Applied and Numerical Harmonic Analysis

$$\widehat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Willi Freeden
Martin Gutting

Special Functions of Mathematical (Geo-)Physics

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Special Functions of Mathematical (Geo-)Physics

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Preface

An essential aim of geomathematics is the investigation of qualitative and quantitative structures of the Earth's system to deepen our understanding of its complexity. In this respect, special functions comprise the essential instruments for mathematical interaction of abstraction and concretization. Special functions enable the formulation of a geoscientific problem by reduction such that a new, more concrete problem can be attacked within a well-structured framework, usually in the context of differential equations. A good understanding of special functions provides the capacity to recognize causality between the abstractness of the geomathematical concept and the impact on, as well as cross-sectional importance to, the geoscientific reality.

Our purpose in this work is to present a textbook that allows the reader to concentrate on special fields such as the geosphere, hydrosphere, or atmosphere. In other words, the special functions to be discussed vary widely, depending on the chosen measurement parameters (gravitation, electric and magnetic fields, deformation, climate observables, fluid flow, etc.) and on the field characteristic (potential field, diffusion field, wave field). The differential equation under consideration determines the type of special functions that are needed in the desired reduction process.

The diversity of geomathematical problems involves such a large number of scientific manifestations that our approach to any of them has to be selective. In consequence, since greater weight has to be given to some topics than to others, we have chosen to restrict ourselves to gravitation, geomagnetism, elasticity, and fluid flow theory. Gravitational field theory defines a canonical need to generate special function systems for the Laplace equation. Geomagnetism and electric current systems are closely related to the (pre-)Maxwell equation; the deformation of the solid Earth leads to function systems solving the Cauchy–Navier equation (at least when linear material behavior is assumed). Oceanic circulation and wind motion have to be handled in terms of vectorial function systems involving the Navier–Stokes equation or modifications of it. Unfortunately, we are confronted with the difficult challenge to characterize special function systems under adequate consistency in terms of less mathematically structured geometric features of a reference model (such as the geoid or the real Earth's surface) as well as the intrinsic structure of underlying differential equations involving the laws of physics. Thus, at the present stage of geoscience, no compendium can be expected that is both geometrically consistent with modern navigation results and geophysically reflected by advanced mathematical settings. The complexity of a real "potato-like" Earth model is a striking obstacle that can only be overcome to some extent in today's mathematics. Accordingly, the principles lie in the suitable transition to a regularly structured geometry for the Earth, namely, the ball in first approximation. This leads us to a prestructured framework, namely, spherically oriented special function systems.

Looking at the special functions available in the geophysical literature today, we find that a spherical shape of the Earth is used in almost all publications. Indeed, by modern satellite positioning methods, the maximum deviation of the actual Earth's surface from the average Earth's radius (6,371 km) can be determined to be less than 0.4 %. Although a *spheriodization*, i.e., a mathematical formulation simply in spherical reference geometry, amounts to a strong restriction, it is at least acceptable for a large number of problems. Standard special functions since the time of C.F. Gauß are polynomial trial functions, conventionally called spherical harmonics. Spherical harmonics represent the analogs of trigonometric functions for orthogonal (Fourier) expansions on the sphere. In consequence, the use of spherical harmonics in diverse areas of geosciences is a well-established method, particularly for the purpose of decomposing scalar potentials. Nowadays, reference models for the Earth's gravitational and magnetic potential, e.g., are widely known by tables of expansion coefficients of the frequency constituents of their potentials. However, it should be mentioned that vectorial potentials—even in a spherical Earth's reference model—have their own nature. Concerning the mathematical modeling of vector fields, one is usually not interested in their separation into scalar Cartesian component functions. Instead, inherent physical properties should be observed. For example, the external gravitational field is curl-free, the magnetic field is divergence-free, the equations for incompressible flow, i.e., the Navier–Stokes equations, imply divergence-free vector solutions. In a spherical nomenclature as intended in our approach, all these physical constraints result in a formulation by certain operators, such as the surface gradient, surface curl gradient, surface divergence, surface curl. Our types of vector spherical harmonics satisfy these requirements by splitting the tangential part into a curl-free and a divergence-free field, thereby avoiding artificial singularities arising from the use of local coordinates. Basically, two transitions are undertaken in our approach to harmonics: first, the extension from the scalar to the vectorial case is strictly realized under physical constraints and, second, the definition of Legendre functions is canonically described under the phenomenon of rotational invariance on the sphere. The Legendre functions act as constituting elements for zonal functions, i.e., one-dimensional functions only depending on the polar distance of their two arguments. Altogether, the concept of spherical harmonics plays the central role in a geomathematical presentation of special functions, reflecting the significance of a polynomial nature in a spherically shaped Earth. In addition, spherical harmonics comprise the canonical candidates to represent the angular part in a radial/angular decomposition of solution systems for Laplace, Helmholtz, Cauchy–Navier, (pre-)Maxwell, and Navier–Stokes equations.

It is surprising that, besides the geometrically implied spheriodization, the methodologically oriented *periodization* should take some space in a modern collection of special function systems of geomathematical importance. The reasons are twofold. First, the periodization leads back to the Fourier transform in Euclidean spaces that has been well understood for a long time and is extremely efficient in numerical computation. Second, the procedure of periodization leads to the Euler summation formula and the Poisson summation formula which show a close relationship to each other. The Green (lattice) functions forming the essential basis of these summation formulas indeed enable us to express key volume integrals in geophysics, such as the Newton integral, Mie potentials, elastic potentials, by mass lattice point conglomerates that discretely fill out the integration domain under consideration in an equidistributed way.

A variety of examples for *combined periodization and spheriodization* occur in the theory of Earth-satellite relations (cf., e.g., Kaula 1966), mixing time-wise obligations on periodic orbits with space-wise approaches on torus and/or sphere. Satellite gravimetry (see, e.g., Pail and Plank (2002), Sneeuw (2000), Xu et al. (2008), and the references therein) is a particularly interesting area of spaceborne technology, where one-dimensional periodization in time is adequately involved in three-dimensional periodization and/or spheriodization in space.

