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Stability of Vector Differential Delay Equations

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Preface

1. The core of this book is an investigation of linear and nonlinear vector differential delay equations, extended to coverage of the topic of causal mappings. Explicit conditions for exponential, absolute and input-to-state stabilities are suggested. Moreover, solution estimates for these classes of equations are established. They provide the bounds for regions of attraction of steady states. We are also interested in the existence of periodic solutions. In addition, the Hill method for ordinary differential equations with periodic coefficients is developed for these equations.

The main methodology presented in the book is based on a combined usage of recent norm estimates for matrix-valued functions with the following methods and results:

a) the generalized Bohl–Perron principle and the integral version of the generalized Bohl–Perron principle;
b) the freezing method;
c) the positivity of fundamental solutions.

A significant part of the book is devoted to a solution of the Aizerman–Myshkis problem and integrally small perturbations of linear equations.

2. Functional differential equations naturally arise in various applications, such as control systems, viscoelasticity, mechanics, nuclear reactors, distributed networks, heat flow, neural networks, combustion, interaction of species, microbiology, learning models, epidemiology, physiology, and many others. The theory of functional differential equations has been developed in the works of V. Volterra, A.D. Myshkis, N.N. Krasovskii, B. Razumikhin, N. Minorsky, R. Bellman, A. Halanay, J. Hale and other mathematicians.

The problem of stability analysis of various equations continues to attract the attention of many specialists despite its long history. It is still one of the most burning problems because of the absence of its complete solution. For many years the basic method for stability analysis has been the use of Lyapunov functionals, from which many strong results have been obtained. We do not discuss this method here because it has been well covered in several excellent books. It should be noted that finding Lyapunov type functionals for vector equations is often connected with serious mathematical difficulties, especially in regard to nonautonomous equations. To the contrary, the stability conditions presented in this book are mainly formulated in terms of the determinants and eigenvalues of auxiliary matrices dependent on a parameter. This fact allows us to apply well-known results of the theory of matrices to stability analysis.

One of the methods considered in the book is the freezing method. That method was introduced by V.M. Alekseev in 1960 forstability analysis of ordinary differential equations and extended to functional differential equations by the author.

We also consider some classes of equations with causal mappings. These equations include differential, differential-delay, integro-differential and other traditional equations. The stability theory of nonlinear equations with causal mappings is in an early stage of development.

Furthermore, in 1949 M.A. Aizerman conjectured that a single input-single output system is absolutely stable in the Hurwitz angle. That hypothesis created great interest among specialists. Counter-examples were set up that demonstrated it was not, in general, true. Therefore, the following problem arose: to find the class of systems that satisfy Aizerman's hypothesis. The author has shown that any system satisfies the Aizerman hypothesis if its impulse function is non-negative. A similar result was proved for multivariable systems.

On the other hand, in 1977 A.D. Myshkis pointed out the importance of consideration of the generalized Aizerman problem for retarded systems. In 2000 it was proved by the author, that a retarded system satisfies the generalized Aizerman hypothesis if its Green function is non-negative.

3. The aim of the book is to provide new tools for specialists in the stability theory of functional differential equations, control system theory and mechanics.

This is the first book that:

i) gives a systematic exposition of an approach to stability analysis of vector differential delay equations based on estimates for matrix-valued functions allowing us to investigate various classes of equations from a unified viewpoint;
ii) contains a solution of the Aizerman–Myshkis problem;
iii) develops the Hill method for functional differential equations with periodic coefficients;
iv) presents an integral version of the generalized Bohl–Perron principle.

It also includes the freezing method for systems with delay and investigates integrally small perturbations of differential delay equations with matrix coefficients.

The book is intended not only for specialists in stability theory, but for anyone interested in various applications who has had at least a first year graduate level course in analysis.

I was very fortunate to have fruitful discussions with the late Professors M.A. Aizerman, M.A. Krasnosel'skii, A.D. Myshkis, A. Pokrovskii, and A.A. Voronov, to whom I am very grateful for their interest in my investigations.

Chapter 1

Preliminaries

1.1 Banach and Hilbert spaces

In Sections 1–3 we recall very briefly some basic notions of the theory of Banach and Hilbert spaces. More details can be found in any textbook on Banach and Hilbert spaces (e.g., [2] and [16]).

Denote the set of complex numbers by $\mathbb{C}$ and the set of real numbers by $\mathbb{R}$.

A linear space X over $\mathbb{C}$ is called *a (complex) linear normed space* if for any $x \in X$ a non-negative number $\|x\|_X = \|x\|$ is defined, called the norm of x, having the following properties:

1. $\|x\| = 0$ iff $x = 0$,
2. $\|\alpha x\| = |\alpha|\|x\|$,
3. $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X, \ \alpha \in \mathbb{C}$.

