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Contents

Introduction

This book originates from a graduate course given at the University of Milan in 2007.

Our goal is twofold: first, to present a self-contained introduction to the geometric and analytic aspects of the Yamabe problem on a complete noncompact Riemannian manifold, treating existence, nonexistence, uniqueness and *a priori* estimates of the solutions. Secondly, we intend to describe in a way accessible to the nonspecialist a range of methods and techniques that can be successfully applied to more general nonlinear equations which arise in applications.

The classical Yamabe problem concerns the possibility of pointwise conformally deforming a metric of scalar curvature $S(x)$ on the manifold M to a new metric with prescribed scalar curvature $K(x)$. In the case where K is constant it is a natural higher dimensional generalization of the Poincaré–Köbe Uniformization Theorem for Riemann surfaces and can be seen as a way to select a privileged metric on the manifold.

If $\langle\ ,\ \rangle$ is the original metric of the Riemannian manifold M and we denote with $\widetilde{\langle\ ,\ \rangle} = \varphi^2 \langle\ ,\ \rangle$, $\varphi > 0$, a conformally deformed metric, then the two scalar curvatures $S(x)$ and $\widetilde{S}(x)$ are related by the equation

$$\varphi^2 \widetilde{S}(x) = S(x) - 2(m-1)\frac{\Delta\varphi}{\varphi} - (m-1)(m-4)\frac{|\nabla\varphi|^2}{\varphi^2}$$

(see equation (2.7) in Chapter 2), where Laplacian, gradient, and norm are those of the metric $\langle\ ,\ \rangle$. In the case where the dimension m of the manifold is greater than or equal to three, it is useful to set

$$\varphi = u^{\frac{2}{m-2}}$$

so that the above equation takes the form

$$\widetilde{S} u^{\frac{m+2}{m-2}} = Su - 4\frac{m-1}{m-2}\Delta u.$$

Thus the *Yamabe problem* amounts to finding a positive solution u of the familiar *Yamabe equation*

$$c_m \Delta u - Su + K u^{\frac{m+2}{m-2}} = 0, \tag{1}$$

where $c_m = 4\frac{m-1}{m-2}$ and $K = \widetilde{S}$, the prescribed scalar curvature of the conformally deformed metric. If M is compact and K is constant, after an initial attempted solution by H. Yamabe [Yam60], the problem was solved thanks to efforts of N. Trudinger [Tru68], T. Aubin [Aub76] and R. Schoen [Sch84] (see the nice survey paper by J.M. Lee and T.H. Parker, [LP87], for a complete and self-contained treatment). The solution was obtained using variational methods, and one of the main analytic difficulties stems from the fact that $\frac{m+2}{m-2}$ is the critical exponent for the Sobolev embedding $W^{1,2} \hookrightarrow L^{\frac{2m}{m-2}}$.

A natural generalization of the classical Yamabe problem is the case where K is nonconstant and/or M is noncompact. In this direction we mention the pioneering work of J. L. Kazdan and F. W. Warner, [KW74a], [KW74b], [KW75a], [KW75b]. It should also be mentioned that even the classical Yamabe problem of deforming the metric to one of constant scalar curvature in the noncompact setting is in general not solvable, as first shown by Z. R. Jin, [Jin88].

The Yamabe problem for noncompact manifolds with variable prescribed curvature is the subject of the present monograph. Indeed, we describe methods which allows us to consider the more general *Yamabe-type equations* (resp. inequalities) of the form

$$\Delta u + a(x)u - b(x)u^{\sigma} = 0 \quad (\text{resp. } \geq 0) \tag{2}$$

where $\sigma > 1$, and we study nonexistence, *a priori* estimates, uniqueness and existence.

Equations of the form (2) and still more general differential inequalities of the form

$$u\Delta u + a(x)u^2 - b(x)u^{\sigma+1} \geq -A|\nabla u|^2 \tag{3}$$

arise in complex analysis (e.g. in the study of the structure of complete Kähler manifolds, [LY90], [Li90] and [LR96], in the Schwarz Lemma for the ratio of volume elements of Kähler manifolds of the same dimension, [Gri76], in the study of pluriharmonic functions on a Kähler manifold, [PRS08]), in the study of harmonic maps with bounded dilation ([EL78] and [PRS08] Chapter 8), in the classification of locally conformally flat manifolds ([PRS07]), in the study of Yang-Mills fields, and in population dynamics, to quote only a few examples.