This textbook presents material used by the Geomathematics Group, University of Kaiserslautern, during the last several years to set up a contemporary theory of special functions of mathematical (geo-)physics. Our work canonically shows a threefold subdivision. Part I provides preparatory material concerning auxiliary functions such as the Gamma function and important classes of orthogonal polynomials. The general concept of orthogonal polynomials is introduced before we start to consider the classical polynomials, in particular the Jacobi polynomials and—as a special and very important case of them—the ultraspherical or Gegenbauer polynomials. Several basic mathematical and physical applications are included, such as quadrature rules, modeling of the electrostatic potential, and the quantum-mechanical description of oscillations. Part II deals with spherically structured function systems. It starts with the scalar theory of spherical harmonics in the Euclidean space $\mathbb{R}^3$ including the addition theorem, the Funk–Hecke formula, as well as the closure and completeness of spherical harmonics in the space of square-integrable functions, i.e., the space of functions with finite signal energy. It follows the physically based theory of vector spherical harmonics. The basic tool to establish divergence-free and curl-free tangential fields is the Helmholtz decomposition theorem. An alternative system of vector spherical harmonics is also constructed in such a way that they can be identified as eigenfunctions of the Beltrami operator. This eigenfunction system plays a particular role in geomagnetism to separate, e.g., the crustal field from other magnetic sources. Both vector spherical harmonic systems are shown to be closed and complete in the space of square-integrable vector fields on the sphere. All properties characterize vector spherical harmonics as suitable trial functions to constitute the angular ingredients in a radial/angular decomposition of solutions of the Cauchy–Navier as well as the Navier–Stokes equation. Part III is devoted to the lattice function as the multi-dimensional,

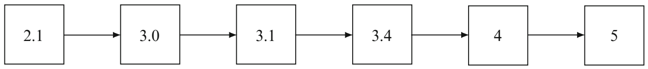

Fig. 1 Selected sections and chapters for a basic one-term course (note that some parts of Sect. 3.3 have to be included to complete Sect. 3.4. Sections 5.5, 5.6, 5.9, and 5.10 can be skipped in the one-term course)

periodic analog to the well-known Bernoulli function. From a physical point of view, the lattice function is interpreted to be the Green function for the Laplace operator corresponding to the boundary conditions of periodicity. It turns out to be the most essential tool for the process of periodization in the context of Euler and Poisson summation formulas. Lattice point sums such as the Zeta and Theta functions, generated by the interaction of point potentials to each other, conclude our multi-periodic theory. It should be remarked that the whole palette of multi-periodic functions is provided in relation to the Laplace operator and arbitrary lattices so that this approach serves as a prototype for further formulations of more general (elliptic) partial differential equations.

Essential ingredients of the textbook are the work of Müller (1952, 1969, 1998), Freeden et al. (1998), Freeden and Schreiner (2009), and Freeden (2011).

Each chapter of the book is followed by exercises related to the presented material. The exercises reflect significant topics, mostly in computational geo-applications. In doing so, they not only confront the reader directly with the contents of the chapter, but also with additional knowledge in geomathematical fields of research, where special functions play a decisive role in applications. Students who wish to continue further studies should consult the literature given as supplements for each topic worked out by exercises. All in all, the content of the book is equally suitable for an education in geomathematics and a study in applied and harmonic analysis.

The book is primarily meant to be a self-consistent introductory text for an advanced undergraduate or graduate course in special functions. The schedule of topics allows a selected subdivision into a one-term course (see Fig. 1) as well as a two-term course. In addition to the proposed sections and chapters in Fig. 1, further contents can be selected from Chap. 3, such as Sect. 3.2 and all details of Sects. 3.3 or 3.5–3.7, if the schedule allows it. The examples of Chap. 1 can be presented at any appropriate time.

A two-term course with special emphasis on particular research fields should include additional material from Chaps. 5 and 7 documenting the special interest of a graduate student in gravitation, geomagnetism, deformation, atmospheric/oceanic flow, respectively. Chapters 6 and 8 give multi-dimensional radial/angular decompositions of harmonic and metaharmonic functions as a reference tool, thereby assuming as preparatory material the whole theory of the Gamma function as presented in Chap. 2. Another separate route going exclusively into the field of lattice functions includes Chaps. 9 and 10 while also requiring all the material of Chap. 2.

In a book of this type, special precautions have been taken to ensure the accuracy of formulas and examples. It is a pleasure to acknowledge with thanks the valuable reading of the manuscript by Dipl.-Math. C. Blick, Dr. C. Gerhards, and Dr. I. Ostermann. We thank our student cand.-phys. Hanna Haug for pointing out some errors and slips in an early version of the manuscript.

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Chapter 1
Introduction: Geomathematical Motivation

In the first chapter we briefly introduce four fields showing strong geophysical background. Thereby, we are naturally led to differential equations which are closely related to solution systems of special functions. Since the Earth is a ball in first approximation, a spherical coordinate frame and spherical functions play a huge role in geomathematics. Concerning the fields of research, we consider the modeling of the gravitational field in Sect. 1.1, the magnetic field of the Earth in Sect. 1.2, the atmospheric flow in Sect. 1.3, and the elastic field in Sect. 1.4. Each example gives a motivation to investigate special function systems that are essential for the analysis of the underlying geoscientific problem. Knowledge of potential theory will be helpful in understanding the mathematical nature of the occurring (boundary value) problems. Although we give all the necessary material in this book, we also recommend the following books on potential theoretic methods, Backus et al. (1996), Blakely (1996), Freeden and Gerhards (2012), Freeden et al. (2010), Gurtin (1972), Helms (1969), Kellogg (1929), Martensen (1968), Miranda (1970), Müller (1969), and Wangerin (1921) for further reading as well as Freeden and Michel (2004) for a multi-scale context.

1.1 Example: Gravitation (Laplace and Poisson Equation)

Since "De mundi systemate" of Newton (1687) it has been an established fact that the motion of any free falling body is determined by the Earth's gravitational field. More precisely, Newton's law about the mutual attraction of two masses informs us that the attractive force, called *gravitation*, is directed along the line connecting the two centers of the objects and is proportional to both masses as well as to the squared inverse of the distance between the two objects. In consequence,

$$\int_{\mathscr{G}} \varrho(x) \frac{a-x}{|a-x|^3}\, \mathrm{d}V(x) = \nabla_a \int_{\mathscr{G}} \frac{\varrho(x)}{|a-x|}\, \mathrm{d}V(x) \qquad (1.1.1)$$

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is the gravitational field of the Earth (with the interior density distribution ϱ),

$$V(a) = \int_{\mathscr{G}} \frac{\varrho(x)}{|x-a|} \, \mathrm{d}V(x) \tag{1.1.2}$$

is the so-called gravitational potential. These formulas explain that the line, along which Newton's body fell, would be indeed a straight line, directed radially and going exactly through the Earth's center of mass if the Earth had a perfectly spherical shape and the mass inside the Earth were distributed homogeneously or rotationally symmetric. The gravitational field obtained in this way would be perfectly symmetric. In reality, however, the situation is much more complicated (see, e.g., Groten 1979; Heiskanen and Moritz 1967; Hofmann-Wellenhof and Moritz 2005; Misner et al. 1973; Torge 2001). The topographic features of $\partial\mathscr{G}$, mountains and valleys, are very irregular, leading to a mathematical simplification by *spheriodization* at global scale. The actual gravitational field is (see Fig. 1.1) influenced by strong irregularities in the density ϱ within the Earth $\mathscr{G}$, implying a mathematical simplification by *periodization* at discrete scale. In spite of both simplifications, a deviation of the gravitational force is caused from one place to the other and two essential phenomena of the gravitational field are modeled in simplified form. Spherical signatures of the gravitational field in frequency domain show an exponential smoothing effect in the exterior with increasing distance from the Earth's body. Density signatures on internal lattices are reflected by gravitational field variations, and vice versa.

