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Egbert Brieskorn
Horst Knörrer

Plane Algebraic Curves

Translated by John Stillwell

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Plane Algebraic Curves

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Reprint of the 1986 Edition

Birkhäuser

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Foreword to the English edition

It is natural that authors should be delighted when their works are translated into other languages, and even more so when the translation is done very nicely – as it is the case with the present translation of our notes on plane algebraic curves by John Stillwell. On the other hand, it is also true that as time goes by one gets more aware of the defects of one's work. One of our friends has criticized us for writing a heavy volume on such an elementary subject, and we have to admit that this criticism is not totally unjustified. However, we would like to say in our defence that a number of people who are now doing research on singularities found the book quite useful as a first introduction, and so it is our hope that readers of the English edition will have the same experience.

We would like to point out that there is now a more effective approach to iterated torus knots than the one presented in this book. This is developed in the beautiful new book by David Eisenbud and Walter Neumann, "Three-dimensional link theory and invariants of plane curve singularities", and in the forthcoming work of Françoise Michel and Claude Weber.

We would like to thank John Stillwell for all his work. We realize that in some instances translating such a book must have been a really difficult task. We feel that he has succeeded very well.

Egbert Brieskorn and Horst Knörrer   Bonn, April 1986

"Es ist die Freude an der Gestalt
in einem höheren Sinne, die den
Geometer ausmacht."
(Clebsch, in memory of Julius
Plücker, Göttinger Abh. Bd. 15).

Foreword

In the summer of 1976 and winter 1975/76 I gave an introductory course on plane algebraic curves to undergraduate students. I wrote a manuscript of the course for them. Since I took some trouble over it, and some colleagues have shown interest in this manuscript, I have now allowed it to be reproduced, in the hope that others may find it useful.

In this foreword I should like to explain what I wanted to achieve with this course. I wanted – above all – to show, by means of beautiful, simple and concrete examples of curves in the complex projective plane, the interplay between algebraic, analytic and topological methods in the investigation of these geometric objects. I did not succeed in developing the theory of algebraic curves as far as is possible – I must even say that in some places the course stops where the theory is just beginning. Rather, I aimed to allow the listeners to develop as much familiarity as possible with the new objects, and the best possible intuition. For this reason I almost always used the most elementary and concrete methods. Also for this reason, I have taken the trouble to make a great number of drawings. I once read a remark of Felix Klein to the effect that what a geometer values in his science is that he sees what he thinks.

Another principle which I have tried to put into effect in this course is that of breaking through the formal lecture style – the style which replaces the development of ideas by a staccato of definitions, theorems and proofs. Thus heuristic, historical and methodological considerations took up a substantial portion of the course, which they likewise occupy in the manuscript. I am well aware of the disadvantages of this method for the reader, and the resulting lack of formal precision, conciseness, clarity and elegance is an annoyance to me too. However, I have accepted this annoyance in order to be able to develop the ideas in a natural way and to promote understanding and thought.

I have developed no new scientific ideas in this course, but have drawn much from other sources. Thus in Chappter II I have depended heavily on notes of a course by R. Remmert on algebraic curves which brought me into contact with algebraic geometry for the first time as a student, and on the book by Walker. I have also used the introduction to algebraic geometry by van der Waerden. In the historical remarks I have relied a lot on the corresponding Enzyclopädie articles and the books of Smith and Struik. My aim was not historical refinement but to give students a picture of the beginnings from which the theory has developed. The whole later history – from the second half of the 19^{th} century onwards – was not so important for this pedagogical purpose, and for it I refer to the new book of Dieudonné or the beginning of the book by Shafarevich. For the local investigations of topology and resolution of singularities I have depended on lecture notes of F. Pham and H. Hironaka as well as original work of A'Campo. Finally, I have used many other sources, which I perhaps have not always acknowledged. I hope the authors will forgive me.

What is lacking in the course? It lacks a chapter on the deformation of singularities, in which I would have liked to introduce the beautiful results of A'Campo and Gusein-Zade on the computation of the monodromy groups of plane curves. For this I refer to the report of A'Campo to the International Congress of Mathematicians in Vancouver in 1974. I have tried to admit the deformation viewpoint at least implicitly in sections 8.5, 9.2 and 9.3. What is not lacking? There is no lack of introductions to modern algebraic geometry. This course is not intended to be such an introduction. For this purpose there are now the beautiful books of D. Mumford, I.R. Shafarevich, P.A. Griffiths and R. Hartshorne. The purpose of the course is to familiarise students, in a natural, intuitive and concrete way, with the various methods for the investigation of singularities, and to lead in this way to my own field of work. I believe that it has reached this goal. This is shown by a series of beautiful <u>Diplomarbeiten</u> which have come into being in the meantime. If the notes presented here can also serve others similarly, then they have fulfilled their purpose, even if only as a source of suggestions or as a collection of material.

In conclusion, I should like to thank all of those who have helped in the production of this extensive piece of work : diploma mathematician Mr. Ebeling and above all Dr. Knörrer for working out some lectures in Chapter III, and for a critical inspection and proof-reading of the manuscript, and help with assembling the references, the three secretaries Mrs. Schmirler, Mrs. Weiss and Mrs. Eligehausen for the laborious and alienating work of typing the text, and the printers at the Mathematische Institut, Mr. Vogt and Mr. Popp, for printing the almost one thousand pages. But above all I should like to thank two students, Mr. Koch and Mr. Scholz, for taking on the enormous job of producing the manuscript, proof-reading, making the index and redrawing more nicely many of the figures. Without them the manuscript would never have been completed. Once again : many thanks!

