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Editors

Spectral Analysis of Quantum Hamiltonians

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Preface

Spectral analysis has been thriving for years as a vital part of mathematical physics and analysis. Conferences dealing with various aspects of spectral analysis are frequently organized, but the spectral community is only beginning to build its own organization. The International Spectral Network was formed in 2010 by a number of research groups around the world with the aim of fostering worldwide collaboration in the field. One of the aims of the Network is to establish Spectral Days as the regular conference on spectral analysis where researchers will meet to discuss and record recent progress.

Spectral Days 2010 was the first conference organized by the Spectral Network. It took place in Santiago, Chile, in September 2010. This volume contains expanded versions of many of the research talks and surveys delivered in Santiago. The second Spectral Days conference took place in Munich in April 2012 and we hope that the reader will soon peruse the second of many volumes covering recent progress in Spectral Analysis.

This volume contains surveys as well as research articles in topics ranging from spectral continuity for magnetic and pseudodifferential operators, localization in random media, stability of matter, to properties of Aharonov–Bohm Hamiltonians and spectral properties of Quantum Hall Hamiltonians. Also included are studies of operators associated to waveguides, resonance in time-dependent systems, supersymmetric models with singular potentials, non-linear equations related to bosonic strings, dissipative fermion systems and conjugate operators in connection with time operators and time delay.

We take the opportunity to heartily thank the Chilean *Iniciativa Científica Milenio* whose generous funding of the Scientific Nucleus ICM P07-027-F "Mathematical Theory of Quantum and Classical Magnetic Systems" made the conference possible. We also thank the Mathematics Faculty of the Pontificia Universidad Católica de Chile for hosting and supporting the conference. Our thanks also go to the International Spectral Network for choosing Santiago as the first venue of Spectral Days and for travel support for several participants.

We also wish to warmly thank Liliya Simeonova. Without her help, Spectral Days 2010 could not have taken place and this volume would not have seen the light of day.

Rafael Benguria   Eduardo Friedman   Marius Mantoiu

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Remarks on Sojourn Time Estimates for Periodic Time-dependent Quantum Systems

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Abstract. We study some solutions of the Schrödinger equation,

$$i\frac{\partial u}{\partial t} = H(t)u$$

where $H(\cdot)$ is a periodic time-dependent Hamiltonian acting on a Hilbert space $\mathcal{H}$. We prove sojourn time estimates, first by means of an extension of the energy-time Uncertainty Principle, and then, for a specific model by explicit computations.

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Keywords. Sojourn time, uncertainty principle, time periodic quantum systems.

1. Introduction

The long-time behavior of a time-dependent quantum system is related to the spectral properties of its Floquet Hamiltonian [5], [6] or its Floquet operator in the periodic case [4]. Among the systems considered in the literature, a particular attention has been devoted to time-periodic perturbations of absolutely continuous systems (see for example [1], [2], [14] and references therein).

The long time behavior is also closely connected with the concept of resonance, understood here in a dynamical way. Roughly speaking, a state is resonant if it exhibits slow exponential decay of its autocorrelation function at least for a certain time range [3], [12]. For precise results in the case of perturbation of a Hamiltonian having a simple embedded bound state, see [3] and references therein. In [8], the author shows this behavior for a truncated resonant state, see also [13].

In this paper we will concentrate on the slowness of the decay instead of its precise nature. To this end we will establish lower bounds on the sojourn time,

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2) Fondecyt 1080675, ECOS-Conicyt 2010-022, ICM PROY-P07-027-F,
3) Fondecyt 1080455,
4) Fondecyt 1100304, ECOS-Conicyt 2010-022.

i.e., an integrated version of the autocorrelation function of "resonant" states. As a thumb rule remark that (a hypothetical decay) $t \mapsto \exp(-\Gamma|t|)$ would lead to a sojourn time of order $\mathcal{O}(\Gamma^{-1})$. In the following, we show an energy-time Uncertainty Principle, which provides a lower bound for the sojourn time. We review Lavine's results for the autonomous case and extend them for abstract periodic time-dependent Hamiltonians. As an application we obtain a time-dependent version of the results in [8]. In Section 3, we illustrate the resonant behavior for a specific non-autonomous model by explicit computations.

2. Energy-time Uncertainty Principle

We establish an energy-time Uncertainty Principle for periodic time-dependent Hamiltonians. To this end we will follow the framework of Floquet theory.

2.1. The autonomous case: a review

We begin by recalling the Uncertainty Principle for the autonomous case, following a result of Lavine [7].

We recall that the solution of the autonomous Schrödinger equation

$$i\frac{\partial u}{\partial t} = Hu \quad , \quad u(s) = \varphi \in \mathcal{D}(H)$$

where $s \in \mathbb{R}$ and $\mathcal{D}(H)$ denotes the domain of the self-adjoint operator H, is expressed by $u(t) = e^{-iH(t-s)}\varphi$.

Definition 2.1. *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and A be a bounded self-adjoint operator acting on $\mathcal{H}$. Let us denote by $\mathcal{T}_H(A)$, the operator whose quadratic form is,*

$$\langle \varphi, \mathcal{T}_H(A)\varphi\rangle = \int_{-\infty}^{\infty} \langle e^{-iH\sigma}\varphi, Ae^{-iH\sigma}\varphi\rangle \, d\sigma \ , \qquad (2.1)$$

defined for all $\varphi \in \mathcal{H}$ such that the integral converges.

If φ in $\mathcal{H}$ is a normalized state, i.e., $\|\varphi\| = 1$, its sojourn time is defined by $\mathcal{T}(\varphi) = \mathcal{T}_H(|\varphi >< \varphi|)$ where $|\varphi >< \varphi|$ is the one-rank projector in the direction of φ:

$$\mathcal{T}(\varphi) = \int_{-\infty}^{\infty} |\langle \varphi, e^{-iH\sigma}\varphi\rangle|^2 \, d\sigma \ .$$

Remark: In Definition 2.1, $\mathcal{T}(\varphi)$ represents the total expected amount of time the system spends in its initial state φ.

In the following, unless otherwise stated, we will suppose that the state φ is normalized and belongs to the domains of the operators acting on it.

Lemma 2.1. *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(H)$. Let $\lambda_0 \in \mathbb{R}$ and $\varphi \in \mathcal{D}(H)$ such that $\|\varphi\| = 1$ and $\epsilon \equiv \|(H-\lambda_0)\varphi\| > 0$. Then*

$$1 \leq 2\,\epsilon\,\|(H-\lambda_0)\varphi\| \; \|(H-\lambda_0-i\epsilon)^{-1}\varphi\|^2 \ . \qquad (2.2)$$

Proof. Let $d\langle\varphi, E_\lambda\varphi\rangle$ denote the spectral measure associated to the self-adjoint operator H. Hence, $1 = \|\varphi\|^2 = |\int d\langle\varphi, E_\lambda\varphi\rangle|^2$. By Hölder's inequality,

$$1 \leq \left(\int_{-\infty}^{\infty} \frac{(\lambda-\lambda_0)^2+\epsilon^2}{\epsilon}\, d\langle\varphi, E_\lambda\varphi\rangle\right)\left(\int_{-\infty}^{\infty} \frac{\epsilon}{(\lambda-\lambda_0)^2+\epsilon^2}\, d\langle\varphi, E_\lambda\varphi\rangle\right) \quad (2.3)$$

for any $\epsilon > 0$. Using the spectral theorem for H we obtain,

$$\int_{-\infty}^{\infty} \frac{(\lambda-\lambda_0)^2+\epsilon^2}{\epsilon}\, d\langle\varphi, E_\lambda\varphi\rangle = \frac{1}{\epsilon}\|(H-\lambda_0)\varphi\|^2 + \epsilon\,.$$

If we choose $\epsilon = \|(H-\lambda_0)\varphi\|$, then

$$\int_{-\infty}^{\infty} \frac{(\lambda-\lambda_0)^2+\epsilon^2}{\epsilon}\, d\langle\varphi, E_\lambda\varphi\rangle = 2\epsilon\,. \quad (2.4)$$

On the other hand, again, applying the spectral theorem we have

$$\int \frac{\epsilon}{(\lambda-\lambda_0)^2+\epsilon^2}\, d\langle\varphi, E_\lambda\varphi\rangle = \epsilon\,\|(H-\lambda_0-i\epsilon)^{-1}\varphi\|^2\,.$$

Clearly the lemma now follows from the above identity together with (2.3) and (2.4). □