Spheriodization

In order to realize the principle of spheriodization for heterogeneous density distribution, we ideally assume that all mass is contained within a sphere of radius R that models approximately the Earth's surface, i.e., $\partial\mathscr{G} = \mathbb{S}^2_R = \{x \in \mathbb{R}^3 : |x| = R\}$. In other words, the heterogenous mass is supposed to be contained in the ball $\mathbb{B}^3_R = \{x \in \mathbb{R}^3 : |x| < R\}$ of radius R around the origin (where R can be taken as the mean Earth's radius). Under these circumstances, we investigate the gravitational field via the Newtonian gravitational potential

$$V(a) = \int_{\mathbb{B}^3_R} \frac{\varrho(x)}{|x-a|} \, \mathrm{d}V(x) \tag{1.1.3}$$

with the density function ϱ for non-terrestrial data $a \in \mathbb{R}^3 \setminus \overline{\mathbb{B}^3_R}$. From potential theory (see, e.g., Kellogg 1929), it is well-known that the potential and the density are related by the Poisson equation which holds in the ball $\mathbb{B}^3_R$, i.e.,

$$\Delta_x V(x) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) V(x) = -4\pi\varrho(x), \quad x \in \mathbb{B}^3_R. \tag{1.1.4}$$

Inserting this formally in (1.1.3) and using Green's second theorem, we find that

$$\begin{aligned} V(a) &= \int_{\mathbb{B}^3_R} \frac{-1}{4\pi|x-a|} \Delta_x V(x)\, \mathrm{d}V(x) \\ &= -\int_{\mathbb{B}^3_R} V(x) \Delta_x \frac{1}{4\pi|x-a|}\, \mathrm{d}V(x) \\ &\quad - \int_{\mathbb{S}^2_R} \left(\frac{1}{4\pi|x-a|} \frac{\partial V}{\partial \nu}(x) - V(x) \frac{\partial}{\partial \nu_x} \frac{1}{4\pi|x-a|} \right) \mathrm{d}S(x), \end{aligned} \tag{1.1.5}$$

where ν denotes the surface unit normal pointing to the exterior, i.e., to $\mathbb{R}^3 \setminus \overline{\mathbb{B}^3_R}$. Obviously, $\Delta_x \frac{1}{4\pi|x-a|} = 0$ for all $x \in \mathbb{B}^3_R$. Moreover, we can use the following bilinear expansion of the single pole:

$$\frac{1}{|x-a|} = \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{R}{|a|}\right)^{n+1} \left(\frac{|x|}{R}\right)^n \frac{4\pi}{2n+1} \sum_{k=-n}^{n} Y_{n,k}\left(\frac{x}{|x|}\right) Y_{n,k}\left(\frac{a}{|a|}\right), \tag{1.1.6}$$

where $|x| \le R < |a|$ and $Y_{n,k}$ denotes the *spherical harmonic of degree* n *and order* k (see Chap. 4). Roughly spoken, (1.1.6) expresses the single pole as a series expansion in terms of multipoles. It is the key structure for modeling the gravitational potential with respect to frequencies, i.e., degree n and order k. The identity (1.1.6) gives us the following representation of the potential

$$\begin{aligned} V(a) = \int_{\mathbb{S}^2_R} & V(x) \frac{\partial}{\partial \nu_x} \frac{1}{R} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \left(\frac{|x|}{R}\right)^n Y_{n,k}\left(\frac{x}{|x|}\right) \left(\frac{R}{|a|}\right)^{n+1} Y_{n,k}\left(\frac{a}{|a|}\right) \\ & - \frac{1}{R} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \left(\frac{|x|}{R}\right)^n Y_{n,k}\left(\frac{x}{|x|}\right) \left(\frac{R}{|a|}\right)^{n+1} Y_{n,k}\left(\frac{a}{|a|}\right) \frac{\partial V}{\partial \nu}(x)\, \mathrm{d}S(x). \end{aligned} \tag{1.1.7}$$

Since the surface is a sphere of radius R, we can easily compute the normal derivative, i.e.,

$$\frac{\partial}{\partial \nu_x} \left(\frac{|x|}{R}\right)^n Y_{n,k}\left(\frac{x}{|x|}\right) = \frac{n}{R} \left(\frac{|x|}{R}\right)^{n-1} Y_{n,k}\left(\frac{x}{|x|}\right), \tag{1.1.8}$$

and on the surface $\mathbb{S}^2_R$ we have $|x| = R$. If we assume V to be sufficiently smooth, we obtain from (1.1.7) with the help of (1.1.8) that

$$V(a) = \frac{1}{R} \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{1}{2n+1} \int_{\mathbb{S}^2_R} \left(\frac{n}{R} V(x) - \frac{\partial V}{\partial \nu}(x) \right) Y_{n,k}\left(\frac{x}{|x|}\right) \mathrm{d}S(x) \left(\frac{R}{|a|}\right)^{n+1} Y_{n,k}\left(\frac{a}{|a|}\right). \tag{1.1.9}$$

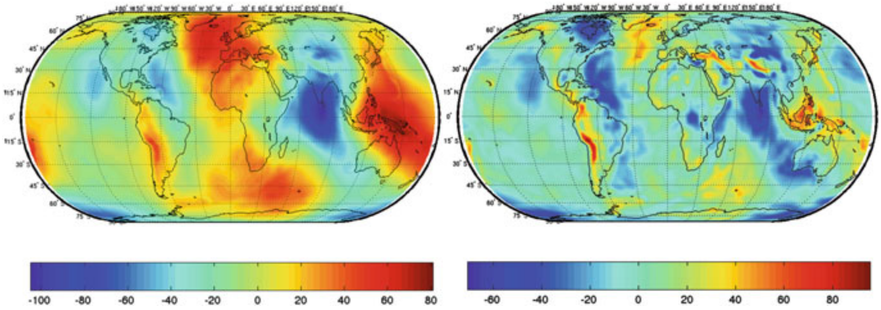

Fig. 1.1 Geoid, i.e., equipotential surface of the Earth at sea level (*left*) and gravity anomalies (*right*). For a detailed overview of the functionals related to the gravitational potential we refer, e.g., to Fengler et al. (2004) and the references therein

Even more,

$$\int_{\mathbb{S}_R^2} \frac{\partial V}{\partial \nu}(x) Y_{n,k}\left(\tfrac{x}{|x|}\right) \, \mathrm{d}S(x) = -\tfrac{n+1}{R} \int_{\mathbb{S}_R^2} V(x) Y_{n,k}\left(\tfrac{x}{|x|}\right) \, \mathrm{d}S(x). \tag{1.1.10}$$

Finally, this leads us to the expansion

$$V(a) = \tfrac{1}{R} \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \int_{\mathbb{S}_R^2} V(x) \tfrac{1}{R} Y_{n,k}\left(\tfrac{x}{|x|}\right) \, \mathrm{d}S(x) \left(\tfrac{R}{|a|}\right)^{n+1} Y_{n,k}\left(\tfrac{a}{|a|}\right). \tag{1.1.11}$$