Existence and nonexistence of positive solutions of (2) clearly depend on the geometry of the underlying manifold, typically encoded by curvature or volume growth of geodesic balls, on properties of the coefficients (typically the relative signs of the coefficients $a(x)$ and $b(x)$) and their asymptotic behavior and on the mutual interplay of the two. This interplay can be taken into account in terms of the relative asymptotic behavior of the coefficients versus the geometry at infinity of the manifold or, at a deeper level, in terms of spectral properties of Schrödinger operators naturally associated to the equation.

From the geometrical interpretation of the equation, it is natural to expect it will be easier to have existence when a and b are "close" enough, for instance they have the same sign, while it will be more difficult to have existence (and therefore it will be easier to prove nonexistence) when a and b are farther apart, for instance when they have opposite sign. This expectation is confirmed by both the existence and the nonexistence results that we will describe.

The geometry of the manifold also plays a natural role in the uniqueness results as well in the *a priori* estimates on the solutions. The latter have a particular geometric interest since they are responsible for the completeness/noncompleteness of the deformed metric.

As mentioned above, we use a variety of techniques adapted to the geometric situation at hand in which the lack of compactness and of symmetry and homogeneity prevents the use of more standard tools typical of compact situations or of the Euclidean setting.

In particular, for existence we will essentially use the method of *sub-super solutions*, [Ama76], [Sat73]. Nonexistence will be obtained using *Liouville-type results* which in turn are obtained using either integral formulas or a method based on the coupling of the supposed solution of the Yamabe-type inequality with that of an appropriate Schrödinger-type inequality associated to it, in a manner reminiscent of the classical generalized maximum principle. Uniqueness will be obtained using variants of the *weak maximum principle* (see, e.g., [PRS05b]) and of clever integration by parts arguments. Finally, *a priori* estimates will be typically obtained using an elaboration of the old idea of the proof of the Schwarz's Lemma by L. H. Ahlfors, [Ahl38], which is at the heart of the maximum principle.

The book is divided into seven chapters.

In the first chapter we give a quick review of Riemannian geometry using the method of *moving frames*. While we assume basic knowledge of Riemannian geometry, several computations will be carried out in full detail in order to acquaint the reader with notation and formalism. We concentrate on derivation of the symmetry properties of the curvature tensors together with a number of other identities that will be repeatedly used in the sequel. In particular, we will describe the commutation rules for covariant derivatives up to fourth order. Then we describe comparison results for the Laplacian of the Riemannian distance function, and for the volume of geodesic balls in terms of lower bounds for the Ricci curvature. We point out that our treatment, which follows that of [PRS05b], does not use Jacobi fields.

In Chapter 2 we first derive equations for the change of curvature tensors under a conformal change of the metric and introduce the Yamabe equation. As a side product of our computations we obtain decomposition of the Riemann curvature tensor in its irreducible components and we exhibit the conformal invariance of the Weyl tensor. Then, we briefly consider the case where M is compact to illustrate the interplay between geometry and analysis, with a few illuminating examples such as the Kazdan-Warner obstruction, a result of Obata on Einstein manifolds and a far-reaching "generalization" due to Véron-Véron, through which we prove further results of Escobar. Along the way we give a detailed proof, which inspires to Petersen's treatise [Pet06a], of a famous rigidity result of Obata. The goal is also to give some geometrical feeling on the subject that will enable us to proceed with the noncompact case: the case of the rest of our investigation.

The core of the monograph begins with Chapter 3, devoted to nonexistence results. As mentioned above, since our methods apply to more general situations which have a wide range of applications, we consider in fact differential inequalities of the form (2) and (3). We describe several nonexistence results; in most of them we assume that u satisfies suitable integrability conditions, that $b(x)$ is nonnegative and that there exists a positive solution φ to the differential inequality