Lemma 2.2. *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(H)$. Let $\lambda_0 \in I\!R$, $\varphi \in \mathcal{D}(H)$ such that $\|\varphi\| = 1$ and $\epsilon > 0$. We have the following lower bounds for the sojourn time $\mathcal{T}(\varphi)$:*

$$\mathcal{T}(\varphi) \geq 2\epsilon\,|\langle\varphi, (H-\lambda_0-i\epsilon)^{-1}\varphi\rangle|^2$$
$$\mathcal{T}(\varphi) \geq 4\epsilon^3\,\|(H-\lambda_0-i\epsilon)^{-1}\varphi\|^4\ \ .$$

Proof. Take $\epsilon > 0$ and consider the Laplace transform representation of $(H-\lambda_0-i\epsilon)^{-1}$, i.e.,

$$(H-\lambda_0-i\epsilon)^{-1} = i\int_0^{\infty} e^{-\epsilon\, t} e^{-i(H-\lambda_0)\, t}\, dt\,. \quad (2.5)$$

Let $\varphi \in \mathcal{D}(H)$, $\phi \in \mathcal{H}$, and P be a bounded operator acting on $\mathcal{H}$. Since $\|\varphi\| = 1$,

$$\begin{aligned} |\langle\phi, P(H-\lambda_0-i\epsilon)^{-1}\varphi\rangle|^2 &\leq \left|\int_0^{\infty} e^{-\epsilon\, t}|\,\langle\phi, Pe^{-iHt}\varphi\rangle|\, dt\right|^2 \\ &\leq \int_0^{\infty} e^{-2\epsilon\, t} dt \int_0^{\infty} |\,\langle\phi, Pe^{-iHt}\varphi\rangle|^2\, dt\,. \end{aligned}$$

Taking supremum over $\|\phi\| = 1$ in the above inequality we have

$$\|P(H-\lambda_0-i\epsilon)^{-1}\varphi\|^2 \leq \frac{1}{2\,\epsilon}\int_0^{\infty} \|Pe^{-iHt}\varphi\|^2\, dt\,.$$

Taking the orthogonal projector $P = |\varphi >< \varphi|$ in the direction of φ, we obtain the first inequality. Concerning the second inequality, by Laplace representation:

$$(H-\lambda_0-i\epsilon)^{-1} - (H-\lambda_0+i\epsilon)^{-1} = i\int_{-\infty}^{\infty} e^{-\epsilon\,|t|} e^{-i(H-\lambda_0)\, t}\, dt\,. \quad (2.6)$$

Using the above representation, we obtain that:

$$\langle\varphi, ((H-\lambda_0-i\epsilon)^{-1}-(H-\lambda_0+i\epsilon)^{-1})\varphi\rangle = i\int_{-\infty}^{\infty} e^{-\epsilon\,|t|}e^{i\lambda_0\,t}\langle\varphi, e^{-iH\,t}\varphi\rangle dt\,.$$

Now by the Schwarz's inequality,

$$|\langle\varphi, \left((H-\lambda_0-i\epsilon)^{-1}-(H-\lambda_0+i\epsilon)^{-1}\right)\varphi\rangle| \leq \frac{1}{\sqrt{\epsilon}}\left(\mathcal{T}(\varphi)\right)^{1/2}\,. \tag{2.7}$$

On the other hand,

$$|\langle\varphi, \left((H-\lambda_0-i\epsilon)^{-1}-(H-\lambda_0+i\epsilon)^{-1}\right)\varphi\rangle| = 2\,\epsilon\|(H-\lambda_0-i\epsilon)^{-1}\varphi\|^2$$

The inequality now follows from the above identity and estimate (2.7). □

The next result follows immediately from Lemma 2.1 and 2.2.

Theorem 2.1 (First Uncertainty Principle). *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(H)$. Let $\lambda_0 \in I\!R$, $\varphi \in \mathcal{D}(H)$ such that $\|\varphi\| = 1$ and $(H-\lambda_0)\varphi \neq 0$. Then*

$$1 \leq \|(H-\lambda_0)\varphi\|\,\mathcal{T}(\varphi)\,. \tag{2.8}$$

This result gives a lower bound for the sojourn time and it can be used to construct long living states. But, a much better lower bound can be obtained in terms of the so-called *energy width*.

Definition 2.2. *Given a self-adjoint operator H on $\mathcal{H}$, a real number λ and a state φ, the energy width $\Delta_H(\lambda,\varphi)$ for the Hamiltonian H is defined by:*

$$\Delta_H(\lambda,\varphi) = \inf_{\epsilon>0}\left\{\epsilon^2\|(H-\lambda-i\epsilon)^{-1}\varphi\|^2 \geq \frac{1}{2}\right\}.$$

Remark: The energy width is zero if and only if $H\varphi = \lambda\varphi$.

Theorem 2.2 (Second Uncertainty Principle).

$$\frac{1}{2} \leq \Delta_H(\lambda,\varphi)\,\mathcal{T}(\varphi)\,. \tag{2.9}$$

We refer to [7] for a proof.

Remark: Let us consider a Hamiltonian of the form $H_\beta = H_0 + \beta V$. Assume λ_0 is a simple, embedded eigenvalue of H_0 with associated eigenvector φ: $H_0\varphi = \lambda_0\varphi$. Under suitable hypotheses, [3] shows that when t tends to infinity, up to small errors,

$$\langle\varphi, e^{-iH_\beta t}\varphi\rangle \approx e^{-i(\lambda_\beta - i\Gamma_\beta)t},$$

where β is small, λ_β is close to λ_0, Γ_β is positive and of order β^2 (Fermi Golden Rule). Therefore, the sojourn time of φ is $\mathcal{O}(\beta^{-2})$. Lavine's second Uncertainty Principle gives precisely a lower bound of this order (with no extra hypothesis), while the one involving the variance only gives $\mathcal{O}(\beta^{-1})$.

2.2. The non-autonomous case

In this section, some of the ideas exposed before are adapted to the non-autonomous context.

For simplicity, the Hamiltonian $(H(t))_{t\in\mathbb{R}}$ will stand for a strongly continuous family of self-adjoint operators with common domain $\mathcal{D}$. We assume it satisfies adequate conditions so that the solution of the associated Schrödinger equation

$$i\,\frac{\partial u}{\partial t} = H(t)u \ , \ u(s) = \varphi \in \mathcal{D} \tag{2.10}$$

is given by a strongly continuous unitary propagator $(U(t,s))$ [10], [11].

We start by an adaptation of the concept of the sojourn time into this framework:

Definition 2.3. *Let $(U(t,s))$ be a strongly continuous unitary propagator defined on some Hilbert space $\mathcal{H}$ and φ in $\mathcal{H}$. The sojourn time associated to the evolution of the state φ starting at time s ($s \in \mathbb{R}$) is defined by:*

$$\mathcal{T}_s(\varphi) = \int_{\mathbb{R}} |\langle \varphi, U(s, s-\sigma)\varphi\rangle|^2 \, d\sigma \ .$$

Since we deal in this manuscript with periodic time-dependent Hamiltonian (with period T, $T > 0$), the mean sojourn time associated to the state φ is denoted by:

$$\langle \mathcal{T} \rangle(\varphi) = \frac{1}{T}\int_0^T \mathcal{T}_s(\varphi)\, ds \ .$$

Remark: If the Hamiltonian is time-independent, i.e., $H(t) \equiv H$ for all t in $\mathbb{R}$, then for all (s,σ) in $\mathbb{R}^2$, $U(s, s-\sigma) = e^{-i\sigma H}$ and

$$\mathcal{T}_s(\varphi) = \langle \mathcal{T} \rangle(\varphi) = \mathcal{T}(\varphi) \ .$$

Our main result is an extension of Theorem 2.1 for periodic time-dependent quantum systems. In the following, the period of the system is denoted by T. Following [5], we write: $\mathcal{K} = L^2(\mathbb{R}/T\mathbb{Z}, \mathcal{H})$.