A sequence $\{h_n\}_{n=1}^{\infty}$ of elements of X converges *strongly* (in the norm) to $h \in X$ if

$$\lim_{n\to\infty} \|h_n - h\| = 0.$$

A sequence $\{h_n\}$ of elements of X is called the fundamental (Cauchy) one if

$$\|h_n - h_m\| \to 0 \text{ as } m, n \to \infty.$$

If any fundamental sequence converges to an element of X, then X is called *a (complex) Banach space.*

In a linear space H over $\mathbb{C}$ for all $x, y \in H$, let a number (x, y) be defined, such that

1. $(x, x) > 0$, if $x \neq 0$, and $(x, x) = 0$, if $x = 0$,
2. $(x, y) = \overline{(y, x)}$,
3. $(x_1 + x_2, y) = (x_1, y) + (x_2, y) \ \ (x_1, x_2 \in H)$,
4. $(\lambda x, y) = \lambda(x, y) \ \ (\lambda \in \mathbb{C})$.

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Then $(.,.)$ is called the scalar product. Define in H the norm by

$$\|x\| = \sqrt{(x,x)}.$$

If H is a Banach space with respect to this norm, then it is called *a Hilbert space*. The Schwarz inequality

$$|(x,y)| \leq \|x\| \, \|y\|$$

is valid.

If, in an infinite-dimensional Hilbert space, there is a countable set whose closure coincides with the space, then that space is said to be *separable*. Any separable Hilbert space H possesses an orthonormal basis. This means that there is a sequence $\{e_k \in H\}_{k=1}^{\infty}$ such that

$$(e_k, e_j) = 0 \text{ if } j \neq k \quad \text{and} \quad (e_k, e_k) = 1 \ \ (j, k = 1, 2, \dots)$$

and any $h \in H$ can be represented as

$$h = \sum_{k=1}^{\infty} c_k e_k$$

with

$$c_k = (h, e_k), \ k = 1, 2, \dots .$$

Besides the series strongly converges.

Let X and Y be Banach spaces. A function $f : X \to Y$ is continuous if for any $\epsilon > 0$, there is a $\delta > 0$, such that $\|x - y\|_X \leq \delta$ implies $\|f(x) - f(y)\|_Y \leq \epsilon$.

Theorem 1.1.1 (The Urysohn theorem)**.** *Let A and B be disjoint closed sets in a Banach space X. Then there is a continuous function f defined on X such that*

$$0 \leq f(x) \leq 1, \ f(A) = 1 \quad \textit{and} \quad f(B) = 0.$$

For the proof see, for instance, [16, p. 15].

Let a function $x(t)$ be defined on a real segment $[0, T]$ with values in X. An element $x'(t_0)$ $(t_0 \in (0, T))$ is the derivative of $x(t)$ at t_0 if

$$\left\| \frac{x(t_0 + h) - x(t_0)}{h} - x'(t_0) \right\| \to 0 \text{ as } |h| \to 0.$$

Let $x(t)$ be continuous at each point of $[0, T]$. Then one can define the Riemann integral as the limit in the norm of the integral sums:

$$\lim_{\max \ |\Delta t_k^{(n)}| \to 0} \sum_{k=1}^{n} x(t_k^{(n)}) \Delta t_k^{(n)} = \int_0^T x(t) dt,$$

$$\left(0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T, \Delta t_k^{(n)} = t_k^{(n)} - t_{k-1}^{(n)}\right).$$

1.2 Examples of normed spaces

The following spaces are examples of normed spaces. For more details see [16, p. 238].

1. The complex n-dimensional Euclidean space $\mathbb{C}^n$ with the norm

$$\|x\|_n = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} \qquad (x = \{x_k\}_{k=1}^{n} \in \mathbb{C}^n)\,.$$

2. The space $B(S)$ is defined for an arbitrary set S and consists of all bounded scalar functions on S. The norm is given by

$$\|f\| = \sup_{s\in S} |f(s)|.$$

3. The space $C(S)$ is defined for a topological space S and consists of all bounded continuous scalar functions on S. The norm is

$$\|f\| = \sup_{s\in S} |f(s)|.$$

4. The space $L^p(S)$ is defined for any real number p, $1 \le p < \infty$, and any set S having a finite Lebesgue measure. It consists of those measurable scalar functions on S for which the norm

$$\|f\| = \left[\int_S |f(s)|^p ds\right]^{1/p}$$

is finite.

5. The space $L^\infty(S)$ is defined for any set S having a finite Lebesgue measure. It consists of all essentially bounded measurable scalar functions on S. The norm is

$$\|f\| = \operatorname{ess\,sup}_{s\in S} |f(s)|.$$

Note that the Hilbert space has been defined by a set of abstract axioms. It is noteworthy that some of the concrete spaces defined above satisfy these axioms, and hence are special cases of abstract Hilbert space. Thus, for instance, the n-dimensional space $\mathbb{C}^n$ is a Hilbert space, if the inner product (x, y) of two elements

$$x = \{x_1, \ldots, x_n\} \quad \text{and} \quad y = \{y_1, \ldots, y_n\}$$

is defined by the formula

$$(x, y) = \sum_{k=1}^{n} x_k \overline{y}_k.$$

In the same way, complex l^2 space is a Hilbert space if the scalar product (x, y) of the vectors $x = \{x_k\}$ and $y = \{y_k\}$ is defined by the formula