$$\Delta\varphi + Ha(x)\varphi \leq -K\frac{|\nabla\varphi|^2}{\varphi}$$

with H, K parameters satisfying $H > 0$, $K > -1$. Note that in the special case where $K = 0$ the latter condition amounts to the fact that the bottom of the spectrum of the operator $-\Delta - Ha(x)$ is nonnegative. Since $-\Delta$ is a nonnegative operator, the condition is trivially satisfied if $a(x) \leq 0$ on M and may be interpreted as a measure of smallness in a spectral sense of the positive part of $a(x)$. This agrees with the heuristic intuition on the effect of the relative signs of $a(x)$ and $b(x)$ on the existence of solutions. The existence of the positive function φ enters the proof in two different ways. In Theorem 3.2 one uses the functions φ to obtain an integral inequality involving u and its gradient from which one concludes that u is constant, and therefore necessarily identically zero. In a second group of results, the function φ is combined with the solution u to give rise to a diffusion-type differential inequality for which we prove a Liouville theorem. This yields the desired triviality. We also show that when σ is greater than or equal to the critical exponent $(m+2)/(m-2)$, then, by performing an appropriate change of the metric and of the solution, the nonexistence results can be improved to allow even some controlled negativity of the coefficient $b(x)$.

Chapter 4 is devoted to establishing *a priori* upper and lower estimates for the asymptotic behavior of solutions of the differential inequalities

$$\Delta u + a(x)u - b(x)u^\sigma \geq 0, \text{ resp. } \Delta u + a(x)u - b(x)u^\sigma \leq 0,$$

under assumptions on $a(x)$ and $b(x)$ related to an assumed radial lower bound for the Ricci curvature. As briefly mentioned above, the results are obtained by applying Alhfors's old idea, namely, one considers an auxiliary function defined in terms of the solution u which by construction attains an extremum, and applies the usual maximum principle. Clearly, the heart of the method consists in finding the best auxiliary function for the problem at hand. We exhibit examples showing that our estimates are essentially sharp. Some further estimates, which cannot be obtained with the previous method, are provided by direct comparison with the aid of the maximum principle (see section 4.3). The chapter ends with some nonexistence results for the Yamabe problem, which complement those described in Chapter 3.

In Chapter 5 we discuss some uniqueness results for positive solutions of Yamabe-type equations (2). The first, the very general Theorem 5.1, states that if the coefficient $b(x)$ is nonnegative and not identically zero, then two solutions whose difference is L^2-integrable are necessarily the same. Although very general, it is sharp, and, remarkably, the assumption on the L^2-integrability cannot be replaced by an L^p condition with $p > 2$. The result is obtained by means of a clever elementary integral inequality. The second result, Theorem 5.2, follows by a comparison argument which relies on a version of the weak maximum principle (see Theorem 5.3) which is interesting in its own. While in most of the results available in the literature, uniqueness is obtained by requiring that solutions have a rather precisely determined asymptotic behavior, our result applies to solutions whose behavior at infinity is specified in a much less stringent manner, see (5.13); moreover, the conclusion is reached assuming only conditions on the volume growth of the manifold. Counterexamples show the sharpness of each result. The chapter ends with a geometric application to the group of conformal diffeomorphisms of a complete manifold and to the uniqueness of solutions of the geometric Yamabe problem.

Chapter 6 deals with existence results for Yamabe-type equations (2) on the complete, noncompact, Riemannian manifold M. The main tool is the *monotone iteration scheme* in various forms, and we give a rather detailed description of it in the appendix at the end of the chapter. The application of the scheme in this context goes back to W. M. Ni, [Ni82], in the Euclidean setting and to P. Aviles and R. C. McOwen, [AM85] and [AM88], for noncompact manifolds. After having introduced some preliminary material on spectral theory, and a useful comparison result, the main body of the chapter is then devoted to the construction of (global and local) super- and subsolutions for the problem. In general terms, supersolutions are obtained under assumptions on the sign of $b(x)$ and of the first eigenvalue of $L = \Delta + a(x)$ on appropriate domains. Because of the combination of signs of the coefficients, subsolutions are harder to find. We give a number of sufficient conditions which ensure that such subsolutions do exist: among them the spectral condition $\lambda_1^L(M) < 0$, for which we provide a new sufficient condition contained in Theorem 6.11. Furthermore, we mention Theorem 6.15, in which existence is guaranteed under a very week growth condition on $b(x)$, and also Theorem 6.16, where a further weakining on the condition on the sign of $b(x)$ is balanced by the necessity of imposing a constant negative lower bound on the Ricci curvature. We explicitly note that the assumptions of our existence theorems match those of the nonexistence results in the previous chapters.