Theorem 2.3. *Let $\lambda_0 \in \mathbb{R}$ and $\varphi \in \mathcal{D}$ such that $\|\varphi\| = 1$ and $(H(t) - \lambda_0)\varphi \neq 0$ for some $t \in [0,T)$. Consider the propagator $(U(t,s))$ generated by the evolution equation* (2.10)*. Then,*

$$1 \leq \langle \mathcal{T} \rangle(\varphi) \cdot \sqrt{\frac{1}{T}\int_0^T \|(H(t) - \lambda_0)\varphi\|^2 \, dt} \ . \tag{2.11}$$

Proof. Following [5], we consider on the Hilbert space $\mathcal{K}$, the self-adjoint Floquet Hamiltonian

$$K = -i\,\partial_t + H(t)$$

with periodic boundary conditions. Let f in $\mathcal{K}$ defined by: $f(t) \equiv \varphi$ for all t. In particular, f belongs to the domain of K, $\mathcal{D}(K)$, $\|f\|_{\mathcal{K}} = 1$ and $(K - \lambda_0)f \neq 0$. Applying Theorem 2.1, we have that:

$$1 \leq \|(K - \lambda_0)f\|_{\mathcal{K}}\, \mathcal{T}_K(f)\,.$$

In addition, we have that for all $t \in \mathbb{R}$,

$$\|((K - \lambda_0)f)(t)\| = \|(H(t) - \lambda_0)\varphi\|$$

$$\text{and} \quad \|(K - \lambda_0)f\|^2_{\mathcal{K}} = \frac{1}{T}\int_0^T \|(H(t) - \lambda_0)\varphi\|^2\, dt\ .$$

On the other hand, the sojourn time $\mathcal{T}_K(f)$ for the state f in $\mathcal{K}$ is defined by

$$\mathcal{T}_K(f) = \int_{-\infty}^{\infty} |\langle f, e^{-iK\sigma} f\rangle_{\mathcal{K}}|^2\, d\sigma\,.$$

Since for all t and σ in $\mathbb{R}$, $(e^{-iK\sigma}f)(t) = U(t, t-\sigma)f(t-\sigma)$, we have that:

$$\begin{aligned}\mathcal{T}_K(f) &= \int_{-\infty}^{\infty} \left|\frac{1}{T}\int_0^T \langle\varphi, U(t, t-\sigma)\varphi\rangle\, dt\right|^2 d\sigma \\ &\leq \frac{1}{T}\int_{-\infty}^{\infty}\int_0^T |\langle\varphi, U(t, t-\sigma)\varphi\rangle|^2\, dt\, d\sigma \\ &\leq \langle\mathcal{T}\rangle(\phi)\ .\end{aligned}$$

Inequality (2.11) follows. □

Corollary 2.1. *Let H_0 be a self-adjoint operator defined on the Hilbert space $\mathcal{H}$ with domain $\mathcal{D}$. Assume that λ_0 is an eigenvalue of H_0 and φ a corresponding normalized eigenvector. Let $(V(t))$ be a strongly continuous periodic time-dependent family of symmetric and bounded operators on $\mathcal{H}$ (with period T, $T > 0$). For $\beta \in \mathbb{R}$, define the Hamiltonian $(H_\beta(t))_{t\in\mathbb{R}}$ by: $H_\beta(t) = H_0 + \beta V(t)$. Then, if $\beta \in \mathbb{R}\setminus\{0\}$ and $H_\beta(t)\varphi \neq 0$ for some $t \in [0, T)$,*

$$|\beta| \sup_{t\in[0,T)} \|V(t)\| \cdot \langle\mathcal{T}_\beta\rangle(\varphi) \geq 1\,.$$

3. A model

In this paragraph, we consider a variation of the two-states quantum model presented in [9].

In the following $\mathcal{H}$ denotes the Hilbert space $\mathbb{C} \oplus L^2(\mathbb{R})$ with inner product

$$\langle(z_1, f_1), (z_2, f_2)\rangle = \bar{z}_1 z_2\, +\, \langle f_1, f_2\rangle\,.$$

M stand for the multiplication operator by x on $L^2(\mathbb{R})$ and E_0 is a fixed real number. Let us define the time-dependent Hamiltonian $(H(t))_{t\in\mathbb{R}}$ on $\mathcal{H}$ by: $H(t) =$

$H_0 + \beta h(t)V$ where $\beta \in \mathbb{R}$, h is a C^1, T-periodic real-valued function ($T > 0$) and

$$H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & M \end{pmatrix} \ . \tag{3.1}$$

The perturbation operator V is defined by:

$$V = \begin{pmatrix} 0 & w^* \\ w & 0 \end{pmatrix} \tag{3.2}$$

with $w : \mathbb{C} \to L^2(\mathbb{R})$ given by $w(z) = z\omega$ where $|\omega(x)|^2 = \pi^{-1}(1+x^2)^{-1}$ and $w^*(f) =< \omega, f >$. The spectrum of the unperturbed operator H_0 may be described as an absolutely continuous component $\mathbb{R}$ with an embedded eigenvalue E_0 of multiplicity one. The associated eigenvector $(1,0)^t$ will be denoted by ψ. For simplicity, we assume that $E_0 = 0$. If the propagator associated to the Hamiltonian $(H(t))$ is denoted by $(U(t,s))$, we write:

$$\begin{pmatrix} z(t,s) \\ f(t,s) \end{pmatrix} \equiv U(t,s) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ . \tag{3.3}$$

In particular, for all $(t,s) \in \mathbb{R}^2$, $\langle \psi, U(t,s)\psi \rangle = z(t,s)$.

We start our discussion by the following lemma:

Lemma 3.1. *Assume that for all t, $h(t) \neq 0$. Then, given s in $\mathbb{R}$, for all $t \geq s$, the function $z : t \mapsto z(t,s)$ is solution of the following Cauchy problem:*

$$\frac{\partial^2 z}{\partial t^2} + \left(1 - \frac{\partial \ln |h|}{\partial t}\right) \frac{\partial z}{\partial t} + \beta^2 h^2 z = 0 \tag{3.4}$$

with $z(s,s) = 1$, $\partial_t z(s,s) = 0$.

Proof. Following our notations, the Schrödinger equation

$$i\partial_t U(t,s)\psi = H(t)U(t,s)\psi$$

may be rewritten as a system:

$$i\frac{\partial z}{\partial t} = \beta h(t)\langle \omega, f \rangle \tag{3.5}$$

$$i\frac{\partial f}{\partial t} = Mf + \beta h(t) z\omega \tag{3.6}$$

with initial conditions: $z(s,s) = 1$, $f(s,s) = 0$. We deduce that

$$f(t,s) = \left(-i\beta \int_s^t e^{ix(\tau - t)} h(\tau) z(\tau,s)\, d\tau\right) \omega \, .$$

Plugging it into equation (3.5) and using the fact that for $\tau \leq t$,

$$\langle \omega, e^{ix(\tau-t)}\omega \rangle = e^{-(t-\tau)}$$

we obtain that:

$$\frac{\partial z}{\partial t}(t,s) = -\beta^2 h(t) e^{-t} \int_s^t h(\tau) z(\tau,s) e^{\tau}\, d\tau \ .$$

The result follows by derivation. □

Remark: If the function h is constant ($h \equiv h_0$ for some $h_0 \in \mathbb{R}$), which corresponds to the autonomous version of the model, equation (3.4) becomes:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial z}{\partial t} + \beta^2 h_0^2 z = 0 \ . \tag{3.7}$$

If $0 < |\beta h_0| < 1/2$, the characteristic values associated to equation (3.7) are negative: $-\gamma_+ < -\gamma_- < 0$ where

$$\gamma_\pm = \frac{1}{2} \pm \frac{\sqrt{1 - 4\beta^2 h_0^2}}{2} \ . \tag{3.8}$$

Rewriting equation (3.7) as a first-order linear system of differential equations,

$$X_t(t,s) = A_0 X(t,s) \ ,$$

$X(s,s) = (1,0)^t$ where $X(t,s) = (z(t,s), \partial_t z(t,s))^t$ and

$$A_0 = \begin{pmatrix} 0 & 1 \\ -\beta^2 h_0^2 & -1 \end{pmatrix} \ , \tag{3.9}$$

we obtain that there exists a positive constant C (independent of s) such that: $\|X(t,s)\| \leq C e^{-\gamma_-(t-s)}$ for all $t \geq s$. One finds that:

$$\mathcal{T}(\psi) \leq \frac{C}{\gamma_-} \ .$$

Since $\gamma_- \simeq \beta^2$ when $|\beta| \ll 1$, we recover an upper bound of order $\mathcal{O}(\beta^{-2})$ for the corresponding sojourn time. Moreover, contrasting with the original model studied in [9], the state ψ exhibits exponential decay for all $t \geq s$. mind that the spectrum of the Hamiltonian $H + \beta h_0 V$ covers the whole real line [12].