This shows us how the gravitational potential can be expanded in a series of spherical harmonics. Moreover, the spheriodization of the Earth implies a frequency dependent description of V outside and on the sphere $\mathbb{S}_R^2$. Our work shows that spherical harmonics form a complete orthonormal basis for the square-integrable functions on the sphere. Therefore, this basis system plays the same role in spherical Fourier analysis as the trigonometric polynomials in one dimension. This aspect is the essential prerequisite for constructive approximation on the sphere (see, e.g., Freeden et al. 1998; Freeden and Gerhards 2012; Michel 2012). It is applied in many areas, for example, in physical geodesy (see, e.g., Groten 1979; Heiskanen and Moritz 1967; Hofmann-Wellenhof and Moritz 2005; Torge 2001), in geomagnetism (see, e.g., Backus et al. 1996), even spaceborne vectorial or tensorial data (see Freeden and Nutz 2011; Freeden and Schreiner 2009; Kotsiaris and Olsen 2012; Rummel and van Gelderen 1992) can be handled in modern satellite geodesy, such as satellite-to-satellite tracking or satellite-gravity-gradiometry. Scalar-valued spherical harmonics are investigated in depth in Chap. 4. Further applications can be found, e.g., in quantum mechanics (see, e.g., Edmonds 1964; Zare 1988), in many other geomathematical problems or in crystallography (see, e.g., Ewald 1921; Hielscher et al. 2010 or Schaeben and van den Boogaart 2003) using also results of higher dimensions as presented in Chap. 6.

Periodization

In order to explain the principle of periodization under general geometry $\partial\mathscr{G}$ we only give a heuristic motivation for the discretization of the Newton integral

$$V(a) = \int_{\mathscr{G}} \frac{\varrho(x)}{|x-a|} \, dV(x) \tag{1.1.12}$$

on internal lattice points of $\mathbb{Z}^3$, i.e., a generation of V by periodically located single poles (with realistic gravitational intensity) under explicit availability of a remainder term in integral form (the accurate formulation follows from Chap. 10). In light of (1.1.12), we consider the auxiliary function

$$F(x) = \begin{cases} \frac{\varrho(x)}{|x-a|} & , \quad x \in \overline{\mathscr{G}} \\ 0 & , \quad x \notin \overline{\mathscr{G}} \end{cases} \tag{1.1.13}$$

with sufficiently often differentiable density function $\varrho : \overline{\mathscr{G}} \to \mathbb{R}$. A first periodization $F_{\mathrm{per}}^{(1)} : \mathbb{R}^3 \to \mathbb{R}$ of F is straightforward:

$$F_{\mathrm{per}}^{(1)}(x) = \sum_{g \in \mathbb{Z}^3} F(x+g). \tag{1.1.14}$$

Since this (formal) sum is extended over all lattice points of the lattice $\mathbb{Z}^3$, it is obviously periodic, i.e.,

$$F_{\mathrm{per}}^{(1)}(x+g') = \sum_{g \in \mathbb{Z}^3} F(x+g+g') = \sum_{g \in \mathbb{Z}^3} F(x+g) = F_{\mathrm{per}}^{(1)}(x) \tag{1.1.15}$$

for all $x \in \mathbb{R}^3$ and $g' \in \mathbb{Z}^3$. A second periodization is based on the (formal) Fourier inversion formula

$$F(x) = \int_{\mathbb{R}^3} F_{\mathbb{R}^3}^{\wedge}(y) e^{2\pi i x \cdot y} \, dV(y), \tag{1.1.16}$$

where the Fourier transform in the Euclidean space $\mathbb{R}^3$ is given by

$$F_{\mathbb{R}^3}^{\wedge}(y) = \int_{\mathbb{R}^3} F(z) e^{-2\pi i y \cdot z} \, dV(z). \tag{1.1.17}$$

This leads to the second periodization given by

$$F_{\mathrm{per}}^{(2)}(x) = \sum_{h \in \mathbb{Z}^3} F_{\mathbb{R}^3}^{\wedge}(h) \Phi_h(x), \tag{1.1.18}$$

where the system $\{\Phi_h\}_{h\in\mathbb{Z}^3}$ is given by

$$\Phi_h(x) = \mathrm{e}^{2\pi \mathrm{i} h\cdot x}, \tag{1.1.19}$$

$x \in \mathbb{R}^3$, $h \in \mathbb{Z}^3$. Note that the system of trigonometric polynomials $\{\Phi_h\}_{h\in\mathbb{Z}^3}$ is an orthonormal basis in the space $\mathrm{L}^2_{\mathbb{Z}^3}(\mathbb{R}^3)$ of periodic functions in $\mathbb{R}^3$ that are square-integrable on the fundamental lattice cell $\mathscr{F} = [-\frac{1}{2}, \frac{1}{2})^3$:

$$\int_{\mathscr{F}} \Phi_h(x)\overline{\Phi_{h'}(x)}\, \mathrm{d}V(x) = \delta_{h,h'}. \tag{1.1.20}$$

Furthermore, $\{\Phi_h\}_{h\in\mathbb{Z}^3}$ is closed and complete in $\mathrm{L}^2_{\mathbb{Z}^3}(\mathbb{R}^3)$ such that any function in $\mathrm{L}^2_{\mathbb{Z}^3}(\mathbb{R}^3)$ can be represented by its Fourier series (of course, understood in the topology of $\mathrm{L}^2_{\mathbb{Z}^3}(\mathbb{R}^3)$). The Poisson summation formula (see, e.g., Benedetto 1996; Butzer and Nessel 1971; Stein and Weiss 1971 and the references therein) tells us that the two approaches to a periodic analog of F, i.e., the two periodizations $F^{(i)}_{\mathrm{per}}$, $i = 1, 2$, are identical under appropriate assumptions. This conclusion can be made precise in many topologies, even in pointwise sense, under geomathematically advantageous criteria (see, e.g., Freeden 2011 and the references therein)

$$F^{(1)}_{\mathrm{per}}(x) = F^{(2)}_{\mathrm{per}}(x), \quad x \in \mathbb{R}^3, \tag{1.1.21}$$

as far as the series on the right-hand side of (1.1.18) is convergent. Therefore, we obtain in this case that

$$\sum_{g\in\mathbb{Z}^3} F(x+g) = \sum_{h\in\mathbb{Z}^3} \int_{\mathbb{R}^3} F(y)\overline{\Phi_h(y)}\, \mathrm{d}V(y)\Phi_h(x). \tag{1.1.22}$$

In particular, for $x = 0$, we get

$$\sum_{g\in\mathbb{Z}^3} F(g) = \sum_{h\in\mathbb{Z}^3} \int_{\mathbb{R}^3} F(y)\overline{\Phi_h(y)}\, \mathrm{d}V(y). \tag{1.1.23}$$