In the last chapter, Chapter 7, we consider some particular cases where the symmetry of the geometry allows one to use special techniques and to obtain stronger results. Typically this happens in Euclidean and Hyperbolic spaces, and more generally in the case of *models* (in the sense of R. Greene and H. Wu, [GW79]), or manifolds with special symmetry. The specific feature of models which make the analysis more precise is that the Laplacian of the distance function from the origin is given explicitly, as opposed to the case of a general manifold where only upper and lower bounds may be obtained under suitable curvature assumptions, by means of the the Laplacian and Hessian comparison theorems, and where the possible presence of the cut locus raises additional difficulties.

We describe refined techniques adapted to the situation at hand and obtain results that, as a by-product, show the degree of sharpness of the general theory and methods we have developed dealing with generic complete Riemannian manifolds. It seems worth remarking that, in the specific case of Hyperbolic space, we provide a nonexistence result with the aid of a Rellich-Pohozaev type formula (see Theorem 7.7) and, even more, in Proposition 7.9 we introduce an integral obstruction to the existence of a conformal deformation which is of a different nature with respect to the Kazdan-Warner condition.

Many of the results presented in this monograph have been obtained over the years by the authors jointly with many collaborators. To all of them we wish to extend our thanks and appreciation. In particular we are indebted to S. Pigola and M. Rimoldi who provided us with the proof of Theorem 2.10 in Chapter 2.

Chapter 1

Some Riemannian Geometry

In this chapter we give a quick review of Riemannian geometry using the moving frame formalism. While we assume basic knowledge of Riemannian geometry, several computations will be carried out in full detail in order to acquaint the reader with notation and formalism. After having introduced frame and coframes, we will describe connection and curvature in terms of the connection and curvature forms. Symmetry properties of the curvature tensors will be described in detail and we will derive a number of identities that will be repeatedly used in the sequel. In particular, we will obtain the commutation rules for covariant derivatives up to fourth order. Along the way, we will introduce Einstein manifolds and prove Schur's lemma. Then we introduce basic results for the Riemannian distance function from a fixed reference point $o \in M$, and discuss briefly the cut locus and some of its properties. We will then describe comparison results for the Laplacian of the Riemannian distance function and for the volume of geodesic balls in terms of lower bounds for the Ricci curvature. We point out that our treatment, which follows that of [PRS05b], does not use Jacobi fields. The chapter ends with a brief section on the geometry of immersed submanifolds to fix notation and terminology used in the second part of Chapter 2.

1.1 Preliminaries

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold of dimension m with metric $\langle\ ,\ \rangle$. The aim of this section is to fix notation and to describe the essential facts of the geometry of $(M, \langle\ ,\ \rangle)$ using É. Cartan's formalism. The usefulness of this approach will become apparent in the sequel, and it will prove to be particularly effective in Chapter 2 in the derivation of the formulae which express the change of the Riemannian, Ricci and scalar curvature under a conformal change of the metric.

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1.1.1 Moving frames and the first structure equations

Let $p \in M$ and (U, φ) a local chart such that $U \ni p$, with coordinate functions $x^1, \dots, x^m$, $m = \dim(M)$. If q is a generic point in U we have, at q,

$$\langle \, , \, \rangle = \langle \, , \, \rangle_{ij} \, dx^i \otimes dx^j, \tag{1.1}$$

where dx^i denotes the differential of the function x^i and $\langle \, , \, \rangle_{ij}$ are the (local) components of the metric, defined by $\langle \, , \, \rangle_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$. In relation (1.1), and throughout the book, we adopt the Einstein summation convention for repeated indices. Applying in q the Gram-Schmidt orthonormalization process we can find linear combinations of the 1-form dx^k, which we will call θ^i, such that

$$\langle \, , \, \rangle = \delta_{ij} \, \theta^i \otimes \theta^j, \tag{1.2}$$

where δ_{ij} is the Kronecker symbol. Since, as q varies in U, the previous process gives rise to coefficients that are C^∞-functions of q, the set of 1-forms $\{\theta^i\}$, $i = 1, \dots, m$, define an orthonormal system on U for the metric $\langle \, , \, \rangle$, i.e., a (local) *orthonormal (o.n.) coframe.* We will sometimes write

$$\langle \, , \, \rangle = \sum_{i=1}^{m} (\theta^i)^2$$

instead of (1.2). We also define the (local) *dual orthonormal frame* $\{e_i\}$, $i = 1, \dots, m$, as the set of m (local) vector fields satisfying, on the open set U,

$$\theta^j(e_i) = \delta_i^j \tag{1.3}$$

(where δ_i^j is just a suggestive way of writing the Kronecker symbol, reflecting the position of the indexes in the pairing of θ^i and e_j). We have the following