Bonn, Spring 1978 E. Brieskorn

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I. HISTORY OF ALGEBRAIC CURVES

The plane algebraic curves have a history of more than 2000 years. At the end of this development stands a definition of algebraic curves, an understanding of the main problems of the theory, and methods for handling them, to the extent that almost all problems about algebraic curves admit methodical treatment and solution. These questions and methods are part of a general field, algebraic geometry, whose development proceeded mainly during the past and present century and still continues. The theory of algebraic curves is a part of this general theory which has the character of a paradigm, and thereby serves as a good introduction to it for the beginner. Today this is the main point of view in introductory treatments of this field, e.g. the book of Fulton [Fl].

This viewpoint will also play a role in our course, but it will not be the only one. Our emphasis will be not so much on the development of the algebraic conceptual apparatus and the algebraic methods, as on the geometric viewpoint, where possible, in working out some of the many connections between this field and analysis and topology. Moreover, I will not, as is almost always done today, completely ignore the two-thousand-year history of algebraic curves.

In the first lectures of this course I shall give an introduction to this history, naturally quite sketchy, in which I omit all proofs to save time. Of course I shall later formulate and prove all assertions with the usual rigour.

The purpose of this historical introduction is to give provisional answers to the following questions :

1. What is the origin of the objects of our investigation, the algebraic curves? Why were mathematicians concerned with them originally?
2. From which viewpoints have they been investigated? What has led

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to the viewpoints and methods which predominate today?

3. What were or are the main objects of investigation?

It will be understood that the answers to these questions, as long as theory has not been given, will be incomplete. For those who are interested in more details, I recommend looking at older books on algebraic curves. I particularly recommend the following literature: the survey articles of Berzolari [B2] and Kohn-Loria [K3] in Band III, 2.1 of the Encyclopädie der Mathematischen Wissenschaften, as well as the books of Loria [L4] and Gomes Teixeira [T2].

1. Origin and generation of curves

First I should like to tell you something of the reasons why the first interesting curves were already considered in antiquity. We shall then see how these were taken up again in the Renaissance, after which there was a permanent enrichment of the theory by new content and methods. We shall see that there were many causes for the origin and generation of algebraic curves: the development of historically important mathematical problems, playful mathematical constructions and the joy of solving problems, but also, and very important in this field, numerous applications of mathematics in other fields: perspective, optics, astronomy, architecture, kinematics, mechanics and technology.

1.1 The circle and the straight line

The simplest curves are the circle and the line. Their origins lie in prehistoric times. Knowledge of them was necessary for the solution of numerous practical problems, such as land measurement (= geometry in the original meaning of the word) and building construction. Straight lines and circles are treated intuitively in the earliest mathematical texts, and the first attempt at strict definition and proof in mathematics, about 600 B.C. (Thales), concerns these objects. The attempt of the Greek mathematicians, such as Euclid (300 B.C.), to define a line was inadequate from a logical standpoint, even though the Greeks of course already knew almost all essential properties of lines in the plane. From our present-day standpoint, the question of the definition of lines depends on the conceptual framework : one can make the line an implicitly defined basic axiomatic concept as, e.g., in Hilbert's axiomatisation of euclidean geometry, or take it to be a derived concept as, e.g., in analytic geometry or linear algebra, where it is a 1-dimensional affine subspace.

The definition of the circle causes no difficulties – and did not cause any for Euclid – when one assumes the euclidean plane as known : it is the locus of all points in the plane which have a given distance from a given point. This type of definition of curves as loci was typical of the way the Greeks handled curves. They were defined as the loci of points having certain distance relationships (specific for each curve) to given points, lines and circles.

To construct, i.e. draw, a circle one uses a simple mechanism, the pair of compasses. To draw a straight line one mostly uses a somewhat problematic instrument, the ruler, and hence a template.

It is very clear that numerous simple practical and mathematical problems can be solved by construction with compasses and ruler. From the analytic-algebraic standpoint, the construction of the intersection of two lines is the solution of two linear equations, the construction of the intersection of a line and a circle, or of two circles, is the solution of a quadratic equation. By iterating such constructions one can therefore solve all problems of the form : from given segments of length $a_1,\dots,a_n$, construct a segment of length a , where a results from $a_1,\dots,a_n$ by repeated rational operations and extractions of square roots. Criteria for this to be the case are obtained from Galois theory. Of course these conditions for the solvability of problems by ruler and compass construction were unknown to the Greeks, so it is understandable why they tried to solve the great problems of antiquity by such constructions.

1.2 The classical problems of antiquity

The classical problems of antiquity were:

1. The trisection of an arbitrary angle.
2. Squaring the circle (before 1500 B.C.?).
3. Duplication of the cube. (Delian problem) (5th century B.C.?)

Squaring the circle means determining the number π. Lindemann (1882) showed π is transcendental, i.e. not the root of any algebraic equation

$$x^n + a_1x^{n-1} + \dots + a_n = 0$$

with rational coefficients a_i. This means, in particular, that π cannot be constructed with compasses and ruler.

Duplication of the cube leads to solving the equation $x^3 - 2 = 0$ which, by Galois theory, likewise cannot be done by ruler and compass construction.

As for the trisection of an arbitrary angle α : the addition theorem for trigonometric functions immediately yields

$$\sin 3\beta = 3 \sin \beta - 4 \sin^3 \beta,$$

hence, setting $\beta = \alpha/3$, $x = \sin \beta$ and $c = \sin \alpha$,

$$4x^3 - 3x + c = 0,$$

and it again follows from Galois theory that the solution of this equation for arbitrary c cannot be found by ruler and compass construction.

Thus the classical problems cannot be solved by ruler and compass constructions. It is true that the Greeks had no proof of this, but they saw the futility of their attempts – and found solutions with the help of curves less simple than the circle and the line. I shall say something about them in what follows.