$$(x, y) = \sum_{k=1}^{\infty} x_k \overline{y}_k.$$

Also the complex space $L^2(S)$ is a Hilbert space with the scalar product

$$(f, g) = \int_S f(s)\overline{g}(s)ds.$$

1.3 Linear operators

An operator A, acting from a Banach space X into a Banach space Y, is called a linear one if

$$A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2$$

for any $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{C}$. If there is a constant a, such that the inequality

$$\|Ah\|_Y \leq a\|h\|_X \text{ for all } h \in X$$

holds, then the operator is said to be bounded. The quantity

$$\|A\|_{X\to Y} := \sup_{h\in X} \frac{\|Ah\|_Y}{\|h\|_X}$$

is called the norm of A. If $X = Y$ we will write $\|A\|_{X\to X} = \|A\|_X$ or simply $\|A\|$.

Under the natural definitions of addition and multiplication by a scalar, and the norm, the set $B(X, Y)$ of all bounded linear operators acting from X into Y becomes a Banach space. If $Y = X$ we will write $B(X, X) = B(X)$. A sequence $\{A_n\}$ of bounded linear operators from $B(X, Y)$ converges *in the uniform operator topology* (in the operator norm) to an operator A if

$$\lim_{n\to\infty} \|A_n - A\|_{X\to Y} = 0.$$

A sequence $\{A_n\}$ of bounded linear operators *converges strongly* to an operator A if the sequence of elements $\{A_n h\}$ strongly converges to Ah for every $h \in X$.

If ϕ is a linear operator, acting from X into $\mathbb{C}$, then it is called a linear functional. It is bounded (continuous) if $\phi(x)$ is defined for any $x \in X$, and there is a constant a such that the inequality

$$\|\phi(h)\|_Y \leq a\|h\|_X \text{ for all } h \in X$$

holds. The quantity

$$\|\phi\|_X := \sup_{h\in X} \frac{|\phi(h)|}{\|h\|_X}$$

is called *the norm of the functional* ϕ. All linear bounded functionals on X form a Banach space with that norm. This space is called the space *dual* to X and is denoted by X^*.

In the sequel $I_X = I$ is the identity operator in $X : Ih = h$ for any $h \in X$.

The operator A^{-1} is the inverse one to $A \in B(X,Y)$ if $AA^{-1} = I_Y$ and $A^{-1}A = I_X$.

Let $A \in B(X,Y)$. Consider a linear bounded functional f defined on Y. Then on X the linear bounded functional $g(x) = f(Ax)$ is defined. The operator realizing the relation $f \to g$ is called the operator A^* *dual* (*adjoint*) to A. By the definition

$$(A^*f)(x) = f(Ax) \ (x \in X).$$

The operator A^* is a bounded linear operator acting from Y^* to X^*.

Theorem 1.3.1. *Let $\{A_k\}$ be a sequence of linear operators acting from a Banach space X to a Banach space Y. Let for each $h \in X$,*

$$\sup_k \|A_k h\|_Y < \infty.$$

Then the operator norms of $\{A_k\}$ are uniformly bounded. Moreover, if $\{A_n\}$ strongly converges to a (linear) operator A, then

$$\|A\|_{X\to Y} \le \sup_n \|A_n\|_{X\to Y}.$$

For the proof see, for example, [16, p. 66].

A point λ of the complex plane is said to be a regular point of an operator A, if the operator $R_\lambda(A) := (A - I\lambda)^{-1}$ (the resolvent) exists and is bounded. The complement of all regular points of A in the complex plane is the *spectrum* of A. The spectrum of A is denoted by $\sigma(A)$.

The quantity

$$r_s(A) = \sup_{s\in\sigma(A)} |s|$$

is the *spectral radius* of A. The Gel'fand formula

$$r_s(A) = \lim_{k\to\infty} \sqrt[k]{\|A^k\|}$$

is valid. The limit always exists. Moreover,

$$r_s(A) \le \sqrt[k]{\|A^k\|}$$

for any integer $k \ge 1$. So

$$r_s(A) \le \|A\|.$$

If there is a nontrivial solution e of the equation $Ae = \lambda(A)e$, where $\lambda(A)$ is a number, then this number is called an eigenvalue of operator A, and $e \in H$ is an eigenvector corresponding to $\lambda(A)$. Any eigenvalue is a point of the spectrum. An eigenvalue $\lambda(A)$ has the (algebraic) multiplicity $r \le \infty$ if

$$\dim(\cup_{k=1}^\infty \ker(A - \lambda(A)I)^k) = r.$$

In the sequel $\lambda_k(A)$, $k = 1, 2, \dots$ are the eigenvalues of A repeated according to their multiplicities.

A vector v satisfying $(A - \lambda(A)I)^n v = 0$ for a natural n, is a root vector of operator A corresponding to $\lambda(A)$.

An operator V is called a quasinilpotent one, if its spectrum consists of zero, only.