In other words, observing the specific definition of F, we are, for $a \in \mathbb{R}^3 \setminus \overline{\mathscr{G}}$, led to

$$\begin{aligned}\sum_{\substack{g+x\in\overline{\mathscr{G}}\\ g\in\mathbb{Z}^3}}{}' \frac{\varrho(x+g)}{|x+g-a|} &= \int_{\mathscr{G}} \frac{\varrho(y)}{|y-a|}\, \mathrm{d}V(y) + \sum_{h\in\mathbb{Z}^3\setminus\{0\}} \int_{\mathscr{G}} \frac{\varrho(y)}{|y-a|}\overline{\Phi_h(y)}\, \mathrm{d}V(y)\Phi_h(x)\\ &= \int_{\mathscr{G}} \frac{\varrho(y)}{|y-a|}\, \mathrm{d}V(y) + R_{\mathbb{Z}^3}(x),\end{aligned} \tag{1.1.24}$$

where

$$\sum_{g\in\overline{\mathscr{G}}\cap\mathbb{Z}^3}{}' F(g) = \sum_{g\in\overline{\mathscr{G}}\cap\mathbb{Z}^3} \alpha(g)\ F(g) \tag{1.1.25}$$

and $\alpha(g)$ (see (6.2.33) in Sect. 6.2) is the solid angle subtended at $g \in \overline{\mathscr{G}}$ by the surface $\partial\mathscr{G}$. The term $R_{\mathbb{Z}^3}(x)$ is called the remainder or error term between the approximating $\mathbb{Z}^3$-lattice sum and the gravitational potential and is given by

$$R_{\mathbb{Z}^3}(x) = \sum_{h\in\mathbb{Z}^3\setminus\{0\}} \int_{\mathscr{G}} \frac{\varrho(y)}{|y-a|}\overline{\Phi_h(y)}\ \mathrm{d}V(y)\Phi_h(x). \tag{1.1.26}$$

This remainder is a $\mathbb{Z}^3$-periodic function. In consequence, its behavior is determined totally on the fundamental cell of the lattice $\mathbb{Z}^3$, i.e., $[-\frac{1}{2},\frac{1}{2})^3$. Even more, for $h \neq 0$, the integral

$$\int_{\mathscr{G}} F(y)\overline{\Phi_h(y)}\ \mathrm{d}V(y) \tag{1.1.27}$$

can be replaced by

$$-\frac{1}{4\pi^2h^2}\int_{\mathscr{G}} F(y)\Delta\overline{\Phi_h(y)}\ \mathrm{d}V(y). \tag{1.1.28}$$

Thus, Green's second theorem in $\mathbb{R}^3$ involving the Laplace operator Δ shows us that

$$\begin{aligned}\int_{\mathscr{G}} F(y)\overline{\Phi_h(y)}\ \mathrm{d}V(y) &= -\frac{1}{4\pi^2h^2}\int_{\mathscr{G}} F(y)\Delta\overline{\Phi_h(y)}\ \mathrm{d}V(y)\\ &= -\frac{1}{4\pi^2h^2}\int_{\mathscr{G}} (\Delta F(y))\overline{\Phi_h(y)}\ \mathrm{d}V(y)\\ &\quad -\frac{1}{4\pi^2h^2}\int_{\partial\mathscr{G}} F(y)\frac{\partial}{\partial\nu}\overline{\Phi_h(y)}\ \mathrm{d}S(y)\\ &\quad +\frac{1}{4\pi^2h^2}\int_{\partial\mathscr{G}} \left(\frac{\partial}{\partial\nu}F(y)\right)\overline{\Phi_h(y)}\ \mathrm{d}S(y),\end{aligned} \tag{1.1.29}$$

provided that $\varrho : \overline{\mathscr{G}} \to \mathbb{R}^3$ is twice continuously differentiable. Note that $\mathrm{d}S$ denotes the surface element and ν_y denotes the outer surface normal at the point y. Observing the Fourier expansion for the lattice function $G(\Delta;\cdot)$, i.e., the Green function with respect to the Laplace operator corresponding to "boundary-conditions" of $\mathbb{Z}^3$-periodicity,

$$G(\Delta; x-y) \sim \sum_{h\in\mathbb{Z}^3\setminus\{0\}} \frac{1}{-4\pi^2h^2}\overline{\Phi_h(y)}\Phi_h(x), \tag{1.1.30}$$

this allows the following reformulation of $R_{\mathbb{Z}^3}(x)$ from (1.1.26):

$$R_{\mathbb{Z}^3}(x) = \int_{\mathscr{G}} G(\Delta; x-y) \left(\Delta_y \frac{\varrho(y)}{|y-a|} \right) \mathrm{d}V(y) \qquad (1.1.31)$$
$$+ \int_{\partial\mathscr{G}} \left(\frac{\partial}{\partial \nu_y} G(\Delta; x-y) \right) \frac{\varrho(y)}{|y-a|} \,\mathrm{d}S(y)$$
$$- \int_{\partial\mathscr{G}} G(\Delta; x-y) \left(\frac{\partial}{\partial \nu_y} \frac{\varrho(y)}{|y-a|} \right) \mathrm{d}S(y),$$

where $a \in \mathbb{R}^3 \setminus \overline{\mathscr{G}}$. In other words, the so-called Euler summation formula involving the Laplace operator

$$\sum_{\substack{g+x\in\overline{\mathscr{G}} \\ g\in\mathbb{Z}^3}}{}' F(x+g) = \int_{\mathscr{G}} F(y)\,\mathrm{d}V(y) + \int_{\mathscr{G}} G(\Delta; x-y) \left(\Delta_y F(y)\right) \mathrm{d}V(y)$$
$$+ \int_{\partial\mathscr{G}} \left(\frac{\partial}{\partial \nu_y} G(\Delta; x-y) \right) F(y)\,\mathrm{d}S(y)$$
$$- \int_{\partial\mathscr{G}} G(\Delta; x-y) \left(\frac{\partial}{\partial \nu} F(y) \right) \mathrm{d}S(y) \qquad (1.1.32)$$

holds true for the function F defined by (1.1.13). Going over to the dilated lattice $\tau\mathbb{Z}^3$ (for sufficiently small $\tau > 0$) it can be verified by techniques as provided in Chap. 10 that

$$\lim_{\tau\to 0+} R_{\tau\mathbb{Z}^3}(x) = 0 \qquad (1.1.33)$$

for all $x \in \mathbb{R}^3$. This illustrates that the volume potential

$$V(a) = \int_{\mathscr{G}} \frac{\varrho(y)}{|y-a|} \,\mathrm{d}V(y), \quad a \in \mathbb{R}^3 \setminus \overline{\mathscr{G}}, \qquad (1.1.34)$$

can be replaced by a sum of single poles

$$\tau^3 \sum_{\substack{\tau g+x\in\overline{\mathscr{G}} \\ g\in\mathbb{Z}^3}}{}' \frac{\varrho(\tau g + x)}{|\tau g + x - a|}, \quad a \in \mathbb{R}^3 \setminus \overline{\mathscr{G}}, \qquad (1.1.35)$$

as far as τ is sufficiently small. This demonstrates the application of lattice point theory in gravitational field modeling (see also Freeden 2011 and the references therein) or, more generally, in numerical integration.