1.3 The conic sections

The discovery of the conic sections is attributed to Menaechmus (c. 350 B.C.). They were intensively investigated by Apollonius of Perga (c. 225 B.C.). These mathematicians generated the conic sections as intersections of cones with planes. Apollonius and, to some extent, Menaechmus could also characterise the conic sections by area application properties, which are expressed in today's notation by

$$y^2 = px + qx^2.$$

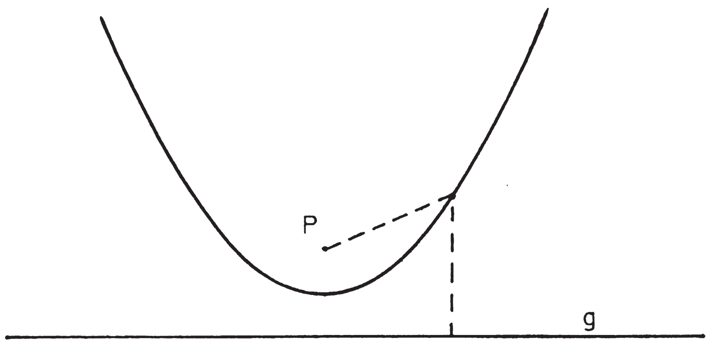
In the terminology of Apollonius we obtain for

$q = 0$ the parabola, i.e. "application"
$q > 0$ the hyperbola, i.e. "application with excess"
$q < 0$ the ellipse, i.e. "application with defect".

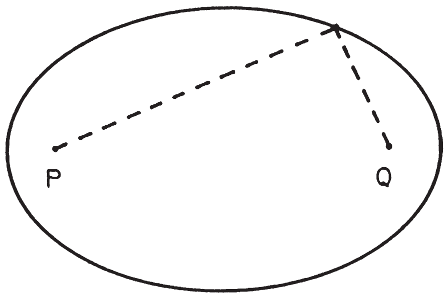
Of course the Greeks also knew the definitions of these curves as loci.

A parabola is the locus of all points having equal distances from a given point P and a given line g.

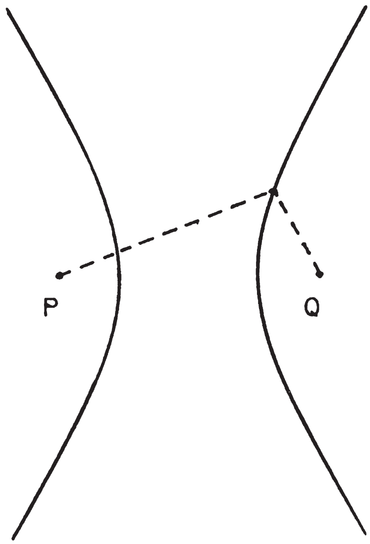
An ellipse (hyperbola) is the locus of all points for which the sum (difference) of distances from two given points P, Q has a fixed value.



Parabola



Ellipse



Hyperbola

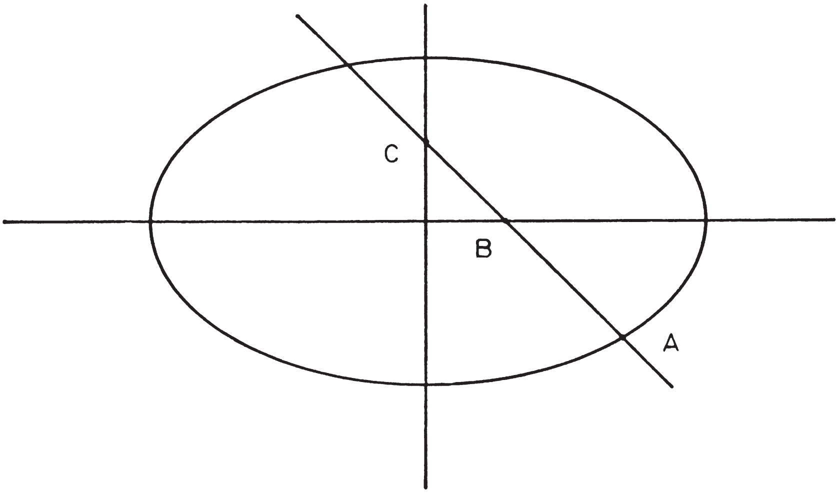
Menaechmus had already seen that one could use conic sections to solve the Delian problem. Determination of the intersection of the parabolas with equations