Proposition 1.1. *Let $\{\theta^i\}$ be a local o.n. coframe on M, defined on an open set U; then there exist unique 1-forms*

$$\{\theta_j^i\}, \quad i, j = 1, \dots, m$$

on U such that, $\forall \, i, j = 1, \dots, m$,

$$d\theta^i = -\theta_j^i \wedge \theta^j, \tag{1.4}$$

$$\theta_j^i + \theta_i^j = 0. \tag{1.5}$$

The forms $\{\theta_j^i\}$ are called the Levi-Civita connection forms *associated to the o.n. coframe $\{\theta^i\}$.*

Remark. Equations (1.4) are classically known as the *first structure equations.*

Proof. Let us assume the existence of the forms $\{\theta^i_j\}$ satisfying (1.5) and (1.4) and determine their expression. Of course

$$\theta^i_j = a^i_{jk}\theta^k$$

for some $a^i_{jk} \in C^\infty(U)$ (where $C^\infty(U)$ denotes the set of smooth functions defined on the open set U), and (1.5) is equivalent to

$$a^i_{jk} + a^j_{ik} = 0. \tag{1.6}$$

The 2-forms $d\theta^i$ can be written, for some (unique) coefficients $b^i_{jk} \in C^\infty(U)$, as

$$d\theta^i = \frac{1}{2} b^i_{jk}\,\theta^j \wedge \theta^k, \quad b^i_{jk} + b^i_{kj} = 0.$$

Since (1.4) must hold we have:

$$\frac{1}{2} b^i_{jk}\theta^j \wedge \theta^k = -a^i_{jk}\theta^k \wedge \theta^j = a^i_{jk}\theta^j \wedge \theta^k = \frac{1}{2}(a^i_{jk} - a^i_{kj})\theta^j \wedge \theta^k.$$

It follows that

$$b^i_{jk} = a^i_{jk} - a^i_{kj}. \tag{1.7}$$

Cyclic permutations of the indices i, j, k and use of (1.6) and (1.7) yield

$$b^k_{ij} = a^k_{ij} - a^k_{ji} = -a^i_{kj} + a^j_{ki}; \tag{1.8}$$

$$b^j_{ki} = a^j_{ki} - a^j_{ik} = a^j_{ki} + a^i_{jk}. \tag{1.9}$$

Adding (1.7) and (1.9) and subtracting (1.8) we get

$$a^i_{jk} = \frac{1}{2}(b^i_{jk} - b^k_{ij} + b^j_{ki}). \tag{1.10}$$

The previous relation determines the expression of the forms θ^i_j and also proves uniqueness. Now define

$$\theta^i_j = \frac{1}{2}(b^i_{jk} - b^k_{ij} + b^j_{ki})\theta^k, \tag{1.11}$$

where the b^i_{jk}'s satisfy

$$b^i_{jk} + b^i_{kj} = 0.$$

It is clear that

$$a^i_{jk} = \frac{1}{2}\left(b^j_{ik} - b^k_{ji} + b^i_{kj}\right) = -\frac{1}{2}\left(b^i_{jk} - b^k_{ij} + b^j_{ki}\right) = -a^i_{jk},$$

thus (1.6) is met, and then the θ^i_j defined in (1.11) satisfy (1.5); it is also immediate to verify that they satisfy (1.4). □

1.1.2 Covariant derivative of tensor fields

The Levi-Civita connection forms are the starting point to define a *covariant derivative of tensor fields* in the following way. Following standard notation we denote with T_pM the tangent space at $p \in M$ and with T_p^*M the cotangent space at p.
We recall that a *tensor field* T *of type* (r,s) is a law that assigns to all points $p \in M$ a multilinear map

$$T_p : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{r \text{ times}} \times \underbrace{T_pM \times \cdots \times T_pM}_{s \text{ times}} \to \mathbb{R}$$

with the usual differentiability request with respect to the variable p (see e.g. [Lee03]). The set of tensor fields of type (r,s) will be denoted by $T_r^s(M)$. Let us begin considering the case of a *vector field* X on M, i.e., a tensor of type $(1,0)$. We denote with $\mathfrak{X}(M)$ the set of all (smooth) vector fields on M.