Our book provides a concise introduction to periodization starting with the one-dimensional case in Chap. 9 and verifies the results that have been briefly introduced here in Chap. 10.

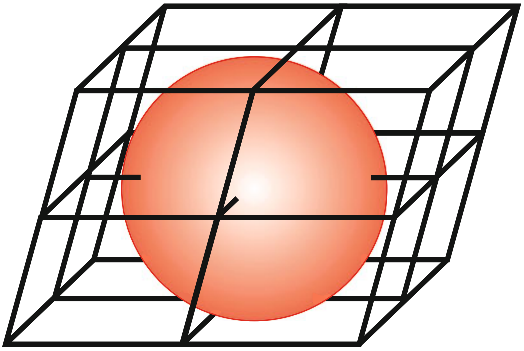

Fig. 1.2 A sphere inside a lattice (for more details on "Spherical Periodization" see Freeden (2011))

The determination of the gravitational potential from airborne or spaceborne data as well as the gravimetry problem of modeling the density distribution in the Earth's interior leads to the rich field of ill-posed or inverse problems for which we refer to, e.g., Anger et al. (1993), Benedetto and Zayed (2004), Colton and Kress (1998), Engl et al. (2000), Engl et al. (1997), Freeden and Michel (2004), Freeden et al. (2010), Kirsch (1996), Louis (1989), Nashed (1976a,b), Nashed and Whaba (1974), and Rieder (2003).

1.2 Example: Geomagnetism (Maxwell's Equations)

The basis of all electromagnetic considerations is the system of *Maxwell's equations* given by

$$\nabla_x \wedge e(x,t) + \frac{\partial}{\partial t} b(x,t) = 0, \qquad \nabla_x \wedge h(x,t) - \frac{\partial}{\partial t} d(x,t) = j_f(x,t),$$
$$\nabla_x \cdot d(x,t) = F_f(x,t), \qquad \nabla_x \cdot b(x,t) = 0, \tag{1.2.1}$$

where the unknowns are defined as follows (note that capital letters are used for scalar fields, lower-case letters for vector fields in $\mathbb{R}^3$):

d	Electric displacement
e	Electric field
h	Magnetic displacement

b	Magnetic field
F_f	Density of free charges
j_f	Density of free currents

As usual, $\nabla_x \wedge$ denotes the curl operator and $\nabla_x \cdot$ stands for the divergence operator. All quantities are understood as averages over a unit volume in space. The electric and magnetic displacement, d and h, can be written as

$$d(x,t) = \varepsilon_0 e(x,t) + p(x,t), \tag{1.2.2}$$

$$h(x,t) = \frac{1}{\mu_0} b(x,t) - m(x,t), \tag{1.2.3}$$

where p is the averaged polarization, m is the (averaged) magnetization, ε_0 is the permittivity of the vacuum and μ_0 is the permeability of vacuum. The total charge and current density, respectively, can be written as the sum of the free charges and currents plus the bounded ones, i.e.,

$$F(x,t) = F_f(x,t) + F_b(x,t), \qquad j(x,t) = j_f(x,t) + j_b(x,t), \tag{1.2.4}$$

where it is well-known that

$$\nabla_x \cdot p(x,t) = -F_b(x,t), \qquad \nabla_x \cdot j_b(x,t) = -\frac{\partial}{\partial t} F_b(x,t) \tag{1.2.5}$$

and

$$\nabla_x \wedge m(x,t) = -\frac{\partial}{\partial t} p(x,t) + j_b(x,t). \tag{1.2.6}$$

We can now reformulate Maxwell's equations:

$$\nabla_x \cdot e(x,t) \overset{(1.2.2)}{=} \frac{1}{\varepsilon_0} \nabla_x \cdot (d(x,t) - p(x,t)) = \frac{1}{\varepsilon_0}(F_f(x,t) + F_b(x,t)), \tag{1.2.7}$$

i.e., we obtain

$$\nabla_x \cdot e(x,t) = \frac{1}{\varepsilon_0} F(x,t). \tag{1.2.8}$$

Furthermore, we have

$$\nabla_x \wedge e(x,t) = -\frac{\partial}{\partial t} b(x,t), \tag{1.2.9}$$

$$\nabla_x \cdot b(x,t) = 0, \tag{1.2.10}$$

as well as by use of (1.2.3)

$$\begin{aligned} \nabla_x \wedge b(x,t) &= \mu_0 \left(\nabla_x \wedge h(x,t) + \nabla_x \wedge m(x,t)\right) \\ &= \mu_0 \left(j_f(x,t) + \frac{\partial}{\partial t} d(x,t) + \nabla_x \wedge m(x,t) \right), \end{aligned} \tag{1.2.11}$$

which becomes

$$\nabla_x \wedge b(x,t) = \mu_0 \left(j_f(x,t) + \nabla_x \wedge m(x,t) + \varepsilon_0 \frac{\partial}{\partial t} e(x,t) + \frac{\partial}{\partial t} p(x,t) \right). \tag{1.2.12}$$

In most geomathematical problems, e.g., satellite geomagnetism, this system of equations is too detailed to describe the occurring phenomena (cf. Backus et al. 1996). They have to be reduced as follows: Let L be the typical length scale of the discussed geomathematical problem and T be the typical time scale. In most problems we have $L = 10^2$ km–10^3 km and $T =$ hours – days, such that we get for the typical velocity of the system

$$\frac{L}{T} \ll c, \tag{1.2.13}$$

where c is the speed of light ($c = 299,792,458$ m/s). Thus, it can be shown, that the term $\varepsilon_0 \frac{\partial}{\partial t} e(x,t) + \frac{\partial}{\partial t} p(x,t)$ can be neglected. Hence, Maxwell's equations partially decouple and the resulting equations for the magnetic field are given by

$$\nabla_x \cdot b(x,t) = 0, \tag{1.2.14}$$

$$\nabla_x \wedge b(x,t) = \mu_0 \left(j_f(x,t) + \nabla_x \wedge m(x,t) \right). \tag{1.2.15}$$

Since div curl $= 0$, we can conclude that by applying $\nabla\cdot$ to (1.2.15)

$$\nabla_x \cdot \left(\mu_0 \left(j_f(x,t) + \nabla_x \wedge m(x,t) \right) \right) = 0. \tag{1.2.16}$$

For solving this system of equations, data of the magnetic field of the Earth are primarily available in the exterior of the Earth, i.e., at the Earth's surface or at satellite altitude. Thus, we can assume that the magnetization m of the surrounding medium can be neglected. Therefore, we arrive at the pre-Maxwell equations

$$\nabla_x \cdot b(x,t) = 0, \qquad \nabla_x \wedge b(x,t) = \mu_0 j_f(x,t). \tag{1.2.17}$$

Furthermore, due to (1.2.16), we have

$$\nabla_x \cdot j_f(x,t) = 0. \tag{1.2.18}$$

In concepts close to the Earth's surface, geoscientists assumed that the current density j is also negligible in the spherical shell $\mathbb{B}^3_{\sigma_1,\sigma_2} = \{x \in \mathbb{R}^3 : \sigma_1 < |x| < \sigma_2\}$, in which the magnetic field is measured. This yields

$$\nabla_x \cdot b(x,t) = 0, \qquad \nabla_x \wedge b(x,t) = 0. \tag{1.2.19}$$

Hence, the magnetic field b can be written as the gradient of a scalar potential U, i.e.,

$$b(x,t) = \nabla_x U(x,t), \quad x \in \mathbb{B}^3_{\sigma_1,\sigma_2}, \tag{1.2.20}$$

where U fulfills the Laplace equation

$$\Delta_x U(x,t) = 0, \quad x \in \mathbb{B}^3_{\sigma_1,\sigma_2}. \tag{1.2.21}$$

This so-called Gauß-representation yields a spherical harmonic expansion of the scalar potential U which is similar to the modeling of the gravitational field of the Earth.