$$y^2 = 2x$$
$$x^2 = y$$

leads to the solution of the equation

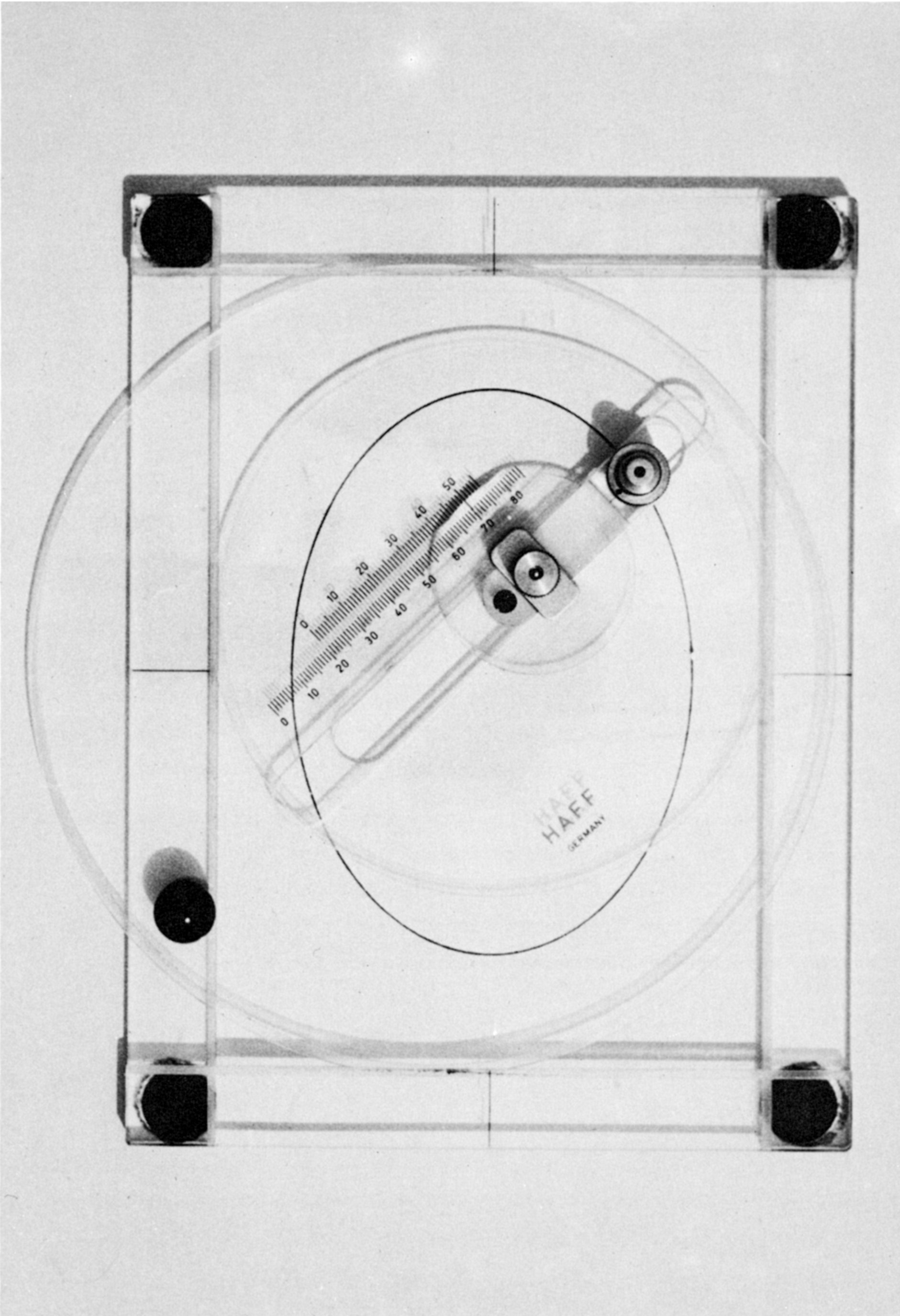
$$2x = x^4$$

and hence to $x = 0$ and $x = \sqrt[3]{2}$, whence one has a method to solve the Delian problem when one has a method to construct parabolas which yields not just particular points but the whole curve. It is not known which constructions the Greeks used to draw the conic sections. Later, many such constructions – so-called "organic generations" - were found for conic sections. The best known is undoubtedly the generation of an ellipse by a point A on a line which moves in the plane so that two other points B, C on the line move on a pair of axes.



By using this manner of generation one can construct an apparatus convenient for drawing ellipses. The next page shows such an ellipsograph. There are similar parabolagraphs and hyperbolagraphs.

Like the Delian problem, the angle trisection problem can also be solved with the help of conic sections, but I shall not go further into this. Of course there is no such solution for squaring the circle, but here the Greeks also found a way : they used certain other curves, such as the quadratrix of Hippias (c. 425 B.C.).

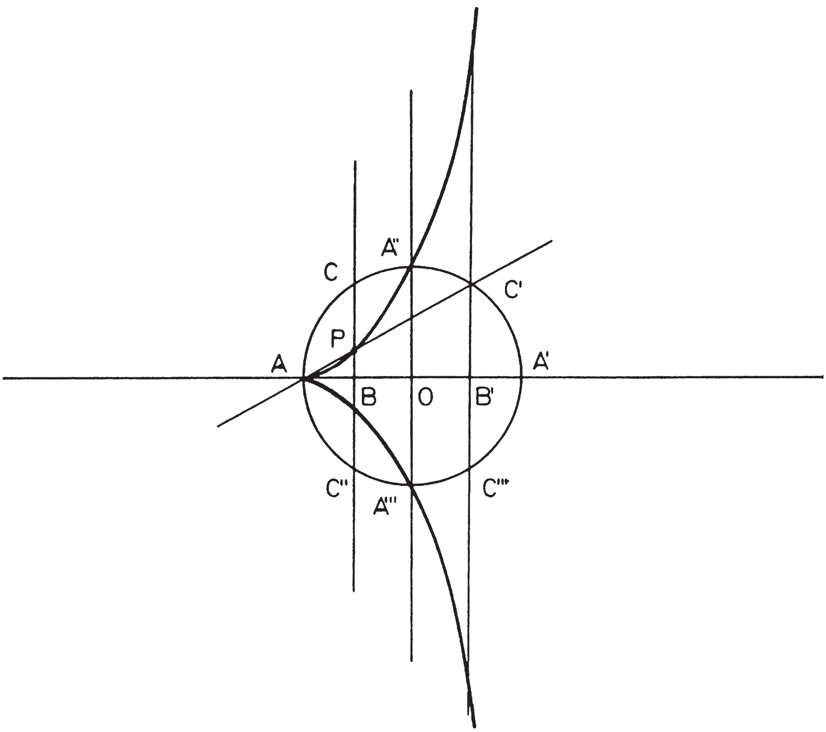


Ellipsograph

1.4 The cissoid of Diocles

Apart from solutions of the problems of doubling the cube and trisecting the angle with the help of conic sections, the Greeks found still other solutions with the help of other curves. The oldest and most important of these curves were the cissoid of Diocles and the conchoid of Nicomedes.