Under some suitable conditions on β and the function h, we show that this decaying behavior is preserved in our periodic setting and give an upper bound on the mean sojourn-time associated to the vector ψ. Let us precise our main result:

Proposition 3.1. *Let $\beta \in \mathbb{R}$ and $h \in C^1(\mathbb{R}, \mathbb{R})$ be a periodic time-dependent function with period T, $T > 0$ such that for all t in $\mathbb{R}$, $h(t) \neq 0$. Its average over one period is denoted by h_0. Assume that: $|\beta h_0| < 1/2$ and that*

$$\gamma \equiv \gamma_- - \sqrt{\beta^4 \sup_t [h(t)^2 - h_0^2]^2 + \sup_t [\partial_t \ln |h(t)|]^2} > 0 \ ,$$

where γ_- has been defined in (3.8). Then, there exists a positive constant C, independent of s, such that for all $t \geq s$:

$$|z(t,s)| \leq C e^{-\gamma(t-s)} \ .$$

Proof. We rewrite equation (3.4) as the following first-order system

$$X_t(t,s) = A(t) X(t,s) \ , \tag{3.10}$$

$X(s,s) = (1,0)^t$ where $X(t,s) = (z(t,s), \partial_t z(t,s))^t$ and

$$A(t) - A_0 = \begin{pmatrix} 0 & 0 \\ -\beta^2 (h(t)^2 - h_0^2) & \partial_t \ln |h(t)| \end{pmatrix} \ .$$

Following the above remark, we know that there exists $C > 0$ (independent of s) such that for all $t \geq s$,

$$\|e^{A_0(t-s)}\| \leq Ce^{-\gamma_-(t-s)} \ .$$

On the other hand, for any $t \geq s$, the solution of (3.10) satisfies,

$$X(t,s) = e^{A_0(t-s)}X(s,s) + \int_s^t e^{A_0(t-\tau)}(A(\tau) - A_0)X(\tau,s)\,d\tau$$

which, given our initial condition, implies that:

$$\|X(t,s)\| \leq Ce^{-\gamma_-(t-s)} + C\int_s^t e^{-\gamma_-(t-\tau)}\|A(\tau) - A_0\|\|X(\tau,s)\|\,d\tau \ .$$

Since for all t in $\mathbb{R}$,

$$\|A(t) - A_0\| = \sqrt{\beta^4(h(t)^2 - h_0^2)^2 + (\partial_t \ln|h(t)|)^2}$$

we obtain, using Gronwall's inequality,

$$\|X(t,s)\| \leq Ce^{-\gamma_-(t-s)}e^{\int_s^t \|A(\tau)-A_0\|\,d\tau} \leq Ce^{-\gamma(t-s)} \ . \qquad \square$$

Corollary 3.1. *Following Proposition* 3.1*, for any* $s \in [0,T)$*,*

$$\mathcal{T}_s(\psi) \leq \frac{C}{\gamma} \quad \text{and} \quad \langle\mathcal{T}\rangle(\psi) \leq \frac{C}{\gamma} \ .$$

Proof. Since U denotes a unitary propagator, for $(s,t) \in \mathbb{R}^2$, $U(s,t) = U(t,s)^*$ and

$$\begin{aligned}
\mathcal{T}_s(\psi) &= \int_{\mathbb{R}} |\langle\psi, U(s,s-\sigma)\psi\rangle|^2\,d\sigma \\
&= \int_{-\infty}^0 |\langle U(s-\sigma,s)\psi,\psi\rangle|^2\,d\sigma + \int_0^\infty |\langle\psi, U(s,s-\sigma)\psi\rangle|^2\,d\sigma \\
&= \int_{-\infty}^0 |z(s-\sigma,s)|^2\,d\sigma + \int_0^\infty |z(s,s-\sigma)|^2\,d\sigma \ .
\end{aligned}$$

Applying Proposition 3.1, the result follows. $\square$

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Continuity of Spectra in Rieffel's Pseudodifferential Calculus

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Abstract. Using the fact that Rieffel's quantization sends covariant continuous fields of C^*-algebras in continuous fields of C^*-algebras, we prove spectral continuity results for families of Rieffel-type pseudodifferential operators.

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Introduction

One naturally expects that topics or tools coming from the standard pseudodifferential theory could make sense and even work in the more general setting of Rieffel's calculus. In [16, 17], some C^*-algebraic techniques of spectral analysis ([3, 4, 10, 15, 18] and references therein) were tuned with Rieffel quantization [24], getting results on spectra and essential spectra of certain self-adjoint operators that seemed to be out of reach by other methods. In the present article we continue the project by studying *spectral continuity.*

Pioneering work on applying C^*-algebraic methods to spectral continuity problems and applications to discrete physical systems may be found in [3, 5, 8]. Results on continuity of spectra for unbounded Schrödinger-like Hamiltonians (especially with magnetic fields) appear in [1, 2, 13, 20] and references therein.

Roughly, the abstract problem can be stated as follows: For each point t of the locally compact space T we are given a self-adjoint element (a classical observable) $f(t)$ of a C^*-algebra $\mathcal{A}(t)$, which is Abelian for most of the applications, and we assume some simple-minded continuity property in the variable t for this family. By quantization, $f(t)$ is turned into a quantum observable $\mathfrak{f}(t)$ belonging to a new, non-commutative C^*-algebra $\mathfrak{A}(t)$ (in spite of the notation, rather often $\mathfrak{f}(t)$ is just $f(t)$ with a new interpretation). We inquire if the family $S(t) := \mathrm{sp}\,[\mathfrak{f}(t)]$

of spectra computed in these new algebras vary continuously with t. Intuitively, outer continuity says that the family cannot suddenly expand: if for some t_0 there is a gap in the spectrum of $f(t_0)$ around a point $\lambda_0 \in \mathbb{R}$, then for t close to t_0 all the spectra $S(t)$ will have gaps around λ_0. On the other hand, inner continuity insures that if $f(t_0)$ has some spectrum in a non-trivial interval of $\mathbb{R}$, this interval will contain spectral points of all the elements $f(t)$ for t close to t_0. Although traditionally $\mathfrak{A}(t)$ is thought to be a C^*-algebra of bounded operators in some Hilbert space, the abstract situation is both natural and fruitful. One can work with abstract C^*-algebras $\mathfrak{A}(t)$ and then, if necessarily, represent them faithfully in Hilbert spaces; the spectrum will be preserved under representation.

It is well known (see Theorem 3.2) that spectral continuity can be obtained from corresponding continuity properties of resolvent families of the elements $\mathfrak{f}(t)$ but this involves both inversion and norm in each complicated C^*-algebra $\mathfrak{A}(t)$. Things are smoothed out if the family $\{\mathfrak{A}(t) \mid t \in T\}$ has a priori continuity properties, that may be connected to the concept of (upper or lower semi)-continuous C^*-bundle, cf. [25, 26] and Definition 1.1. We are going to investigate the case in which $\{\mathfrak{A}(t) \mid t \in T\}$ in obtained from another family $\{\mathcal{A}(t) \mid t \in T\}$ of simpler (classical) C^*-algebras by Rieffel quantization.

Rieffel's calculus [24, 14] is a method that transforms "simpler" C^*-algebras and morphisms into more complicated ones. The ingredients to do this are an action of the vector group $\Xi := \mathbb{R}^d$ by automorphisms of the "simple" algebra as well as a skew symmetric linear operator of Ξ. The initial data are naturally defining a Poisson structure, regarded as a mathematical modelization of the observables of a classical physical system. After applying the machine to this classical data one gets a C^*-algebra seen as the family of observables of the same system, but written in the language of Quantum Mechanics.

In simple situations the multiplication in the initial C^*-algebra is just pointwise multiplication of functions defined on some locally compact topological space Σ, on which Ξ acts by homeomorphisms. The non-commutative product in the quantized algebra can be interpreted as a symbol composition of a pseudodifferential type. Actually the concrete formulae generalize and are motivated by the usual Weyl calculus [9].

The basic technical fact is that *by Rieffel quantization an upper semi-continuous fields of C^*-algebras is turned into an upper semi-continuous fields of C^*-algebras and the same is true if upper semi-continuity is replaced by lower semi-continuity.* This is shown in [6]; a partial result without proof is announced in [11] (see also [12]). For the convenience of the readers, we are going to sketch a new proof in Section 1, relying on results from [7, 23].

As said before the most interesting cases, those which are closer to the initial spirit of Weyl quantization, involve Abelian initial algebras $\mathcal{A}$. In this situation the information is encoded in a topological dynamical system with locally compact space Σ and the upper semi-continuous field property can be read in the existence of a continuous covariant surjection $q : \Sigma \to T$; if this one is open, then lower semi-continuity also holds. This is explained in Section 2.