On a linear manifold $D(A)$ of a Banach space X,let there be defined a linear operator A, mapping $D(A)$ into a Banach space Y. Then $D(A)$ is called the domain of A. A linear operator A is called a closed operator, if from $x_n \in X \to x_0$ and $Ax_n \to y_0$ in the norm, it follows that $x_0 \in D(A)$ and $Ax_0 = y_0$.

Theorem 1.3.2 (The Closed Graph theorem). *A closed linear map defined on the all of a Banach space, and with values in a Banach space, is continuous.*

For the proof see [16, p. 57].

Theorem 1.3.3 (The Riesz–Thorin theorem). *Assume T is a bounded linear operator from $L^p(\Omega_1)$ to $L^p(\Omega_2)$ and at the same time from $L^q(\Omega_1)$ to $L^q(\Omega_2)$ $(1 \le p, q \le \infty)$. Then it is also a bounded operator from $L^r(\Omega_1)$ to $L^r(\Omega_2)$ for any r between p and q. In addition the following inequality for the norms holds:*

$$\|T\|_{L^r(\Omega_1) \to L^r(\Omega_2)} \le \max\{\|T\|_{L^p(\Omega_1) \to L^p(\Omega_2)}, \|T\|_{L^q(\Omega_1) \to L^q(\Omega_2)}\}.$$

For the proof (in a more general situation) see [16, Section VI.10.11].

Theorem 1.3.4. *Let $f \in L^1(\Omega)$ be a fixed integrable function and let T be the operator of convolution with f, i.e., for each function $g \in L^p(\Omega)$ $(p \ge 1)$ we have*

$$(Tg)(t) = \int_\Omega f(t - s) g(s) ds.$$

Then

$$\|Tg\|_{L^p(\Omega)} \le \|f\|_{L^1(\Omega)} \|g\|_{L^p(\Omega)}.$$

For the proof see [16, p. 528].

Now let us consider operators in a Hilbert space H. A bounded linear operator A^* is adjoint to A, if

$$(Af, g) = (f, A^* g) \text{ for every } h, g \in H.$$

The relation $\|A\| = \|A^*\|$ is true. A bounded operator A *is a selfadjoint* one, if $A = A^*$. A is *a unitary operator*, if $AA^* = A^*A = I$. Here and below $I \equiv I_H$ is the identity operator in H. A selfadjoint operator A is positive (negative) definite, if

$$(Ah, h) \ge 0 \;\; ((Ah, h) \le 0) \text{ for every } h \in H.$$

A selfadjoint operator A is strongly positive (strongly negative) definite, if there is a constant $c > 0$, such that

$$(Ah, h) \ge c\,(h, h) \;\; ((Ah, h) < -c\,(h, h)) \text{ for every } h \in H.$$

A bounded linear operator satisfying the relation $AA^* = A^*A$ is called *a normal operator*. It is clear that unitary and selfadjoint operators are examples of normal ones. The operator $B \equiv A^{-1}$ is the inverse one to A, if $AB = BA = I$. An operator P is called *a projection* if $P^2 = P$. If, in addition, $P^* = P$, then it is called *an orthogonal projection* (*an orthoprojection*). The spectrum of a selfadjoint operator is real, the spectrum of a unitary operator lies on the unit circle.

1.4 Ordered spaces and Banach lattices

Following [91], let us introduce an inequality relation for normed spaces which can be used analogously to the inequality relation for real numbers.

A non-empty set M with a relation $\leq$ is said to be an ordered set, whenever the following conditions are satisfied.

i) $x \leq x$ for every $x \in M$,
ii) $x \leq y$ and $y \leq x$ implies that $x = y$ and
iii) $x \leq y$ and $y \leq z$ implies that $x \leq z$.

If, in addition, for any two elements $x, y \in M$ either $x \leq y$ or $y \leq x$, then M is called a totally ordered set. Let A be a subset of an ordered set M. Then $x \in M$ is called an upper bound of A, if $y \leq x$ for every $y \in A$. $z \in M$ is called a lower bound of A, if $y \geq z$ for all $y \in A$. Moreover, if there is an upper bound of A, then A is said to be bounded from above. If there is a lower bound of A, then A is called bounded from below. If A is bounded from above and from below, then we will briefly say that A is order bounded. Let

$$[x, y] = \{z \in M : \ x \leq z \leq y\}.$$

That is, $[x, y]$ is an order interval.

An ordered set $(M, \leq)$ is called *a lattice*, if any two elements $x, y \in M$ have a least upper bound denoted by $\sup(x, y)$ and a greatest lower bound denoted by $\inf(x, y)$. Obviously, a subset A is order bounded, if and only if it is contained in some order interval.

Definition 1.4.1. A real vector space E which is also an ordered set is called an ordered vector space, if the order and the vector space structure are compatible in the following sense: if $x, y \in E$, such that $x \leq y$, then $x + z \leq y + z$ for all $z \in E$ and $ax \leq ay$ for any positive number a. If, in addition, $(E, \leq)$ is a lattice, then E is called a Riesz space (or a vector lattice).