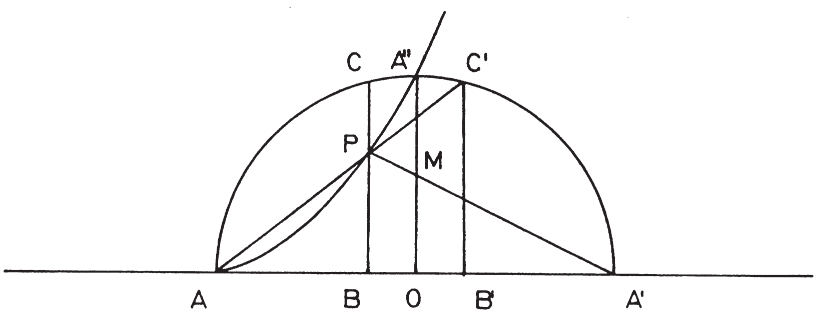
The cissoid is constructed pointwise as follows. We draw a circle about the intersection 0 of two perpendicular lines $\overline{AA'}$ and $\overline{A''A'''}$. Parallel to $\overline{A''A'''}$ we draw two lines equidistant from it, meeting $\overline{AA'}$ in B and B' respectively. Let their intersections with the circle be C, C', C'', C''' as in the figure. The line through one of these points, say C', and A cuts the other parallel, $\overline{CC''}$, in a point P. The totality of points constructed in this way is the cissoid of Diocles.



The name "cissoid" was used for this curve from the early seventeenth century on. The Greek word κισσοειδής means "ivy-shaped". We know from Proclus and Pappus that there were curves called κισσοειδής (e.g. Pappus, Collection III, 20 and IV, 58). We do not know which curves were so called, but probably the cissoid of Diocles was not one of them ([T3], p. 24). We may assume that the word κισσοειδής refers to the way the curve comes to a singular point, "making an angle with itself", as the Greeks put it, reminding us in a way of the edge of an ivy leaf.

The curve obviously goes through the points A, A'', A''' of the circle and has the tangent to the circle at A' as asymptote. The cissoid has a cusp or, as one says, a return point. Such singular points will be carefully studied in this course, and the cissoid is perhaps the oldest example (c. 180 B.C.) of a curve with such a singularity.

We now suppose that the cissoid is already drawn, as described above. Then Diocles solves the Delian problem by the following construction :



Let M be the midpoint of $\overline{OA''}$ and let P be the intersection of $\overline{A'M}$ with the cissoid, with $\overline{CB}$ and $\overline{C'B'}$ as above. Let x,y,z be the segments x = A'B, y = BC, z = AB. Then obviously

$$\frac{A'B}{PB} = \frac{A'O}{MO} = 2$$

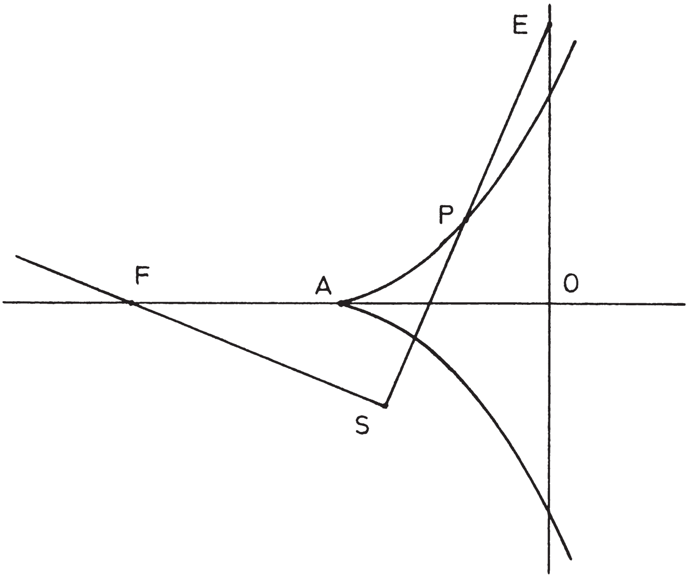
$$\frac{A'B}{BC} = \frac{CB}{AB} = \frac{C'B'}{B'A'} = \frac{AB'}{B'C'} = \frac{AB}{BP}$$

hence

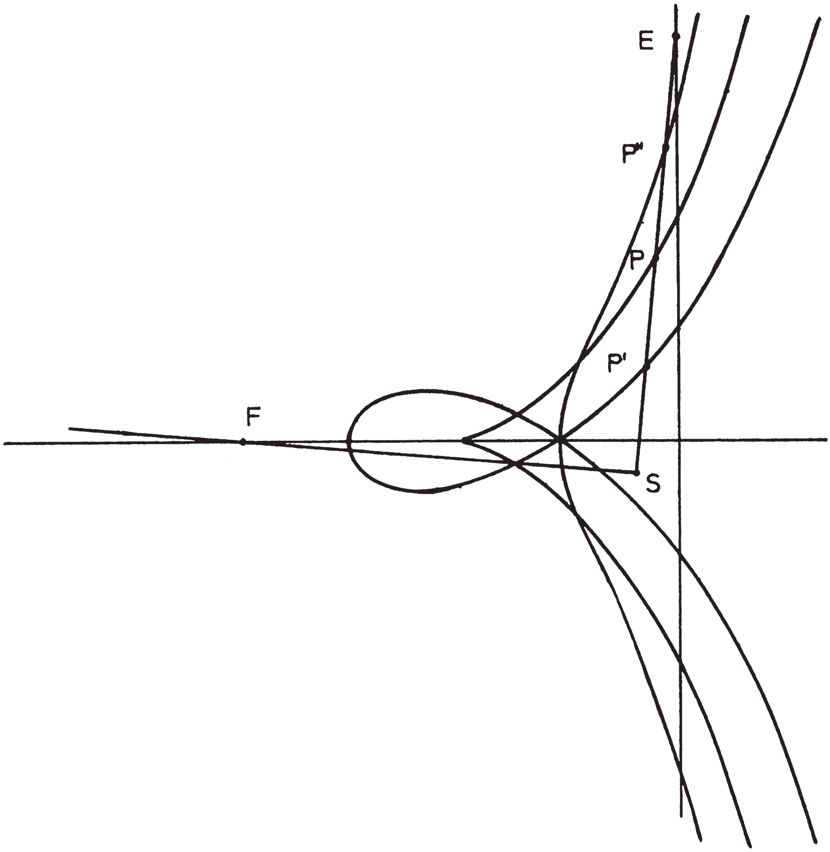
$$\frac{x}{y} = \frac{y}{z} = \frac{z}{\frac{1}{2}x} ,$$