Using these facts, in the final sections, we prove spectral continuity. We start with families of elements belonging to the abstract Rieffel algebras. Then we outline a setting in which these algebras admit interesting faithful representations in a unique Hilbert space, thus getting spectral continuity for families of pseudodifferential-like operators. Making suitable adaptations of the dynamical system, we also include an outer continuity result for *essential spectra* of Rieffel pseudodifferential operators. As an example, we are going to show that our results cover families of zero order standard pseudodifferential operators and this is new up to or knowledge. Spectral continuity for families of elliptic strictly positive order Hamiltonians (even including variable magnetic fields) is known; see [1, 2, 13, 20]. But the methods of these articles do not extend in some obvious way to zero-order operators. The resolvent of an elliptic operator of order $m > 0$ is a pseudodifferential operator of strictly negative order and this helps a lot. In the framework of [1] for instance, it allows using a certain form of crossed product C^*-algebras, which form semi-continuous fields by well-known results [19, 22, 23]; this is not available if $m = 0$. Continuity in Planck's constant $\hbar$, treated in [24] and in [16], is also special case of our general results but we shall not repeat this here.

The full strength of these spectral techniques would require an extension of Rieffel's calculus to suitable families of unbounded elements. Hopefully this will be achieved in the future and this would be the right opportunity to present detailed examples, which could include non-elliptic positive order pseudodiferential operators with variable magnetic fields.

1. Families of Rieffel quantized C^*-algebras

Let T be a locally compact space (always supposed Hausdorff); we denote dy $\mathcal{C}(T)$ the space of all complex continuous functions defined on T and vanishing at infinity.

Definition 1.1. (*see* [19, 23, 26] *and references therein*) *By* upper semi-continuous field of C^*-algebras *we mean a family* $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ *of epimorphisms of C^*-algebras indexed by the locally compact topological space T and satisfying:*

1. *For every $b \in \mathcal{B}$ one has $\| b \|_{\mathcal{B}} = \sup_{t\in T} \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$.*
2. *For every $b \in \mathcal{B}$ the map $T \ni t \mapsto \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$ is upper semi-continuous and vanishes at infinity.*
3. *There is a multiplication $\mathcal{C}(T) \times \mathcal{B} \ni (\varphi, b) \to \varphi * b \in \mathcal{B}$ such that*

$$\mathcal{P}(t)[\varphi * b] = \varphi(t)\,\mathcal{P}(t)b\,, \qquad \forall\, t \in T\,,\ \varphi \in \mathcal{C}(T)\,,\ b \in \mathcal{B}\,.$$

If the map $t \mapsto \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$ is upper semi-continuous for every $b \in \mathcal{B}$, we say that $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ *is* an upper semi-continuous field of C^*-algebras.

One can identify $\mathcal{B}$ with a C^*-algebra of sections of the field. It will always be assumed that $\mathcal{B}(t) \neq \{0\}$ for all $t \in T$.

We go on by describing briefly Rieffel quantization [24]. Let $(\Xi, [\![\cdot,\cdot]\!])$ be a $2n$-dimensional symplectic vector space and $(\mathcal{A}, \Theta, \Xi)$ a C^*-*dynamical system*, meaning that Ξ acts strongly continuously by automorphisms of the C^*-algebra $\mathcal{A}$. We denote by $\mathcal{A}^\infty$ the family of elements f such that the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is C^∞; it is a dense *-algebra of $\mathcal{A}$. Inspired by Weyl's pseudodifferential calculus, one keeps the involution unchanged but introduce on $\mathcal{A}^\infty$ the product

$$f \# g := \pi^{-2n} \int_\Xi \int_\Xi dY dZ \, e^{2i[\![Y,Z]\!]} \, \Theta_Y(f) \, \Theta_Z(g) \, , \tag{1.1}$$

defined by oscillatory integral techniques. One gets a *-algebra $(\mathcal{A}^\infty, \#, ^*)$, which admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\| \cdot \|_{\mathfrak{A}}$ as described in [24]. The action Θ leaves $\mathcal{A}^\infty$ invariant and extends [24, Prop. 5.11] to a strongly continuous action of the C^*-algebra $\mathfrak{A}$, that will also be denoted by Θ. The space $\mathfrak{A}^\infty$ of C^∞-vectors coincide with $\mathcal{A}^\infty$, cf [24, Th. 7.1].

Let $(\mathcal{A}_j, \Theta_j, \Xi, [\![\cdot,\cdot]\!])$, $j = 1, 2$, be two data as above and let $\mathcal{R} : \mathcal{A}_1 \to \mathcal{A}_2$ be a Ξ-morphism, *i.e.*, a (C^*-)morphism intertwining the two actions Θ_1, Θ_2. Then $\mathcal{R}$ sends $\mathcal{A}_1^\infty$ into $\mathcal{A}_2^\infty$ and extends to a morphism $\mathfrak{R} : \mathfrak{A}_1 \to \mathfrak{A}_2$ that also intertwines the corresponding actions.

Let now T be a locally compact Hausdorff space and let $\left\{ \mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T \right\}$ be a field of C^*-algebras. We are given actions Θ of Ξ on $\mathcal{A}$ and $\Theta(t)$ of Ξ on $\mathcal{A}(t)$ satisfying $\Theta(t)_X \circ \mathcal{P}(t) = \mathcal{P}(t) \circ \Theta_X$ for each $t \in T$ and $X \in \Xi$. One can say that $\left\{ \mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T \right\}$ is *a covariant field of C^*-algebras.* Then, by Rieffel quantization, one constructs the new covariant field $\left\{ \mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t) \mid t \in T \right\}$.

Theorem 1.2. *Rieffel quantization transforms covariant semi-continuous fields of C^*-algebras into covariant semi-continuous fields of C^*-algebras.*

It is understood that the statement holds separately for upper and for lower semi-continuity. In the remaining part of this section we are going to present a proof of this result, different from that of [6].

First define

$$\kappa : \Xi \times \Xi \to \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \, , \qquad \kappa(X, Y) := \exp\left(-\frac{i}{2} [\![X, Y]\!] \right) \tag{1.2}$$

and notice that it is a group 2-cocycle, i.e., for all $X, Y, Z \in \Xi$ one has

$$\kappa(X, Y) \, \kappa(X + Y, Z) = \kappa(Y, Z) \, \kappa(X, Y + Z) \, , \qquad \kappa(X, 0) = 1 = \kappa(0, X) \, .$$

Thus the initial data is converted into $(\mathcal{A}, \Theta, \Xi, \kappa)$, a very particular case of *twisted C^*-dynamical system* [21, 22]. To any twisted C^*-dynamical system one associates canonically a C^*-algebra $\mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi$ (called *twisted crossed product*). This is the enveloping C^*-algebra of the Banach *-algebra $\left(L^1(\Xi;\mathcal{A}), \diamond, ^\diamond, \|\cdot\|_1\right)$, where

$$\| F \|_1 := \int_\Xi dX \, \| F(X) \|_{\mathcal{A}}, \qquad F^\diamond(X) := F(-X)^*$$

and (symmetrized version of the standard form)

$$(F_1 \diamond F_2)(X) := \int_\Xi dY \, \kappa(X,Y) \, \Theta_{(Y-X)/2} \left[F_1(Y)\right] \, \Theta_{Y/2} \left[F_2(X-Y)\right]. \tag{1.3}$$

In the same way, for each $t \in T$, to $(\mathcal{A}(t), \Theta(t), \Xi, \kappa)$ one associates the twisted crossed product $\mathcal{A}(t) \rtimes_{\Theta(t)}^{\kappa} \Xi$. Let us use the abbreviations $\mathfrak{C} := \mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi$ and $\mathfrak{C}(t) := \mathcal{A}(t) \rtimes_{\Theta(t)}^{\kappa} \Xi$. The epimorphism $\mathcal{P}(t) : \mathcal{A} \to \mathcal{A}(t)$ raises canonically to an epimorphism $\mathcal{P}(t)^{\rtimes} : \mathfrak{C} \to \mathfrak{C}(t)$. As a consequence of results in [19, 22, 23] (see [23, Sect. 3] for instance), if $\left\{\mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T\right\}$ is an upper (or lower, respectively) semi-continuous field, then $\left\{\mathfrak{C} \xrightarrow{\mathcal{P}(t)^{\rtimes}} \mathfrak{C}(t) \mid t \in T\right\}$ is also an upper (resp. lower) semi-continuous field.