Let E be a Riesz space. The *positive cone* E_+ of E consists of all $x \in E$, such that $x \geq 0$. For every $x \in E$ let

$$x^+ = \sup\,(x, 0), \ \ x^- = \inf\,(-x, 0), \ \ |x| = \sup\,(x, -x)$$

be the *positive part, the negative part and the absolute value of* x, respectively.

Example 1.4.2. Let $E = \mathbb{R}^n$ and

$$R_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_k \geq 0 \text{ for all } k\}.$$

Then R_+^n is a positive cone and for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we have

$$x \leq y \text{ iff } x_k \leq y_k \quad \text{and} \quad |x| = (|x_1|, \ldots, |x_n|).$$

Example 1.4.3. Let X be a non-empty set and let $B(X)$ be the collection of all bounded real-valued functions defined on X.

It is a simple and well-known fact that $B(X)$ is a vector space ordered by the positive cone

$$B(X)_+ = \{f \in B(X) : f(t) \geq 0 \text{ for all } t \in X\}.$$

Thus $f \geq g$ holds, if and only if $f - g \in B(X)_+$. Obviously, the function $h_1 = \sup\,(f, g)$ is defined by

$$h_1(t) = \max\,\{f(t), g(t)\}$$

and the function $h_2 = \inf\,(f, g)$ is defined by

$$h_2(t) = \min\,\{f(t), g(t)\}$$

for every $t \in X$ and $f, g \in B(X)$. This shows that $B(X)$ is a Riesz space and the absolute value of f is $|f(t)|$.

Definition 1.4.4. Let E be a Riesz space furnished with a norm $\|.\|$, satisfying $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. In addition, let the space E be complete with respect to that norm. Then E is called a Banach lattice.

The norm $\|.\|$ in a Banach lattice E is said to be *order continuous*, if

$$\inf\{\|x\| : x \in A\} = 0$$

for any down-directed set $A \subset E$, such that $\inf\{x \in A\} = 0$, cf. [91, p. 86].

The real spaces $C(K), L^p(K) \quad (K \subseteq \mathbb{R}^n)$ and $l^p \quad (p \geq 1)$ are examples of Banach lattices.

A bounded linear operator T in E is called *a positive one*, if from $x \geq 0$ it follows that $Tx \geq 0$.

1.5 The abstract Gronwall lemma

In this section E is a Banach lattice with the positive cone E_+.

Lemma 1.5.1 (The abstract Gronwall lemma)**.** *Let T be a bounded linear positive operator acting in E and having the spectral radius*

$$r_s(T) < 1.$$

Let $x, f \in E_+$. Then the inequality

$$x \leq f + Tx$$

implies $x \leq y$ where y is a solution of the equation

$$y = f + Ty.$$

Proof. Let $Bx = f + Tx$. Then $x \leq Bx$ implies $x \leq Bx \leq B^2x \leq \cdots \leq B^m x$. This gives

$$x \leq B^m x = \sum_{k=0}^{m-1} T^k f + T^m x \to (I - T)^{-1} f = y \text{ as } m \to \infty.$$

Since $r_s(T) < 1$, the von Neumann series converges. □

We will say that $F : E \to E$ is *a non-decreasing mapping* if $v \leq w \ \ (v, w \in E)$ implies $F(v) \leq F(w)$.

Lemma 1.5.2. *Let $F : E \to E$ be a non-decreasing mapping, and $F(0) = 0$. In addition, let there be a positive linear operator T in E, such that the conditions*

$$|F(v) - F(w)| \leq T|v - w| \ \ (v, w \in E), \tag{5.1}$$

and $r_s(T) < 1$ hold. Then the inequality

$$x \leq F(x) + f \ \ (x, f \in E_+)$$

implies that $x \leq y$ where y is a solution of the equation

$$y = F(y) + f.$$

Moreover, the inequality

$$z \geq F(z) + f \ \ (z, f \in E_+)$$

implies that $z \geq y$.

Proof. We have $x = F(x) + h$ with an $h < f$. Thanks to (5.1) and the condition $r_s(T) < 1$, the mappings $F_f := F + f$ and $F_h := F + h$ have the following properties: F_f^m and F_h^m are contracting for some integer m. So thanks to the generalized contraction mapping theorem [115], $F_f^k(f) \to x, F_h^k(f) \to y$ as $k \to \infty$. Moreover, $F_f^k(f) \geq F_h^k(f)$ for all $k = 1, 2, \ldots,$ since F is non-decreasing and $h \leq f$. This proves the inequality $x \geq y$. Similarly the inequality $x \leq z$ can be proved. □

1.6 Integral inequalities

Let $C(J,\mathbb{R}^n)$ be a space of real vector-valued functions defined, bounded and continuous on a finite or infinite interval J. The inequalities are understood in the coordinate-wise sense.

To receive various solution estimates, we essentially use the following lemma.

