vary depending on context. We would like to leave this exciting task for further work and discussions.

4. Overview of the Volume

This volume contains 10 essays on visual reasoning with diagrams. These original essays come from three main sources. First, some essays were presented at the "Logic diagrams" workshop that took place in Lisbon (Portugal), during the third Universal Logic Congress, from 22 to 25 April 2010. Second, it happened that the event coincided with the famous Eyjafjallajökull eruptions in Iceland, which substantially disturbed transportation systems in Western Europe. This prevented other contributors from attending the workshop. Happily, some of those contributions are now included in this volume. Finally, further essays have been collected thanks to a call for papers that was spread in summer 2010. Papers from these three categories have been reviewed, selected and revised between 2011 and 2012, before being offered here to the reader.

The opening essay by Catherine Legg discusses the complex question of defining what a logic diagram is. This is a classical problem that all diagram users and scholars have faced, whether in the domain of logic or not. It might look intuitive in most cases to agree on whether a given "inscription" on a page is a diagram or not. However, consensus is not always reached. And even when agreement is obtained, it is not necessarily clear what makes us consider such a representation to be diagrammatic or not. One thing is for sure: the sign itself doesn't suffice to tell what its nature is. If you consider for instance the sign "=" on a page, it could legitimately be seen as a diagram with two parallel segments. However, if you add two letters on both sides to get: $x = x$, then the sign "=" is likely to be understood as a symbol for equality. In this essay, Legg develops an expressivist view of logic diagrams and draws on Peirce's concept on iconicity. Diagrams, as signs, are thus inspected in their relationship to the objects that they represent, and in the use we make of them, not just as "pictures on the page".

The second and third essays in this volume introduce new diagrammatic systems for syllogistic calculus. Traditionally, solving syllogisms has been a necessary (but not sufficient) test for new schemes that were offered for consideration. Indeed, syllogisms are often seen as the simplest pieces of formal reasoning. As such, it is expected from any diagrammatic system to be able to solve such problems, before making further claims. New diagrammatic systems regularly appear in logical literature. Two such systems are presented here and are of special interest. The first system, known as Sophie diagrams, is developed by Richard Bosley who argues that a demonstrative syllogism depends upon its figure. As such, his diagrams represent syllogistic figures rather than specific syllogisms proper. Consequently, handling diagrammatically the second and third figures of syllogisms prevents their conversion into the first figure, as is usually done. The second system, devised by Ruggero Pagnan, seems to combine symbolic and diagrammatic elements, and is said by its author to incorporate both "a graphical appearance and an algebraic nature". Pagnan uses here linear diagrams to support a systematic treatment of syllogistic calculus and extends them to handle n-term syllogisms and to syllogisms with completed terms.

The fourth and fifth essays explore similar paths as they discuss the use of logic diagrams for problems that goes beyond traditional syllogistic. In the fourth essay, Amirouche Moktefi discusses a diagrammatic solution to a 4-term problem provided by

Lewis Carroll. This example illustrates the difficulties that are raised from working problems more complex than syllogisms, and also throws some light on what was considered to be a logic problem for early symbolic logicians. Finally, the fifth essay by Ferdinando Cavaliere introduces a new diagrammatic scheme of his invention, known as the numerical segment. As it might be guessed from its name, the purpose is to take into account quantitative considerations as one finds in non-classical logics.

The next essays, sixth and seventh, provide two instances of a formal diagrammatic system, in the style that has been pursued for about two decades. Indeed, diagram studies witnessed a revival in recent years, as has been explained in the previous section. Several diagrammatic systems have been elaborated, with syntax, semantic and manipulation rules formally defined, and have been proved to be sound and complete. The sixth essay by Jørgen Fischer Nilsson provides an example of a diagrammatic system which carefully pays attention to computing needs, notably for computer assisted reasoning. Nilsson develops a diagrammatic visualization and reasoning language, considering various logical relationships between classes. For this purpose, he uses diagrams that are transformations of Euler diagrams, augmented with higraphs. The diagrammatic system presented in the next essay, the seventh, also makes use of extended diagrams, known as spider diagrams. This essay, co-authored by Gem Stapleton, John Howse, Simon Thompson, John Taylor and Peter Chapman, provides an illustration of the type of work carried out within the Visual Modelling Group (University of Brighton, UK). In this essay, the previously published spider diagrams are augmented with constants to mark individuals. The new system is then proved to be sound, complete and decidable.