Thus it would be enough to have an efficient connection between Rieffel quantized C^*-algebras and twisted crossed products. We present some consequences of results from [7]. We recall first that the Schwartz space $\mathcal{S}(\Xi)$ is a *-algebra under complex conjugation and the Weyl product

$$(h \sharp k)(X) := \pi^{-2n} \int_\Xi \int_\Xi dY dZ \, e^{2i[\![Y,Z]\!]} \, h(X+Y) \, k(X+Z). \tag{1.4}$$

Fix now an element $h \in \mathcal{S}(\Xi) \setminus \{0\}$ satisfying $h \sharp h = h = \overline{h}$ and define for each $f \in \mathcal{A}^\infty = \mathfrak{A}^\infty$ and any $X \in \Xi$

$$\left[M_h(f)\right](X) := \int_\Xi dY \, e^{-i[\![X,Y]\!]} h(Y) \Theta_Y(f). \tag{1.5}$$

It is shown in [7] that M_h can be extended as an injective C^*-morphism $M_h : \mathfrak{A} \to \mathfrak{C} \equiv \mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi$. We recall that injective C^*-morphisms are isometric. The construction can be repeated with $(\mathcal{A}, \Theta)$ replaced by $(\mathcal{A}(t), \Theta(t))$, so for each $t \in T$ one gets an isometry $M(t)_h : \mathfrak{A}(t) \to \mathfrak{C}(t) \equiv \mathcal{A}(t) \rtimes_{\Theta(t)}^{\kappa} \Xi$. In addition, by [7] one has $M(t)_h \circ \mathfrak{P}(t) = \mathcal{P}(t)^{\rtimes} \circ M_h$. Then, for any $f \in \mathfrak{A}$

$$\| \mathfrak{P}(t) f \|_{\mathfrak{A}(t)} = \| M(t)_h [\mathfrak{P}(t) f] \|_{\mathfrak{C}(t)} = \| \mathcal{P}(t)^{\rtimes} \left[M_h(f)\right] \|_{\mathfrak{C}(t)} .$$

Therefore, under the right assumption, the mapping $t \mapsto \| \mathfrak{P}(t) f \|_{\mathfrak{A}(t)}$ has the desired semi-continuity properties. The first condition in the definition of a semi-continuous field of C^*-algebras is checked analogously:

$$\begin{aligned} \| f \|_{\mathfrak{A}} = \| M_h(f) \|_{\mathfrak{C}} &= \sup_t \| \mathcal{P}(t)^{\rtimes} \left[M_h(f)\right] \|_{\mathfrak{C}(t)} \\ &= \sup_t \| M(t)_h [\mathfrak{P}(t) f] \|_{\mathfrak{C}(t)} = \sup_t \| \mathfrak{P}(t) f \|_{\mathfrak{A}(t)} . \end{aligned}$$

Finally, one must define the mapping $\star : \mathcal{C}(T) \times \mathfrak{A} \to \mathfrak{A}$ that should be deduced from the already existing $* : \mathcal{C}(T) \times \mathcal{A} \to \mathcal{A}$. Let $\varphi \in \mathcal{C}(T)$ and $f \in \mathfrak{A}$. There is a sequence $(f_n)_{n\in\mathbb{N}} \in \mathfrak{A}^\infty = \mathcal{A}^\infty$ with $\| f - f_n \|_{\mathfrak{A}} \to 0$ for $n \to \infty$. One sets $\varphi \star f := \lim_n \varphi * f_n$. We leave to the reader the easy task to check that this limit exists in $\mathfrak{A}$ and that the identity $\mathfrak{P}(t)[\varphi \star f] = \varphi(t)\mathfrak{P}(t)f$ holds for every $t \in T$.

2. The Abelian case

We denote by $\mathcal{C}(\Sigma)$ the Abelian C^*-algebra of all complex continuous functions on the locally compact Hausdorff space Σ that are arbitrarily small outside large compact subsets. When Σ is compact, $\mathcal{C}(\Sigma)$ is unital. We indicate a framework leading naturally to fields of C^*-algebras.

We assume given a continuous surjection $q : \Sigma \to T$. Then we have the disjoint decomposition of Σ in closed subsets $\Sigma = \sqcup_{t\in T}\Sigma_t$, where $\Sigma_t := q^{-1}(\{t\})$. One has the canonical injections $j_t : \Sigma_t \to \Sigma$ and the restriction epimorphisms $\mathcal{R}(t) : \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma_t)$, with $\mathcal{R}(t)f := f|_{\Sigma_t} = f \circ j_t$, $\forall\, t \in T$. This is the right setting to get semi-continuous fields of Abelian C^*-algebras.

Proposition 2.1. *In the setting above* $\left\{\mathcal{C}(\Sigma) \xrightarrow{\mathcal{R}(t)} \mathcal{C}(\Sigma_t) \mid t \in T\right\}$ *is an upper semi-continuous field of commutative* C^**-algebras. If* q *is also open, the field is continuous.*

Proof. Obviously $\cap_{t\in T} \ker[\mathcal{R}(t)] = \{0\}$, since $f|_{\Sigma_t} = 0$, $\forall\, t \in T$ implies $f = 0$. On the other hand, setting

$$\varphi * f := (\varphi \circ q)f\,, \qquad \forall\, \varphi \in \mathcal{C}(T)\,,\ f \in \mathcal{C}(\Sigma)\,, \tag{2.1}$$

we get immediately $\mathcal{R}(t)(\varphi * f) = \varphi(t)\,\mathcal{R}(t)f$, $\forall\, t \in T$.

We need to study continuity properties of the mapping

$$\begin{aligned} T \ni t \mapsto n_f(t) := \|\mathcal{R}(t)f\|_{\mathcal{C}(\Sigma_t)} &= \sup_{\sigma\in\Sigma_t} |f(\sigma)| \\ &= \inf\left\{\|f + h\|_{\mathcal{C}(\Sigma)} \ \mid\ h \in \mathcal{C}(\Sigma),\ h|_{\Sigma_t} = 0\right\} \in \mathbb{R}_+ \,. \end{aligned}$$

The last expression for the norm can be justified directly easily, but it also follows from the canonical isomorphism $\mathcal{C}(\Sigma_t) \cong \mathcal{C}(\Sigma)/\mathcal{C}_{\Sigma_t}(\Sigma)$, where $\mathcal{C}_{\Sigma_t}(\Sigma)$ is the ideal of functions $h \in \mathcal{C}(\Sigma)$ such that $h|_{\Sigma_t} = 0$.

We first assume that q is only continuous. For every $S \subset T$ we set $\Sigma_S := q^{-1}(S)$. It is easy to see by the Stone-Weierstrass Theorem that

$$\mathcal{C}_{(t)}(\Sigma) := \{h \in \mathcal{C}(\Sigma) \mid \exists \text{ an open neighborhood } U \text{ of } t \text{ such that } h|_{\Sigma_{\overline{U}}} = 0\}$$

is a self-adjoint 2-sided ideal dense in $\mathcal{C}_{\Sigma_t}(\Sigma)$. Let $t_0 \in T$ and $\varepsilon > 0$; by density and the definition of inf

$$\exists\, h \in \mathcal{C}_{(t_0)}(\Sigma) \ \text{ such that } \ n_f(t_0) + \varepsilon \ \geq\ \|f + h\|_{\mathcal{C}(\Sigma)} \,.$$

Let U be the open neighborhood of t_0 for which $h|_{\Sigma_{\overline{U}}} = 0$. For any $t \in U$ one also has $h \in \mathcal{C}_{(t)}(\Sigma)$, so

$$n_f(t) = \inf \left\{ \|f+g\|_{\mathcal{C}(\Sigma)} \;\mid\; g \in \mathcal{C}_{(t)}(\Sigma) \right\} \leq \|f+h\|_{\mathcal{C}(\Sigma)} \leq n_f(t_0) + \varepsilon$$

and this is upper semi-continuity.

Let us also suppose q open, let $t_0 \in T$ and $\varepsilon > 0$. By the definition of sup, there exists $\sigma_0 \in \Sigma_{t_0}$ such that $|f(\sigma_0)| \geq n_f(t_0) - \varepsilon/2$. Since f is continuous, there is a neighborhood V of σ_0 in Σ such that

$$|f(\sigma)| \geq |f(\sigma_0)| - \varepsilon/2 \geq n_f(t_0) - \varepsilon, \qquad \forall\, \sigma \in V.$$

Since q is open, $U := q(V)$ is a neighborhood of t_0. For every $t \in U$ we have $\Sigma_t \cap V \neq \emptyset$, so for such t

$$n_f(t) \geq \sup\{|f(\sigma)| \mid \sigma \in \Sigma_t \cap V\} \geq n_f(t_0) - \varepsilon$$

and this is lower semi-continuity. □

Suppose now that a continuous action Θ of Ξ by homeomorphisms of Σ is also given. For $(\sigma, X) \in \Sigma \times \Xi$ we are going to use all the notations

$$\Theta(\sigma, X) = \Theta_X(\sigma) = \Theta_\sigma(X) \in \Sigma \tag{2.2}$$

for the X-transformed of the point σ. The function Θ is continuous and the homeomorphisms Θ_X, Θ_Y satisfy $\Theta_X \circ \Theta_Y = \Theta_{X+Y}$ for every $X, Y \in \Xi$.