Modern satellite missions like CHAMP, measuring the Earth's magnetic field, are located in the ionosphere, a region where the assumption $j_f = 0$ is not valid anymore. Therefore, we have to deal with the pre-Maxwell equations

$$\nabla_x \cdot b(x,t) = 0, \qquad \nabla_x \wedge b(x,t) = \mu_0 j_f(x,t), \quad x \in \mathbb{B}^3_{\sigma_1,\sigma_2}. \tag{1.2.22}$$

A concept of reflecting this situation has to be applied which is the so-called Mie-representation. This also yields the need for basis systems for vector-valued functions on the sphere, i.e., for vector spherical harmonics. For further information on geomagnetic field modeling see, e.g., Backus et al. (1996), Bayer et al. (2001), Freeden and Gerhards (2012), Gerhards (2011), Maier (2003, 2005), and Mayer (2003) and the references in these publications (see Fig. 1.3 for a graphical illustration).

Reconsidering the set of Maxwell's equation (1.2.1) under the assumption that the fields can be modeled as time-harmonic, i.e., their behavior with respect to t is described by $e^{-i\omega t}$. Moreover, we assume that the density of the free charges F_f and the density of the free currents j_f are both zero. This gives us the following set of equations:

$$\nabla_x \wedge e(x) - i\omega b(x) = 0, \tag{1.2.23}$$

$$\nabla_x \wedge h(x) + i\omega d(x) = 0, \tag{1.2.24}$$

where b and d are both divergence-free. Next, we make use of (1.2.3) as well as (1.2.2) to get rid of the displacements h and d. In doing so, we include the assumption that the polarization p and the magnetization m can both be neglected. Therefore, (1.2.24) becomes

$$\nabla_x \wedge b(x) + i\varepsilon_0\mu_0\omega e(x) = 0, \tag{1.2.25}$$

and e is also divergence-free. Now, we apply the curl operator to (1.2.23) such that we obtain:

$$\nabla_x \wedge (\nabla_x \wedge e(x)) - i\omega \nabla_x \wedge b(x) = 0. \tag{1.2.26}$$

We insert (1.2.25) which gives us:

$$\begin{aligned} 0 &= \nabla_x \wedge (\nabla_x \wedge e(x)) - i\omega(-i\varepsilon_0\mu_0\omega e(x)) \\ &= -\Delta_x e(x) + \nabla_x(\nabla_x \cdot e(x)) - \omega^2\varepsilon_0\mu_0 e(x) \\ &= -\Delta_x e(x) - \omega^2\varepsilon_0\mu_0 e(x), \end{aligned} \tag{1.2.27}$$

where $\nabla_x \cdot e(x) = 0$ has been used. By setting $k^2 = \omega^2\mu_0\varepsilon_0$, we arrive at the Helmholtz equation for the electric field:

$$\Delta_x e + k^2 e = 0. \tag{1.2.28}$$

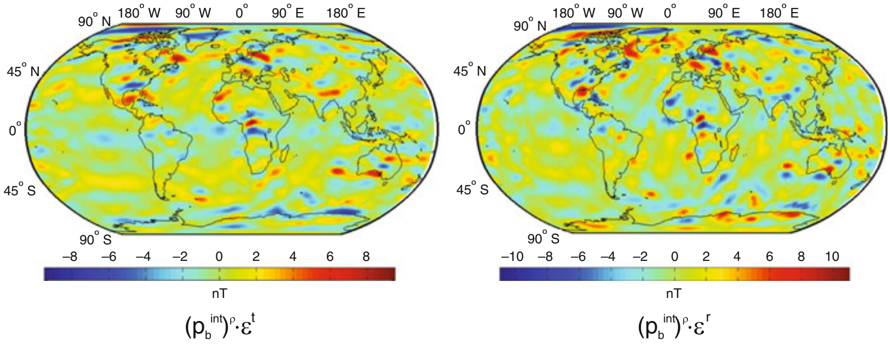

Fig. 1.3 The crustal geomagnetic field of the Earth, north-south component (*left*) and radial component (*right*), see, e.g., Gerhards (2011) and the references therein for more details

Analogously, a Helmholtz equation for the magnetic field b can be found. The analysis of the Helmholtz equation naturally leads to Bessel functions which are discussed in Chaps. 7 and 8. For further details on the theory of electromagnetic waves the reader is referred to, e.g., Colton and Kress (1998), Engl et al. (2000), Jackson (1998), and Müller (1969). For operator-theoretic and computational approaches to ill-posed problems with applications to antenna theory the reader is referred to, e.g., Nashed (1981).

1.3 Example: Fluid Flow (Navier–Stokes Equation)

Our interest is to give a brief derivation of the equations of thermodynamics and fluid dynamics which are used to forecast an atmospheric state. For further details of the deduction of the fundamental equations the reader is referred to, e.g., Ansorge and Sonar (2009), Norbury and Roulstone (2002a,b), Pedlowsky (1979), and Teman (1979, 1983). We consider a meteorological field $\mathscr{F} = \mathscr{F}(t, x)$ (scalar or vectorial) that depends both on time t and space x and assume its differentiability with respect to both arguments. By the Taylor expansion

$$\mathscr{F}(t + \Delta t, x + \Delta x) = \mathscr{F}(t, x) + \frac{\partial \mathscr{F}}{\partial t}\Delta t + (\Delta x \cdot \nabla_x)\mathscr{F}, \tag{1.3.1}$$

where Δt and Δx are displacements in time and space, such that the flow velocity u of the air is given by $\frac{\Delta x}{\Delta t} = u$, we derive

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = (u \cdot \nabla)\mathscr{F} + \frac{\partial \mathscr{F}}{\partial t}. \tag{1.3.2}$$

The term on the left-hand side is the Lagrangian time derivative of $\mathscr{F}$ (the rate of change following a small parcel of air), the term $\frac{\partial \mathscr{F}}{\partial t}$ on the right-hand side is the Eulerian time derivative of $\mathscr{F}$ (rate of change at a fixed point). For the purpose of weather forecasts, the interesting quantity is the local change, i.e.,

$$\frac{\partial \mathscr{F}}{\partial t} = \frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} - (u \cdot \nabla)\mathscr{F}. \tag{1.3.3}$$

If $\mathscr{F}$ is conserved on particles of fluids, i.e., $\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = 0$, the local value at a fixed point can very well be changing because of the advection brought in by the term $-(u \cdot \nabla)\mathscr{F}$. Now, we consider several choices for $\mathscr{F}$ and the corresponding physical laws.