The last three essays in this volume might be connected to what has been recently known as the philosophy of mathematical practice. This new trend aims at paying more attention to the real practices of mathematicians and logicians, rather than standing with the classical problems as to the foundations of mathematics and logic alone. In the eighth essay, Valeria Giardino argues that ambiguity is an inherent feature of diagrams. This ambiguity should not be tamed because it makes diagrammatic reasoning productive by opening the way to interpretation and imagination. As such, it is the manipulation practices shared by the community that fix the meaning of the diagrams on each occasion.

The next two essays continue to consider the role of diagrams in mathematics from the viewpoint of practices, providing concrete examples from different mathematical disciplines. In the ninth paper, Zach Weber relies on category theory to look for a mathematical answer to the philosophical question that he is investigating. There, he uses the idea of natural transformation to describe what is to be considered as a good mathematical representation, be it a formula or a figure. Finally, in the last paper of this volume, Mitsuko Wate-Mizuno discusses the development of the diagrams used in graph theory. She examines Dénes König's representations in *Theorie der endlichen und unendlichen Graphen* (1936) and compares them with his predecessors to provide an account of how those diagrams evolved and took shape.

We hope that the essays in this volume will convince the reader of the richness and energy of current research in diagram studies. Contributions to this volume are made by scholars from various disciplines: philosophy, mathematics, logic, history, etc. As such, they demonstrate the need and interest in fostering interdisciplinary work. This could be carried out only by further exchanges and collaborations between scholars from different disciplines and countries, as is now the case in several international meetings such as the *Diagrams* and the *Universal Logic* congresses. We offer this volume as one further step in that direction.

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Contents

What is a Logical Diagram?

Catherine Legg

Abstract Robert Brandom's expressivism argues that not all semantic content may be made fully explicit. This view connects in interesting ways with recent movements in philosophy of mathematics and logic (e.g. Brown, Shin, Giaquinto) to take diagrams seriously—as more than a mere 'heuristic aid' to proof, but either proofs themselves, or irreducible components of such. However what exactly is a diagram in logic? Does this constitute a cleanly definable semiotic kind? The paper will argue that such a kind does exist in Charles Peirce's conception of iconic signs, but that fully understood, logical diagrams involve a structured array of normative reasoning practices, as well as just a 'picture on a page'.

Keywords Logic · Mathematics · Diagram · Proof · Icon · Existential graphs · Expressivism · Pragmatism · Peirce · Brandom · Ayer

Mathematics Subject Classification Primary 00A66; Secondary 03A05

1 Introduction: 19th Century "Picture Shock"

20th century mainstream analytic philosophy was almost entirely neglectful of diagrams in its theorizing about *semantic content*, and *proof*. It is worth understanding the historical background to this arguably contingent state of philosophical affairs.

The trend began in mathematics. In the 19th century this field was revolutionized by an arithmetization movement, and some of the key developments foregrounded ways in which our "visual expectations in mathematics"[1] might deliver the wrong answer about mathematical fact. A famous example is the claim that a function which is everywhere continuous must be differentiable, which is in fact false. Attempting to evaluate this using visual imagination, one may imagine that if a function is continuous then it contains no 'gaps' or 'breaks', and then one seems to 'see' that at some sufficiently fine-grained level it must present a smooth surface, which would have a gradient, and thus a derivative. However, to the surprise of many, Weierstrass and Bolzano proved that certain functions are infinitely finely jagged, yet still gap-free in a way that fits the formal definition of continuity.[2] Another example is whether a 1-dimensional line might fill a 2-dimensional

[1]This phrase is taken from Marcus Giaquinto [17, p. 3].