The action Θ of Ξ on Σ induces an action of Ξ on $\mathcal{C}(\Sigma)$ (also denoted by Θ) given by $\Theta_X(f) := f \circ \Theta_X$. This action is strongly continuous, *i.e.*, for any $f \in \mathcal{C}(\Sigma)$ the mapping

$$\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{C}(\Sigma) \tag{2.3}$$

is continuous; thus we are placed in the setting presented in the first section. We denote by $\mathcal{C}(\Sigma)^\infty \equiv \mathcal{C}^\infty(\Sigma)$ the set of elements $f \in \mathcal{C}(\Sigma)$ such that the mapping (2.3) is C^∞; it is a dense *-algebra of $\mathcal{C}(\Sigma)$. The general theory supplies a non-commutative C^*-algebra $\mathfrak{A} \equiv \mathfrak{C}(\Sigma)$, acted continuously by the group Ξ, with smooth vectors $\mathfrak{C}^\infty(\Sigma) = \mathcal{C}^\infty(\Sigma)$.

To insure covariance for the emerging families of C^*-algebras, we impose a condition of compatibility between the action Θ and the mapping q.

Definition 2.2. *We say that the surjection q is* Θ-covariant *if it satisfies the equivalent conditions:*

1. *Each Σ_t is Θ-invariant.* 2. *For each $X \in \Xi$ one has $q \circ \Theta_X = q$.*

Recall now the Rieffel-quantized C^*-algebras $\mathfrak{C}(\Sigma)$ and $\mathfrak{C}(\Sigma_t)$ as well as the epimorphisms $\mathfrak{R}(t) : \mathfrak{C}(\Sigma) \to \mathfrak{C}(\Sigma_t)$. Applying Theorem 1.2 and Proposition 2.1, one gets

Corollary 2.3. *Assume that $q : \Sigma \to T$ is a Θ-covariant continuous surjection. Then the covariant field of non-commutative C^*-algebras $\left\{\mathfrak{C}(\Sigma) \xrightarrow{\mathfrak{R}(t)} \mathfrak{C}(\Sigma_t) \,|\, t \in T\right\}$ is upper semi-continuous.*

If q is also open, then the field is continuous.

3. Spectral continuity for symbols

Let us introduce the concept of continuity for families of sets that will be useful below.

Definition 3.1. *Let T be a Hausdorff locally compact topological space and let $\{S(t) \mid t \in T\}$ be a family of compact subsets of $\mathbb{R}$.*

1. *The family is called* outer continuous *if for any $t_0 \in T$ and any compact subset K of $\mathbb{R}$ such that $K \cap S(t_0) = \emptyset$, there exists a neighborhood V of t_0 with $K \cap S(t) = \emptyset$, $\forall t \in V$.*
2. *The family $\{S(t) \mid t \in T\}$ is called* inner continuous *if for any $t_0 \in T$ and any open subset A of $\mathbb{R}$ such that $A \cap S(t_0) \neq \emptyset$, there exists a neighborhood W of t_0 with $A \cap S(t) \neq \emptyset$, $\forall t \in W$.*
3. *If the family is both inner and outer continuous, we say simply that it is* continuous.

In applications the sets $S(t)$ are spectra of some self-adjoint elements $f(t)$ of (non-commutative) C^*-algebras $\mathfrak{A}(t)$. The next result states technical conditions under which one gets continuity of such families of spectra. It is taken from [1] and it has been inspired by the treatment in [3]. We include the proof for the convenience of the reader.

Proposition 3.2. *For any $t \in T$ let $f(t)$ be a self-adjoint element in a C^*-algebra $\mathfrak{A}(t)$ with norm $\| \cdot \|_{\mathfrak{A}(t)}$ and inversion $g \mapsto g^{(-1)_{\mathfrak{A}(t)}}$. We denote by $S(t) \subset \mathbb{R}$ the spectrum of $f(t)$ in $\mathfrak{A}(t)$.*

1. *Assume that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping*
$$T \ni t \mapsto \left\| (f(t) - z)^{(-1)_{\mathfrak{A}(t)}} \right\|_{\mathfrak{A}(t)} \in \mathbb{R}_+ \tag{3.1}$$
is upper semi-continuous. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.
2. *Assume that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping (3.1) is lower semi-continuous. Then the family $\{S(t) \mid t \in T\}$ is inner continuous.*

Proof. We use the functional calculus for self-adjoint elements in the C^*-algebra $\mathfrak{A}(t)$ to define $\chi[f(t)]$ for every continuous function $\chi : \mathbb{R} \to \mathbb{C}$ vanishing at infinity. Notice that
$$(f(t) - z)^{(-1)_{\mathfrak{A}(t)}} = \chi_z[f(t)], \quad \text{with} \quad \chi_z(\lambda) := (\lambda - z)^{-1}.$$
By a standard argument relying on Stone-Weierstrass Theorem, one deduces that the map $t \mapsto \|\chi[f(t)]\|_{\mathfrak{A}(t)}$ has the same continuity properties (upper or lower semi-continuity, respectively) as (3.1).

Let us suppose now upper semi-continuity in t_0 and assume that $S(t_0) \cap K = \emptyset$ for some compact set K. By Urysohn's Lemma, there exists $\chi \in \mathcal{C}(\mathbb{R})_+$ with $\chi|_K = 1$ and $\chi|_{S(t_0)} = 0$, so $\chi\,[f(t_0)] = 0$. Choose a neighborhood V of t_0 such that for $t \in V$
$$\| \chi[f(t)] \|_{\mathfrak{A}(t)} \leq \| \chi[f(t_0)] \|_{\mathfrak{A}(t_0)} + \frac{1}{2} = \frac{1}{2}.$$

If for some $t \in V$ there exists $\lambda \in K \cap S(t)$, then

$$1 = \chi(\lambda) \leq \sup_{\mu \in S(t)} \chi(\mu) = \| \chi[f(t)] \|_{\mathfrak{A}(t)} \leq \frac{1}{2},$$

which is absurd.

Let us assume now lower semi-continuity in t_0. Pick an open set $A \subset \mathbb{R}$ such that $S(t_0) \cap A \neq \emptyset$ and let $\lambda \in S(t) \cap A$. By Urysohn's Lemma there exist a positive function $\chi \in \mathcal{C}(\mathbb{R})$ with $\chi(\lambda) = 1$ and $\operatorname{supp}(\chi) \subset A$; thus $\| \chi [f(t_0)] \|_{\mathfrak{A}(t_0)} \geq 1$. Suppose moreover that for any neighborhood $W \subset T$ of t_0 there exists $t \in W$ such that $S(t) \cap A = \emptyset$ and thus $\chi [f(t)] = 0$. This clearly contradicts the lower semi-continuity of $t \mapsto \| \chi [f(t)] \|_{\mathfrak{A}(t)}$. We conclude thus the inner continuity property for the family $S(t)$. □

Proving these properties of the resolvents is a priori a difficult task, since this involves working both with norms and composition laws that depend on t. But putting together the information obtained until now, we get our abstract result concerning spectral continuity:

Theorem 3.3. *Let* $\left\{ \mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T \right\}$ *be a covariant upper semi-continuous field of* C^**-algebras indexed by a Hausdorff locally compact space* T *and let* f *be a smooth self-adjoint element of* $\mathcal{A}$. *For any* $t \in T$ *we denote by* $\mathfrak{A}(t)$ *the Rieffel quantization of* $\mathcal{A}(t)$ *and consider* $f(t) := \mathcal{P}(t) f$ *as an element of* $\mathcal{A}(t)^\infty = \mathfrak{A}(t)^\infty \subset \mathfrak{A}(t)$, *with spectrum* $S(t)$ *computed in* $\mathfrak{A}(t)$. *Then the family* $\{S(t) \mid t \in T\}$ *is outer continuous.*

If the field is continuous, the family of subsets will also be continuous.

Proof. Theorem 1.2 allows us to conclude that the quantized field

$$\left\{ \mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t) \mid t \in T \right\}$$

has the same continuity properties as the original one.

For any $z \in \mathbb{C} \setminus \mathbb{R}$ one has $(f - z)^{(-1)_{\mathfrak{A}}} \in \mathfrak{A}$ and $(f(t) - z)^{(-1)_{\mathfrak{A}(t)}} = \mathfrak{P}(t) \left[(f - z)^{(-1)_{\mathfrak{A}}} \right]$. Therefore the assumptions of Proposition 3.2 are fulfilled both in the upper semi-continuous and in the lower semi-continuous case, so we obtain the desired continuity properties for the family of sets $\{S(t) \mid t \in T\}$. □

Of course, the conclusion also holds for non-smooth self-adjoint elements $f \in \mathfrak{A}$. Very often they are much less "accessible" than the smooth elements, being obtained by an abstract completion procedure, so we only make the statements for C^∞ vectors.