which immediately implies $x^3 = 2y^3$, and hence the Delian problem is solved.

For this to really solve the problem it is of course necessary to have not just a pointwise construction of the cissoid, but a process which continuously draws a whole piece of the curve, and hence an organic generation. The following organic generation of the cissoid was given by Newton.



A right angle with an arm of fixed length 2r is moved in the plane in such a way that its endpoint E moves on a line and the other arm always goes through a fixed point F at distance 2r from the line. The midpoint P of the arm $\overline{SE}$ then describes the cissoid. It is interesting, incidentally, to use other points P', P'' on the arm $\overline{SE}$ in place of P. These describe curves like the following:

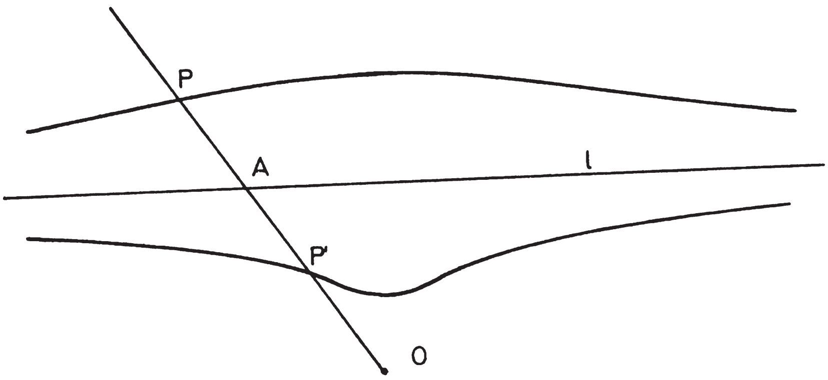


In this way one obtains a whole 1-parameter family of curves which are in a sense deformations of the cissoid.

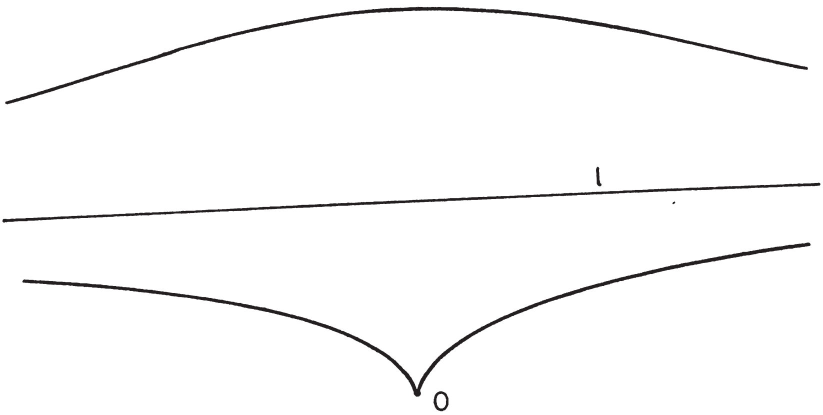
1.5 The conchoid of Nicomedes

The conchoids of Nicomedes are curves, found c. 180 B.C., which solve the Delian problem and also the problem of trisecting the angle.

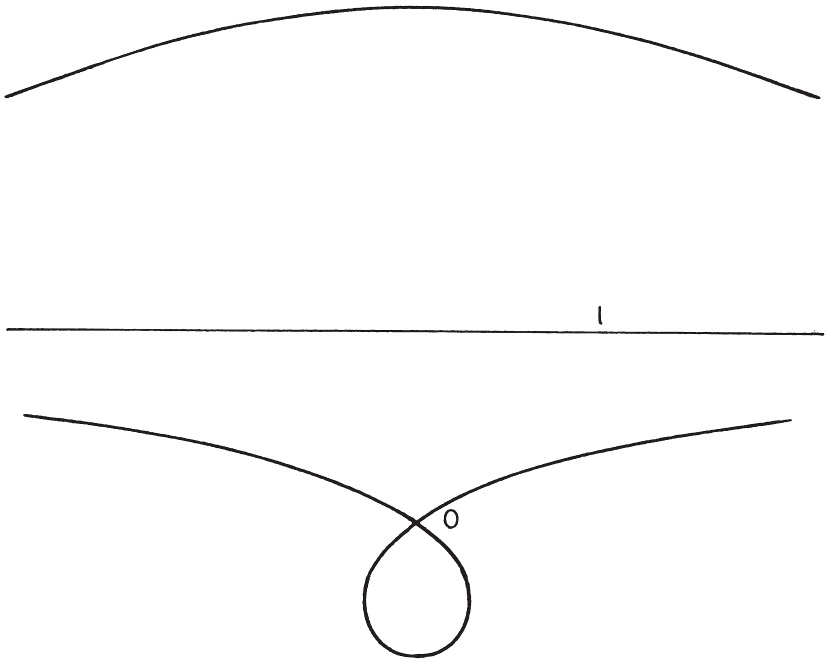
They are constructed as follows. Given a line ℓ, a point 0 at distance d from ℓ, and a segment k, let A be an arbitrary point on ℓ and P, P' the points on the line $\overline{OA}$ at distance k from A. The locus of all these points P, P' is a conchoid. Its form depends on the relation between d and k as follows:



$k < d$



$k = d$



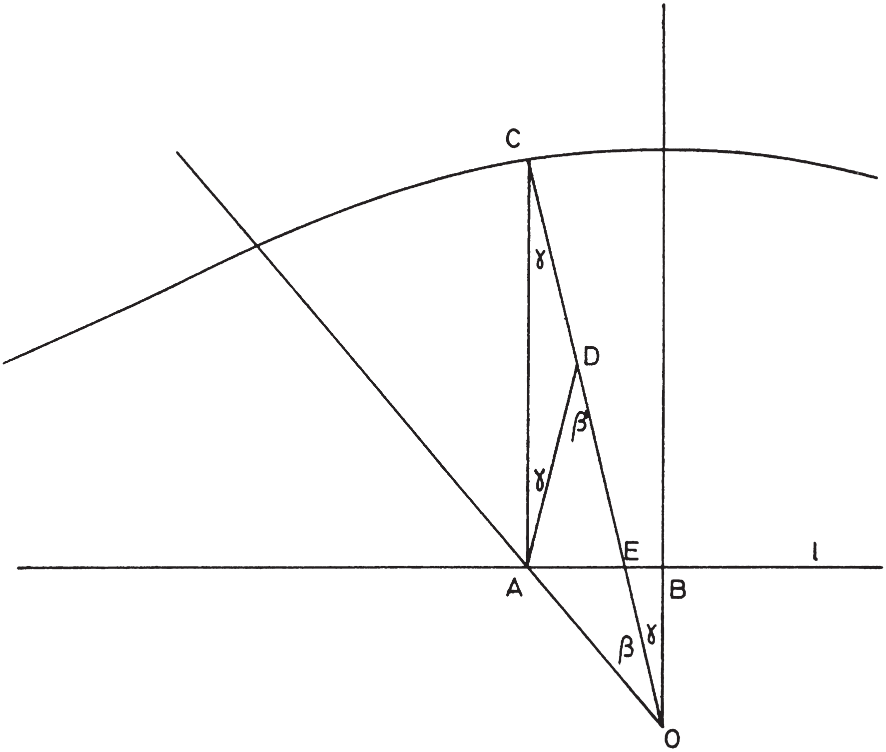
k > d

Thus conchoids have two branches, one of which can have a cusp or double point. The name stems from the shell-like shape (κόγχη = concha = seashell).

In antiquity the conchoid was also used for the construction of vertical sections of columns.

We remark that the construction just described yields an organic generation. From it we obtain the following process of Nicomedes for the trisection of an arbitrary angle α.

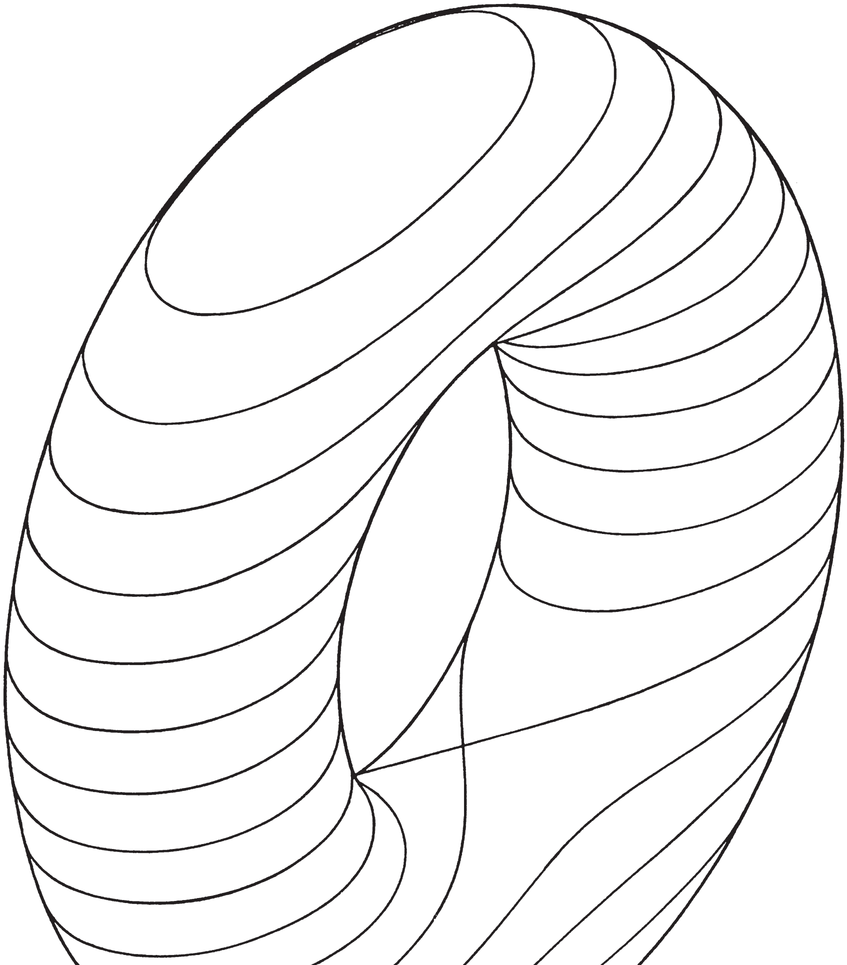
Let $\sphericalangle AOB$ be the given acute angle α, where B is the foot of the perpendicular ℓ from A onto $\overline{OB}$. One draws the conchoid for ℓ and O with $k = 2\overline{OA}$. The parallel to $\overline{OB}$ through A cuts the conchoid on the side away from O at C. Let $\gamma = \sphericalangle BOC$. Then $\gamma = \frac{\alpha}{3}$ and hence the problem is solved.