[2]This example is nicely discussed in [17, pp. 3–4], and [22, pp. 3–4].

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region. Any attempt to mentally picture something resembling an infinitely thin thread unspooling into a finite area and thereby 'filling it in' seems to show that the claim is false, but Peano proved it true.[3] Such examples prompted some of the most influential mathematicians of the 19th century to draw strong morals about the potential for error in diagrammatic reasoning. As Marcus Giaquinto writes:

> Such cases seemed to show not merely that we are prone to make mistakes when thinking visually…but also that visual understanding actually conflicts with the truths of analysis [17, pp. 4–5].

Hilbert famously wrote, "a theorem is only proved when the proof is completely independent of the diagram" [17, p. 8], drawing on an almost identical remark by Moritz Pasch in his influential *Lectures in Modern Geometry* (1882). So, remarkably, even the field of *geometry*, it came to be seen, needed to be purged of diagrams.[4] The end result was a "prevailing conception of mathematical proof" which John Mumma describes as "purely sentential", as follows:

> A proof…is a sequence of sentences. Each sentence is either an assumption of the proof, or is derived via sound inference rules from sentences preceding it. The sentence appearing at the end of the sequence is what has been proven [22, p. 1].

This suspicion of 'visual expectations' then flowed into Frege's work on the foundations of mathematics. Cognizant of the errors which his fellow mathematicians had learned to skirt, Frege attempted to entirely remove 'intuition' from the logic with which he set to put mathematics on an entirely new and more rigorous foundation. Famously, he remarked of his own concept-script:

> So that nothing intuitive could intrude here unnoticed, everything had to depend on the chain of inference being free of gaps [16, p. 48].

Frege argued against the empiricism of John Stuart Mill that numbers were not properties abstracted from the physical world, but definable purely analytically.

Frege in turn was an enormous influence on *logical positivism* (Carnap studied under him, for instance), which in turn set the scene for mainstream analytic philosophy's aims and methodologies in many ways that are still being worked out today. The movement's early strict focus on clarifying *meaning* owed much to Frege's vision of an ideal language all of whose inferential steps are explicitly stated, and use a set of rules specified in advance.[5] Thus A.J. Ayer laid down a strict definition of "literal significance" as confined to claims which have "factual content" by virtue of offering "empirical hypotheses" [1, p. 2]. Thus, to illustrate by way of a simple example, "The cat is on the mat" is literally significant because there is a cat-being-on-the-mat type of *experience* which might be had—or not—in the relevant situations.

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[3] Discussed in [17, pp. 4–5].

[4] "A body of work emerged in the late 19th century which grounded elementary geometry in abstract axiomatic theories…This development is now universally regarded as a methodological breakthrough. Geometric relations which previously were logically free-floating, because they were understood via diagrams, were given a firm footing with precisely defined primitives and axioms" [22, p. 6]. Non-Euclidean geometries are another key example, and I am grateful to an anonymous referee for pointing this out.

[5] Although transposed into a rigidly empiricist setting which truth be told sits oddly with Frege's thinking—and arguably has caused significant problems in the philosophy of mathematics.

Claims which lack "literal significance" fall into two camps. Either they can be "literally false" but somehow "the creation of a work of Art" which is gestured towards as valuable, though Ayer is somewhat vague about how. Or, worse, claims might be "pseudo-propositions"—disguised nonsense. Any claim lacking literal significance is not the purview of philosophy [1, p. 2]. It is hard to see how a diagram could offer an empirical hypothesis, and thus have literal significance in Ayer's sense. And he briskly dismisses the idea that a philosopher might be "endowed with a faculty of intellectual intuition which enabled him to know facts that could not be known through sense-experience" [1, p. 1]. Likewise, the early Carnap [10] claimed that statements were meaningful if syntactically well-formed and their non-logical terms reducible to observational terms in the natural sciences.