Lemma 1.6.1. *Let $\hat{K}(t,s)$ be a matrix kernel with non-negative entries, such that the integral operator*

$$(Kx)(t) = \int_J \hat{K}(t,s)x(s)ds$$

maps $C(J,\mathbb{R}^n)$ into itself and has the spectral radius $r_s(K) < 1$. Then for any non-negative continuous vector function $v(t)$ satisfying the inequality

$$v(t) \le \int_J \hat{K}(t,s)v(s)ds + f(t)$$

where f is a non-negative continuous on J vector function, the inequality $v(t) \le u(t)$ $(t \in J)$ is valid, where $u(t)$ is a solution of the equation

$$u(t) = \int_J \hat{K}(t,s)u(s)ds + f(t).$$

Similarly, the inequality

$$v(t) \ge \int_J \hat{K}(t,s)v(s)ds + f(t)$$

implies $v(t) \ge u(t)$ $(t \in J)$.

Proof. The lemma is a particular case of the abstract Gronwall lemma. □

If $J = [a,b]$ is an arbitrary finite interval and

$$(Kx)(t) = \int_a^t \hat{K}(t,s)x(s)ds \ (t \le b),$$

and the condition

$$\sup_{t\in\ [a,b]} \int_a^t \|\hat{K}(t,s)\|ds < \infty$$

is fulfilled with an arbitrary matrix norm, then it is simple to show that $r_s(K) = 0$.

The same equality for the spectral radius is true, if

$$(Kx)(t) = \int_t^b \hat{K}(t,s)x(s)ds \ (t \ge a),$$

provided

$$\sup_{t\in\ [a,b]} \int_t^b \|\hat{K}(t,s)\|ds < \infty.$$

1.7 Generalized norms

In this section nonlinear equations are considered in a space furnished with a vector (generalized) norm introduced by L. Kantorovich [108, p. 334]. Note that a vector norm enables us to use information about equations more complete than a usual (number) norm.

Throughout this section E is a Banach lattice with a positive cone E_+ and a norm $\|.\|_E$.

Let X be an arbitrary set. Assume that in X a vector metric $M(.,.)$ is defined. That is, $M(.,.)$ maps $X \times X$ into E_+ with the usual properties: for all $x, y, z \in X$

a) $M(x,y) = 0$ iff $x = y$;
b) $M(x,y) = M(y,x)$ and
c) $M(x,y) \leq M(x,z) + M(y,z)$.

Clearly, X is a metric space with the metric $m(x,y) = \|M(x,y)\|_E$. That is, a sequence $\{x_k \in X\}$ converges to x in the metric $m(.,.)$ iff $M(x_k, x) \to 0$ as $k \to \infty$.

Lemma 1.7.1. *Let X be a space with a vector metric $M(.,.)$: $X \times X \to E_+$, and $F(x)$ map a closed set $\Phi \subseteq X$ into itself with the property*

$$M(F(x), F(y)) \leq QM(x,y) \quad (x, y \in \Phi), \tag{7.1}$$

where Q is a positive operator in E whose spectral radius $r_s(Q)$ is less than one: $r_s(Q) < 1$. Then, if X is complete in generalized metric $M(.,.)$ (or, equivalently, in metric $m(.,.)$), F has a unique fixed point $\overline{x} \in \Phi$. Moreover, that point can be found by the method of successive approximations.

Proof. Following the usual proof of the contracting mapping theorem we take an arbitrary $x_0 \in \Phi$ and define the successive approximations by the equality

$$x_k = F(x_{k-1}) \quad (k = 1, 2, \dots).$$

Hence,

$$M(x_{k+1}, x_k) = M(F(x_k), F(x_{k-1})) \leq QM(x_k, x_{k-1}) \leq \dots \leq Q^k M(x_1, x_0).$$

For $m > k$ we thus get

$$\begin{aligned} M(x_m, x_k) &\leq M(x_m, x_{m-1}) + M(x_{m-1}, x_k) \\ &\leq \dots \leq \sum_{j=k}^{m-1} M(x_{j+1}, x_j) \leq \sum_{j=k}^{m-1} Q^j M(x_1, x_0). \end{aligned}$$

Inasmuch as $r_s(Q) < 1$,

$$M(x_m, x_k) \leq Q^k (I - Q)^{-1} M(x_1, x_0) \to 0, \ (k \to \infty).$$

Here and below I is the unit operator in a corresponding space. Consequently, points x_k converge in the metric $M(.,.)$ to an element $\overline{x} \in \Phi$. Since

$$\lim_{k\to\infty} F(x_k) = F(\overline{x}),$$

$\overline{x}$ is the fixed point due to (2.2). Thus, the existence is proved.