Specializing to the Abelian case and using the notations of Section 2, one gets

Corollary 3.4. *Assume that* $q : \Sigma \to T$ *is a* Θ*-covariant continuous surjection. Let* $f \in \mathcal{C}^\infty(\Sigma)$ *a real function and for each* $t \in T$ *denote by* $S(t)$ *the spectrum of* $f(t) := f|_{\Sigma_t} \in \mathcal{C}^\infty(\Sigma_t) = \mathfrak{C}^\infty(\Sigma_t)$ *seen as an element of the non-commutative* C^**-algebra* $\mathfrak{C}(\Sigma_t)$.

Then the family $\{S(t) \mid t \in T\}$ of compact subsets of $\mathbb{R}$ is outer continuous. If q is also open, the family of subsets is continuous.

Remark 3.5. One can use [24, Ex. 10.2] to identify quantum tori as Rieffel-type quantizations of usual tori. One is naturally placed in the setting above and can reproduce some known spectral continuity results [8, 3] on generalized Harper operators.

4. Spectral continuity for operators

The standard approach of Quantum Mechanics asks for Hilbert space operators. This can be achieved by representing faithfully the C^*-algebras $\mathfrak{A}(t)$ in a Hilbert space of L^2-functions in a way that generalizes the Schrödinger representation. We are going to get continuity results for both spectra and essential spectra of the emerging self-adjoint operators. We work in the following

Framework

1. $(\mathcal{C}(\Sigma), \Theta, \Xi)$ is an Abelian C^*-dynamical system, with Σ compact.
2. Ξ is symplectic, given in a Lagrangean decomposition $\Xi = \mathscr{X} \times \mathscr{X}^* \ni X = (x, \xi)$, $Y = (y, \eta)$, where $\mathscr{X}$ is a n-dimensional real vector space, $\mathscr{X}^*$ is its dual and the symplectic form on Ξ is given in terms of the duality between $\mathscr{X}$ and $\mathscr{X}^*$ by $[\![(x,\xi),(y,\eta)]\!] := y \cdot \xi - x \cdot \eta$.
3. $q : \Sigma \to T$ is a Θ-covariant continuous surjection. We also assume that each $\Sigma_t := q^{-1}(\{t\})$ is *a quasi-orbit*, i.e., there is a point $\sigma \in \Sigma_t$ such that the orbit $\mathcal{O}_\sigma := \Theta_\Xi(\sigma)$ is dense in Σ_t (we say that σ *generates the quasi-orbit* Σ_t).
4. We fix a real element $f \in \mathcal{C}^\infty(\Sigma)$. For each $t \in T$ and for any point σ generating the quasi-orbit Σ_t we define $f(t) := f|_{\Sigma_t}$ and $f_\sigma(t) := f(t) \circ \Theta_\sigma : \Xi \to \mathbb{R}$.
5. We set $H_\sigma(t) := \mathfrak{Op}\,[f_\sigma(t)]$ (self-adjoint operator in the Hilbert space $\mathcal{H} := L^2(\mathscr{X})$), by applying to $f_\sigma(t)$ the usual Weyl pseudodifferential calculus. We denote by $S(t)$ the spectrum of $H_\sigma(t)$.

Some explanations are needed. It is easy to see that each $f_\sigma(t)$ belongs to $BC^\infty(\Xi)$, i.e., it is a smooth function with bounded derivatives of any order. Therefore, using oscillatory integrals, one can define the self-adjoint operator in $L^2(\mathscr{X}) \ni u$

$$
\begin{aligned}
[H_\sigma(t)u](x) &\equiv \left[\mathfrak{Op}(f_\sigma(t))u\right](x) \\
&:= (2\pi)^{-n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi\, e^{i(x-y)\cdot\xi}\, [f_\sigma(t)]\left(\frac{x+y}{2}, \xi\right) u(y)\,.
\end{aligned}
$$

This operator is bounded by the Calderón-Vaillancourt Theorem [9]. Using the notation (2.2), we see that for every $X \in \Xi$ one has $[f_\sigma(t)](X) := f[\Theta_X(\sigma)]$; this depends on $t \in T$ through σ and only involves the values of f on the dense subset $\mathcal{O}_\sigma$ of Σ_t. The same is true about $H_\sigma(t)$, which can be written

$$[H_\sigma(t)u](x) = (2\pi)^{-n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi \, e^{i(x-y)\cdot\xi} f\left[\Theta_{\left(\frac{x+y}{2},\xi\right)}(\sigma)\right] u(y) \,. \tag{4.1}$$

It is shown in [16] that if σ and σ' are both generating the same quasi-orbit Σ_t, then the operators $H_\sigma(t)$ and $H_{\sigma'}(t)$ are isospectral (but not unitarily equivalent in general). Thus the compact set $S(t)$ only depends on t and not on the choice of the generating element σ.

Theorem 4.1. *Assume the Framework above. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.*

If q is also open, than the family is continuous.

Proof. By Corollary 3.4, it would be enough to show for every t that $S(t)$ coincides with the spectrum of $f(t) \in \mathfrak{C}(\Sigma_t)$. For this we define

$$\mathcal{N}_\sigma : \mathcal{C}^\infty(\Sigma_t) \to BC^\infty(\Xi), \qquad \mathcal{N}_\sigma(g) := g \circ \Theta_\sigma$$

and then set

$$\mathfrak{Op}_\sigma := \mathfrak{Op} \circ \mathcal{N}_\sigma : \mathcal{C}^\infty(\Sigma_t) \to \mathbb{B}(\mathcal{H}) \,.$$

Then one has $H_\sigma(t) := \mathfrak{Op}\,[f_\sigma(t)] = \mathfrak{Op}_\sigma\,[f(t)]$. It is not quite trivial, but it has been shown in [16], that $\mathfrak{Op}_\sigma$ extends to a faithful representation of the Rieffel quantized C^*-algebra $\mathfrak{C}(\Sigma_t)$ in $\mathcal{H}$. Faithfulness is implied by the fact that σ generates the quasi-orbit Σ_t, which results in the injectivity of $\mathcal{N}_\sigma$, conveniently extended to $\mathfrak{C}(\Sigma_t)$. It follows then that $\mathrm{sp}\,[H_\sigma(t)] = \mathrm{sp}\,[f(t)]$, as required, so the family $\{S(t) \mid t \in T\}$ has the desired continuity properties. □

We recall that *the essential spectrum* of an operator is the part of the spectrum composed of accumulation points or infinitely-degenerated eigenvalues. Let us denote by $S^{\mathrm{ess}}(t)$ the essential spectrum of $H_\sigma(t)$; once again this only depends on t. To discuss the continuity properties of this family of sets we are going to need some preparations relying mainly on results from [16].

First we write each Σ_t as a disjoint Θ-invariant union $\Sigma_t = \Sigma_t^{\mathrm{g}} \sqcup \Sigma_t^{\mathrm{n}}$. The elements σ_1 of Σ_t^{g} are *generic points for* Σ_t, meaning that each of them is generating Σ_t. The points $\sigma_2 \in \Sigma_t^{\mathrm{n}}$ are *non-generic*, i.e., the closure of the orbit $\mathcal{O}_{\sigma_2}$ is strictly contained in Σ_t.

Let us now fix a point $t \in T$ and a generating element $\sigma \in \Sigma_t$. The monomorphism $\mathcal{N}_\sigma$ extends to an isomorphism between $\mathcal{C}(\Sigma_t)$ and a C^*-subalgebra $\mathcal{B}_\sigma(t)$ of the C^*-algebra $BC_{\mathrm{u}}(\Xi)$ of all the bounded uniformly continuous complex functions on Ξ. It is shown in Lemma 2.2 from [16] that only two possibilities can occur, and this is independent of σ: either $\mathcal{C}(\Xi) \subset \mathcal{B}_\sigma(t)$ (and then t is called *of the first type*), or $\mathcal{C}(\Xi) \cap \mathcal{B}_\sigma(t) = \{0\}$ (and then we say that t is *of the second type*). Correspondingly, one has the disjoint decomposition $T = T_I \sqcup T_{II}$.

Theorem 4.2. *Assume the Framework above. Then the family $\{S^{\mathrm{ess}}(t) \mid t \in T\}$ is outer continuous.*