It is well-known that crisp criteria for what constitutes a genuine empirical hypothesis were much more difficult to find than Ayer imagined they would be. Carnap dropped back from demanding verifiability to requiring "partial testability" [11], and confirmation became a more and more holistic affair, until finally Quine acknowledged that what meets the tribunal of experience is in an important sense the whole of science. By way of consolation for thus sounding verificationism's death-knell, Quine offered a new criterion of what might be called 'factuality': if we could imagine our science collated and regularized into a single theory expressed in first-order logic, its bound variables would have values. In a pseudo-science such as witchcraft they would not [33]. Now we can say that "The cat is on the mat" is factual because in the logical formula $\exists \mathtt{x}\ (\mathtt{Cx}\ \&\ \mathtt{Oxm})$ suitably interpreted, the variable $\mathtt{x}$ binds to George.[6]

Philosophers' banishment of diagrams from semantics and theories of inference arguably reached a high-water mark in the 1970s with the publication of Quine's colleague Nelson Goodman's *Languages of Art*. Here Goodman made an influential argument that resemblance plays no interesting or important role in signification. Rather, he claimed that denotation, "is the core of representation and is independent of resemblance" [18, p. 5]. His reasoning was that while the resemblance relation is symmetric (if $\mathtt{X}$ resembles $\mathtt{Y}$ then $\mathtt{Y}$ resembles $\mathtt{X}$), the representation relation is not.[7]

However, a profound challenge to this more than century-long neglect of diagrams is 'in the air'. It seeks to reconceive diagrams as more than a mere 'heuristic aid' to proof in mathematics and logic. Rather diagrams may be understood as capable of serving either as proofs themselves, or irreducible components of such. Thus James R. Brown writes:

> …the prevailing attitude is that pictures are really no more than heuristic devices…I want to oppose this view and to make a case for pictures having a legitimate role to play as evidence and justification—a role well beyond the heuristic. In short, pictures can prove theorems [9, p. 96].

John Mumma writes:

> In the past 15 years, a sizable literature consciously opposed to [the attitude that pictures do not prove anything in mathematics] has emerged. The work ranges from technical presentations of formal diagrammatic systems of proof…to philosophical arguments for the mathematical legitimacy of pictures… [22, p. 8].

[6] (A cat.)

[7] Randall Dipert has argued against this that it no more follows that resemblance is 'entirely independent of' representation because the former relation is symmetric and the latter is not, than that the brother relation is 'entirely independent of' the uncle relation as the former is symmetric and the latter is not [15].

Meanwhile Marcus Giaquinto writes:

> …a time-honoured view, still prevalent, is that the utility of visual thinking in mathematics is only psychological, not epistemological…The chief aim of this work is to put that view to the test [17, p. 1].

Other authors have returned to ancient Greek mathematical texts to argue that one cannot understand them fully without taking their diagrams more seriously [13].[8]

Meanwhile, in logic, Sun-Joo Shin argues that although, "[f]or more than a century, symbolic representation systems have been the exclusive subject for formal logic" [36, p. 1], this should be widened to also consider "heterogeneous systems", which "employ both symbolic and diagrammatic elements" [36, p. 1]. "Heterogeneous systems" is an influential term which derives from Jon Barwise [2]. Shin argues that symbolic and heterogeneous reasoning systems have different strengths and weaknesses, and we should do a thorough study to get the best out of both, bearing in mind that different disciplines which might draw on such systems (such as logic, artificial intelligence and philosophy of mind) might have different needs.

This paper seeks to join these authors while at the same time to put this goal in a broader context, namely a movement which is also aimed at unbuilding the simple picture of "literal significance" that has been so influential in the 20th century—*expressivism*.

2 Expressivism: Saying, Doing and Picturing

Expressivism has a metaethical incarnation, as a view that, "…claims some interesting disanalogy between…evaluations and descriptions of the world" [12, p. 1]. By contrast, Robert Brandom has put forward a semantic expressivism whose main point is that not all semantic content may be made fully *explicit*. This view contrasts with a widespread view often thought to be intuitively obvious, and arguably a downstream specter of Ayer's notion of literal significance. I will call it a *metaphysical realist semantics*. The juxtaposition here is deliberately somewhat controversial, given that many metaphysical realists take great pains to make a clear separation between metaphysical and semantic questions, and to claim that their view lies firmly on the metaphysical side. An argument will be put forward later in the paper that this self-assessment is problematic.