The *first law of thermodynamics* tells us the following relation between temperature and (specific) volume/density of a moving parcel of air

$$c_v \frac{\mathrm{d}T}{\mathrm{d}t} + p \frac{\mathrm{d}\alpha}{\mathrm{d}t} = T \frac{\mathrm{d}s}{\mathrm{d}t} = Q, \tag{1.3.4}$$

where we introduce the following quantities

T	Temperature	α	(Specific) volume
c_v	Specific heat at constant volume	s	(Specific) entropy
p	Pressure	Q	Total heating per unit mass

Combining (1.3.4) with the density $\varrho = \frac{1}{\alpha}$, we obtain

$$c_v \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{p}{\varrho} \frac{\mathrm{d}\varrho}{\mathrm{d}t} = Q. \tag{1.3.5}$$

The *law of mass conservation* says that

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho u) = 0 \tag{1.3.6}$$

such that, by use of (1.3.2),

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho \nabla \cdot u = 0. \tag{1.3.7}$$

Now, we combine (1.3.5), (1.3.7), and (1.3.2) applied to T to get

$$c_v \frac{\partial T}{\partial t} + c_v (u \cdot \nabla) T + \frac{p}{\varrho} \nabla \cdot u = Q. \tag{1.3.8}$$

Moreover, we remember the ideal gas law, i.e.,

$$p = \varrho c_R T \tag{1.3.9}$$

with c_R being the gas constant per unit mass. With the help of Newton's second law of motion we can relate the inertial acceleration of an element of air and the net forces acting on it, i.e., the pressure gradient, gravity, boundary-induced friction, and viscosity. Velocities and accelerations are measured relative to the rotating frame of the solid Earth. Therefore, we introduce Coriolis/centrifugal forces. We find that the Lagrangian rate of change of the velocity u relative to the rotating Earth is governed by the following equation (all forces are expressed per unit mass of air)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -2\omega \wedge u - \frac{1}{\varrho}\nabla p - \nabla\Phi + \nu\Delta u + f, \tag{1.3.10}$$

where we use the following notation

$2\omega \wedge u$	Coriolis force
ω	Rotation vector of the Earth
∇p	Pressure gradient
$\nabla\Phi$	Apparent gravity

$\nu\Delta u$	Friction
ν	Kinematic viscosity
f	Other forces.

Note that the apparent gravity $\nabla\Phi$ is due to the gravitational potential and the centrifugal force $-\omega \wedge (\omega \wedge x)$ at the position x in a frame rotating with the Earth and having its origin at the Earth's center. Equation (1.3.10) is the Navier–Stokes equation of motion and acceleration relative to the Earth. Applying (1.3.2) to the flow velocity u, we can summarize (1.3.6) and (1.3.8)–(1.3.10) to the full set of forecasting equations:

$$\frac{\partial u}{\partial t} = -(u\cdot\nabla)u - 2\omega \wedge u - \frac{1}{\varrho}\nabla p - \nabla\Phi + \nu\Delta u + f, \tag{1.3.11}$$

$$c_v\frac{\partial T}{\partial t} = -c_v(u\cdot\nabla)T - \frac{p}{\varrho}\nabla\cdot u + Q, \tag{1.3.12}$$

$$\frac{\partial \varrho}{\partial t} = -(u\cdot\nabla)\varrho - \varrho\nabla\cdot u, \tag{1.3.13}$$

$$p = \varrho c_R T. \tag{1.3.14}$$

Note that in many climate simulation models and many weather prediction models prognostic equations for local water substance concentration are included. The distribution of water substance greatly effects the heating rate Q. We neither discuss water conservation equations nor the corresponding variations of the gas constant c_R and the specific heat constant c_v. For these considerations the reader is referred to, e.g., Gill (1982).

We now assume *incompressibility* of the fluid, i.e., $\nabla\cdot u = 0$ or $\frac{\mathrm{d}\varrho}{\mathrm{d}t} = 0$. This is justified if the flow is not strongly heated and the density ϱ only slightly varies around the characteristic density ϱ_0, i.e., $\frac{\varrho}{\varrho_0} \approx 0.1$ (see, e.g., Lesieur 1997). This leads us to the simplified equations

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u - 2\omega \wedge u - \frac{1}{\varrho_0}\nabla p - \nabla \Phi + \nu \Delta u + f, \tag{1.3.15}$$

$$\nabla \cdot u = 0. \tag{1.3.16}$$

Finally, we concentrate on tangential streams and assume the Earth to possess a spherical shape, i.e., we discuss tangential, surface divergence free vector fields which we will later call type 3 vector fields (see Chap. 5). This gives us the tangential variants of (1.3.15) and (1.3.16):

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla^*)u - 2\omega \wedge u - \frac{1}{\varrho_0}\nabla^* p - \nabla^* \Phi + \nu \Delta^* u + f, \tag{1.3.17}$$

$$\nabla^* \cdot u = 0. \tag{1.3.18}$$

Note that we denote the tangential parts of differential operators with a star. Going over to the weak formulation of these equations using type 3 test functions, e.g., vector spherical harmonics of type 3, we find that both the pressure surface gradient $\nabla^* p$ and the gravity term $\nabla^* \Phi$ are of type 2 (see Chap. 5). Thus, they both are orthogonal to our test functions. Therefore, we only have to deal with the following Navier–Stokes equation of motion (in the weak sense)

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla^*)u - 2\omega \wedge u + \nu \Delta^* u + f, \tag{1.3.19}$$

$$\nabla^* \cdot u = 0, \tag{1.3.20}$$

where f is projected to the space of type 3 vector fields. Further details on the construction and numerical implementation of the corresponding Galerkin scheme (see Fig. 1.4) can be found in Fengler (2005), Fengler and Freeden (2005), and Marion and Teman (1989). It should be noted that a Poisson equation for the pressure can be derived that involves tensor spherical harmonics (see Freeden and Schreiner 2009 and the references therein) and the surface curl/surface vorticity (see Fengler 2005 and the references therein).

Geostrophic simplifications of the Navier–Stokes equations leading over to the surface modeling of ocean flow by means of the surface curl gradient equation are described in Pedlowsky (1979), Freeden and Schreiner (2009), and Freeden et al. (2010) (see also the references therein).

1.4 Example: Elastic Field (Cauchy–Navier Equation)

First, we are interested in deriving the equations describing the displacement of a solid from the physical laws of conservation. The momentum of the solid enclosed by the sphere $\mathbb{S}_R^2$ of radius R, i.e., within the ball $\mathbb{B}_R^3$ of radius R, at time t is given by