To prove the uniqueness let us assume that $y \neq \overline{x}$ is a fixed point of F as well. Then by (7.1) $M(\overline{x}, y) = M(F(\overline{x}), F(y)) \leq QM(\overline{x}, y)$. Or

$$(I - Q)^{-1}M(\overline{x}, y) \leq 0.$$

But $I - Q$ is positively invertible, because $r_s(Q) < 1$. In this way, $M(\overline{x}, y) \leq 0$. This proves the result. □

Now let X be a linear space with a vector (generalized) norm $M(.)$. That is, $M(.)$ maps X into E_+ and is subject to the usual axioms: for all $x, y \in X$

$$M(x) > 0 \text{ if } x \neq 0;\ M(\lambda x) = |\lambda| M(x)\ \ (\lambda \in \mathbf{C});\ M(x + y) \leq M(x) + M(y).$$

Following [108], we shall call E *a norming lattice, and* X *a lattice-normed space*. Clearly, X with a generalized (vector) norm $M(.) : X \to E_+$ is a normed space with the norm

$$\|h\|_X = \|M(h)\|_E\ (h \in X). \tag{7.2}$$

Now the previous lemma implies

Corollary 1.7.2. *Let X be a space with a generalized norm $M(.) : X \to E_+$ and $F(x)$ map a closed set $\Phi \subseteq X$ into itself with the property*

$$M(F(x) - F(y)) \leq QM(x - y)\ \ (x, y \in \Phi),$$

where Q is a positive operator in E with $r_s(Q) < 1$. Then, if X is complete in the norm defined by (7.2), F has a unique fixed point $\overline{x} \in \Phi$. Moreover, that point can be found by the method of successive approximations.

1.8 Causal mappings

Let $X(a, b) = X([a, b]; Y)$ $(-\infty < a < b \leq \infty)$ be a normed space of functions defined on $[a, b]$ with values in a normed space Y and the unit operator I. For example $X(a, b) = C([a, b], \mathbb{C}^n)$ or $X(a, b) = L^p([a, b], \mathbb{C}^n)$.

Let P_τ $(a < \tau < b)$ be the projections defined by

$$(P_\tau w)(t) = \begin{cases} w(t) & \text{if } a \leq t \leq \tau, \\ 0 & \text{if } \tau < t \leq b \end{cases} \qquad (w \in X(a, b)),$$

and $P_a = 0$, and $P_b = I$.

Definition 1.8.1. Let F be a mapping in $X(a,b)$ having the following properties:

$$F0 \equiv 0, \tag{8.1}$$

and for all $\tau \in [a,b]$, the equality

$$P_\tau F P_\tau = P_\tau F \tag{8.2}$$

holds. Then F will be called a causal mapping (operator).

This definition is somewhat different from the definition of the causal operator suggested in [13]; in the case of linear operators our definition coincides with the one accepted in [17]. Note that, if F is defined on a closed set $\Omega \ni 0$ of $X(a,b)$, then due to the Urysohn theorem, F can be extended by zero to the whole space. Put $X(a,\tau) = P_\tau X(a,b)$. Note that, if $X(a,b) = C(a,b)$, then $P_\tau f$ is not continuous in $C(a,b)$ for an arbitrary $f \in C(a,b)$. So P_τ is defined on the whole space $C(a,b)$ but maps $C(a,b)$ into the space $B(a,b) \supset C(a,b)$, where $B(a,b)$ is the space of bounded functions. However, if $P_\tau F f$ is continuous on $[a,\tau]$, that is $P_\tau F f \in C(a,\tau)$ for all $\tau \in (a,b]$, and relations (8.1) and (8.2) hold, then F is causal in $C(a,b)$.

Let us point an example of a causal mapping. To this end consider in $C(0,T)$ the mapping

$$(Fw)(t) = f(t,w(t)) + \int_0^t k(t,s,w(s))ds \ \ (0 \le t \le T;\ w \in C(0,T))$$

with a continuous kernel k, defined on $[0,T]^2 \times \mathbb{R}$ and a continuous function

$$f : [0,T] \times \mathbb{R} \to \mathbb{R}, \text{ satisfying } k(t,s,0) \equiv 0 \quad \text{and} \quad f(t,0) \equiv 0.$$

For each $\tau \in (0,T)$, we have

$$(P_\tau F w)(t) = f_\tau(t,w(t)) + P_\tau \int_0^t k(t,s,w(s))ds,$$

where

$$f_\tau(t,w(t)) = \begin{cases} f(t,w(t)) & \text{if } \ 0 \le t \le \tau, \\ 0 & \text{if } \ \tau < t \le T. \end{cases}$$

Clearly,

$$f_\tau(t,w(t)) = f_\tau(t,w_\tau(t)) \quad \text{where} \quad w_\tau = P_\tau w.$$

Moreover,

$$P_\tau \int_0^t k(t,s,w(s))ds = P_\tau \int_0^t k(t,s,w_\tau(s))ds = 0, t > \tau$$

and

$$P_\tau \int_0^t k(t,s,w(s))ds = \int_0^t k(t,s,w(s))ds = \int_0^t k(t,s,w_\tau(s))ds, t \le \tau.$$

Hence it follows that the considered mapping is causal. Note that, the integral operator

$$\int_0^c k(t,s,w(s))ds$$

with a fixed positive $c \le T$ is not causal.

1.9 Compact operators in a Hilbert space

A linear operator A mapping a normed space X into a normed space Y is said to be completely continuous (compact) if it is bounded and maps each bounded set in X into a compact one in Y. The spectrum of a compact operator is either finite, or the sequence of the eigenvalues of A converges to zero, any non-zero eigenvalue has the finite multiplicity.