Proof : Let E be the intersection of AB and CO and let D be the midpoint of the segment $\overline{CE}$. Then $\overline{DA} = \overline{AO}$ by construction of the conchoid. Then the base angles of the isosceles triangle ODA satisfy $\beta = \beta'$. Thus it follows from $\beta + \gamma = \alpha$ and $\beta' = 2\gamma$ that $\alpha = 3\gamma$.

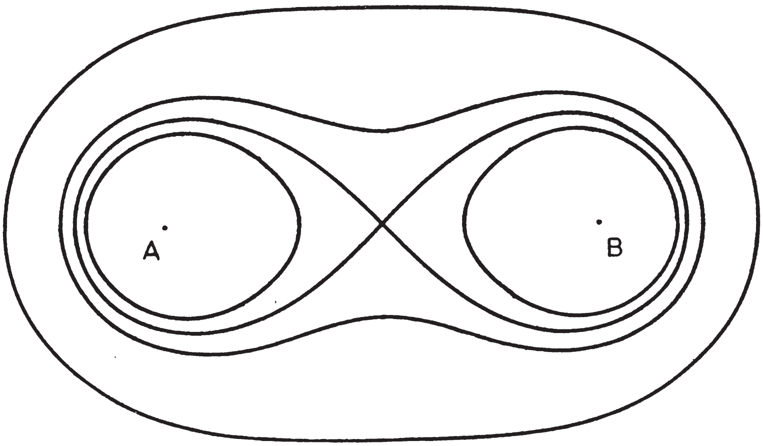
1.6 The spiric sections of Perseus

Following Menaechmus' construction of the ellipse, hyperbola and parabola by cutting a cone by a plane, around 150 B.C. the Greek mathematician Perseus had the idea of cutting a torus by a plane parallel to the axis of rotation, and thereby obtained interesting curves. Because the Greeks called the torus the spira (σπεῖρα), these curves are called the spiric sections of Perseus. We shall not construct these curves exactly here, but only give a rough picture of their form by a drawing.



Perhaps this look at the spiric sections of Perseus will remind some of you of the Cassini curves. Rightly so!

The Cassini curves are defined as follows : one is given two points A, B distance 2a apart, and a positive number c. Then the associated Cassini curve is the locus of all points P such that $\overline{PA} \cdot \overline{PB} = c^2$. For fixed A, B and different values of c these curves look like the following. These curves were found by the astronomer Giovanni Domenico Cassini (c. 1650-1700).



He believed that the sun travelled around the earth on one such convex curve, with the earth at a focus.

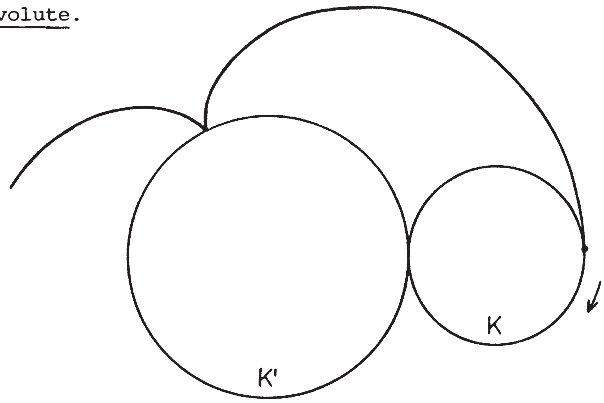
For $a = c$ the curve has an ordinary double point and forms a double loop. This is the lemniscate of Jacob Bernoulli (1694). (ληµνίσκος = loop in the form of an 8). This curve played an important role in the development of the theory of elliptic functions. When the torus on which we consider the spiric sections of Perseus results from rotation of a circle of radius r about an axis at distance R from the centre of the circle, and when the plane of intersection is distance d from the axis, then the spiric sections of Perseus are Cassini curves precisely when $d = r$. Then $a = R$ and $2rR = c^2$. Thus the Cassini curves are special cases of the spiric sections of Perseus.

1.7 From the epicycles of Hipparchos to the Wankel motor

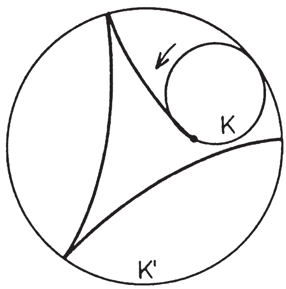
Mathematics is indebted to astronomy for the knowledge of a series of interesting special curves. We have met one example already : the Cassini curves. A more important example was the epicycle, used by the ancient astronomers Hipparchos (c. 180-125 B.C.) and Ptolemy (c. 150 A.D.) to describe the paths of the planets. Such a curve is the path of a point on a circle which turns with constant angular velocity about its centre, while the centre at the same time travels with constant angular velocity on another circle.

During the Renaissance another interesting class of curves, the wheel curves, was found, and it was noticed only later that these wheel curves were included among the classical epicyclic curves.

A wheel curve is the path of a point on a circle K which rolls on a fixed circle K' without slipping. One distinguishes between epicycloids, hypocycloids and pericycloids according to the position of the circles. If K' degenerates to a line, then the resulting curve is called a cycloid. If K degenerates to a line, then the curve is called a circle involute.



Epicycloid



Hypocycloid