A metaphysical realist semantics holds that the purpose of language is to state "facts" which, if the propositions stating them are true, form part of language-independent reality. Thus, to return to our earlier example, "The cat is on the mat" (suitably disambiguated as to cats and mats) is thought to present a 'content' which it is sufficient to know the meaning of the statement's words to fully understand. Brandom calls the view *representationalism*. By contrast, he argues that the primary purpose of language is to transform what we *do* into something that we can *say*:

> By expressivism I mean the idea that discursive practice makes us special in enabling us to make *explicit*, in the form of something we can *say* or *think*, what otherwise remains *implicit* in what we *do* [30, p. 7].

[8] See also, from a more philological perspective, the work of Reviel Netz, e.g. [23].

Crucially, this renders the explicit statement semantically parasitic on the implicit practice, in that one cannot fully understand the statement without antecedently understanding the practice which it "expresses". Thus Brandom writes:

> ...we need not yield to the temptation...to think of what is expressed and the expression of it as individually intelligible independently of consideration of the relations between them...And the explicit may not be specifiable apart from consideration of what is made explicit [8, pp. 8–9].

Consider for example, the invention of *musical notation*. This freed musicians from having to learn music by directly copying a live musician's *actions*. Instead a musical score substitutes dots on a page for string-pluckings, key tappings, and all other actions which might produce a note. In this way a musical score can *say what musicians do* (with added bonuses such as that the score can be indefinitely copied, survive longer than any living musician, and be readily compared and contrasted with other scores). However, it is not possible to fully understand a musical score without having some antecedent understanding of the practices of music which it is expressing. For instance, if aliens were to stumble upon the score for Beethoven's 5th symphony, it is highly unlikely they could perform it without some observation of human musical performance.

This commitment to a parasitism of the explicit statement on the implicit practice renders expressivism a form of *pragmatism*. It claims that certain practices are not fully explicated in language, but presupposed by it [7]. Pragmatism is frequently seen as a form of antirealism, merely internal realism,[9] non-cognitivism,[10] non-factualism,[11] or as some would put it "quasi-realism".[12] But the conclusion of this paper will consider other views on this.

Such an expressivism may make sense for musical notation, but might it be generalized? For Brandom wishes it to be a global view, concerning all language. In particular, might expressivism be applied to talk about *logic*? Surely the matters of truth-preservingness and validity are a paradigm of practice-independent fact? Not so, according to Brandom. He claims that logic also should be seen as a way of *saying* what we are *doing* when we actually make inferences, in ways that can guide our reasoning in systematic and useful ways. In fact he self-consciously highlights the practice of philosophy itself as a particularly sophisticated pulling of unselfconscious implicit *practices* into explicit *statements* that might be critically appraised [8, pp. 56–57].

Brandom's expressivism may be linked in interesting ways with Wittgenstein's Picture Theory of Meaning.[13] In the *Tractatus* Wittgenstein drew a famous distinction between what is *said* (namely atomic facts, and truth-functional combinations of them) and what is *shown* (the laws of logic, the limits of the world and, interestingly in the expressivist context, ethics). In the spirit of Brandom we might describe the former as "explicit" and the latter as "implicit". However, having drawn this distinction between saying and showing, Wittgenstein made the further claim that what is shown *cannot be said*.[14] Early Wittgenstein and Brandom stand out amongst mainstream semantics in their bold claim that not

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[9] [5, 32].

[10] Price suggests Rorty approaches a global non-cognitivism in [29].

[11] This term derives from [6].

[12] The term was coined by Simon Blackburn, see in particular [4]. Its links with pragmatism are explored in [28], though [21] argues that the two views share important similarities *and* differences.

[13] I have argued this previously elsewhere: [19, 20].

[14] Thus for instance *Tractatus* 6.42 states, "...there can be no ethical propositions..." [37].