This section deals with completely continuous operators acting in a separable Hilbert space H. All the results presented in this section are taken from the books [2] and [65].

Any normal compact operator can be represented in the form

$$A = \sum_{k=1}^{\infty} \lambda_k(A) E_k,$$

where E_k are eigenprojections of A, i.e., the projections defined by $E_k h = (h, d_k)d_k$ for all $h \in H$. Here d_k are the normal eigenvectors of A. Recall that eigenvectors of normal operators are mutually orthogonal.

A completely continuous quasinilpotent operator sometimes is called a Volterra operator.

Let $\{e_k\}$ be an orthogonal normal basis in H, and the series

$$\sum_{k=1}^{\infty} (Ae_k, e_k)$$

converges. Then the sum of this series is called *the trace of A:*

$$\text{Trace}\, A = \text{Tr}\ A = \sum_{k=1}^{\infty} (Ae_k, e_k).$$

An operator A satisfying the condition

$$\text{Tr}\ (A^*A)^{1/2} < \infty$$

is called *a nuclear operator*. An operator A, satisfying the relation

$$\text{Tr } (A^*A) < \infty$$

is said to be *a Hilbert–Schmidt operator*.

The eigenvalues $\lambda_k((A^*A)^{1/2})$ $(k = 1, 2, \dots)$ of the operator $(A^*A)^{1/2}$ are called *the singular numbers* (s-numbers) of A and are denoted by $s_k(A)$. That is,

$$s_k(A) := \lambda_k((A^*A)^{1/2}) \ (k = 1, 2, \dots).$$

Enumerate singular numbers of A taking into account their multiplicity and in decreasing order. The set of completely continuous operators acting in a Hilbert space and satisfying the condition

$$N_p(A) := \left[\sum_{k=1}^{\infty} s_k^p(A)\right]^{1/p} < \infty,$$

for some $p \geq 1$, is called *the von Schatten–von Neumann ideal and is denoted by* SN_p. $N_p(.)$ *is called the norm of the ideal* SN_p. It is not hard to show that

$$N_p(A) = \sqrt[p]{\text{Tr } (AA^*)^{p/2}}.$$

Thus, SN_1 is the ideal of nuclear operators (*the Trace class*) and SN_2 is the ideal of Hilbert–Schmidt operators. $N_2(A)$ is called the *Hilbert–Schmidt norm*. Sometimes we will omit index 2 of the Hilbert–Schmidt norm, i.e.,

$$N(A) := N_2(A) = \sqrt{\text{Tr } (A^*A)}.$$

For any orthonormal basis $\{e_k\}$ we can write

$$N_2(A) = \left(\sum_{k=1}^{\infty} \|Ae_k\|^2\right)^{1/2}.$$

This equality is equivalent to the following one:

$$N_2(A) = \left(\sum_{j,k=1}^{\infty} |a_{jk}|^2\right)^{1/2},$$

where $a_{jk} = (Ae_k, e_j)$ $(j, k = 1, 2, \dots)$ are entries of a Hilbert–Schmidt operator A in an orthonormal basis $\{e_k\}$.

For all finite $p \geq 1$, the following propositions are true (the proofs can be found in the book [65, Section 3.7]).

If $A \in SN_p$, then also $A^* \in SN_p$. If $A \in SN_p$ and B is a bounded linear operator, then both AB and BA belong to SN_p. Moreover,

$$N_p(AB) \leq N_p(A)\|B\| \quad \text{and} \quad N_p(BA) \leq N_p(A)\|B\|.$$

In addition, the inequality

$$\sum_{j=1}^{n} |\lambda_j(A)|^p \leq \sum_{j=1}^{n} s_j^p(A) \quad (n = 1, 2, \dots)$$

is valid, cf. [65, Theorem II.3.1].

Lemma 1.9.1. *If* $A \in SN_p$ *and* $B \in SN_q$ $(1 < p, q < \infty)$, *then* $AB \in SN_s$ *with*

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q}.$$

Moreover,

$$N_s(AB) \leq N_p(A)N_q(B).$$

For the proof of this lemma see [65, Section III.7]. Recall also the following result.

Theorem 1.9.2 (Lidskij's theorem)**.** *Let* $A \in SN_1$. *Then*

$$\text{Tr } A = \sum_{k=1}^{\infty} \lambda_k(A).$$

The proof of this theorem can be found in [65, Section III.8].

1.10 Regularized determinants

The regularized determinant of $I - A$ with $A \in SN_p$ $(p = 1, 2, \dots)$ is defined as

$$\det_p(I - A) := \prod_{j=1}^{\infty} E_p(\lambda_j(A)),$$

where $\lambda_j(A)$ are the eigenvalues of A with their multiplicities arranged in decreasing order, and

$$E_p(z) := (1 - z) \exp\left[\sum_{m=1}^{p-1} \frac{z^m(A)}{m}\right], \ p > 1 \quad \text{and} \quad E_1(z) := 1 - z.$$

As shown below, regularized determinant are useful for the investigation of periodic systems.

The following lemma is proved in [57].