Let $\{\theta^i\}$ a local o.n. coframe and $\{e_i\}$ the dual frame.

Definition 1.2. *The* covariant derivative of the vector field X*,* ∇X*, is the tensor field of type* $(1,1)$ ∇X *defined in the following way: if* $X = X^i e_i$*,*

$$\nabla X = (dX^i) \otimes e_i + X^i \nabla e_i,$$

having defined

$$\nabla e_i = \theta_i^j \otimes e_j.$$

Setting

$$X_k^i \theta^k = dX^i + X^j \theta_j^i,$$

∇X *can be then written as*

$$\nabla X = (dX^i + X^j \theta_j^i) \otimes e_i = X_k^i \theta^k \otimes e_i,$$

and X_k^i *is said to be the covariant derivative of the coefficient* X^i*.*

If $Y \in \mathfrak{X}(M)$ we define the *covariant derivative of* X *in the direction of* Y as the vector field

$$\nabla_Y X = \nabla X(Y),$$

which in components reads as

$$\nabla_Y X = X_k^i \theta^k(Y) e_i = X_k^i Y^k e_i.$$

Definition 1.3. *The* divergence *of the vector field* $X \in \mathfrak{X}(M)$ *is the trace of* ∇X*, that is,*

$$\operatorname{div} X = \operatorname{tr}(\nabla X) = \langle \nabla e_i X, e_i \rangle = X_i^i. \tag{1.12}$$

Analogously, for a 1-form ω (i.e., a tensor of type $(0,1)$), we have:

Definition 1.4. *The* covariant derivative of the 1-form ω, $\nabla\omega$, *is the tensor field of type* $(0,2)$ *defined in the following way: if* $\omega = \omega_i \theta^i$,

$$\nabla\omega = (d\omega_i) \otimes \theta^i + \omega_i \nabla\theta^i,$$

with

$$\nabla\theta^i = -\theta^i_j \otimes \theta^j.$$

Note that, setting

$$\omega_{ik}\theta^k = d\omega_i - \omega_j \theta^j_i,$$

it follows that

$$\nabla\omega = \omega_{ik}\theta^k \otimes \theta^i.$$

If $Y \in \mathfrak{X}(M)$ we define the *covariant derivative of* ω *in the direction of* Y as the 1-form

$$\nabla_Y\omega = \nabla\omega(Y),$$

which in components reads as

$$\nabla_Y\omega = \omega_{ik}\theta^k(Y)\theta^i = \omega_{ik}Y^k\theta^i.$$

For a 0-form $f \in C^\infty(M)$ we set

$$\nabla f = df \quad \text{(exterior differential of } f\text{)}.$$

We point out that this notation may give rise to some ambiguity; indeed, in the literature (and also in this book) ∇f often denotes the *gradient* of f, i.e., the vector field dual to the 1-form df: in this case, using standard notation (see for instance [Lee97]), we can write $\nabla f = (df)^\sharp$, where $\sharp$ is the *sharp map* from the cotangent bundle T^*M to the tangent bundle TM defined by

$$\left\langle (df)^\sharp, Y \right\rangle = \langle \nabla f, Y \rangle = df(Y) = Y(f),$$

for all $Y \in \mathfrak{X}(M)$. Note that, in components, setting $df = f_j\theta^j$ for some smooth coefficients f_j, we have $(\nabla f)^i = \delta^{ij}(df)_j = \delta^{ij}f_j = f_i$ (that is, in an orthonormal frame, differential and gradient of a function have the same coefficients with respect to the (dual) basis $\{\theta^i\}$ and $\{e_i\}$).

Finally, ∇ can be extended in a natural way to a generic tensor T, in order to define a connection on each tensor bundle $T^s_r(M)$: this extension of ∇ satisfies the Leibniz rule and some other nice properties, like the commutativity with the trace on any pair of indices (see again [Lee97]).

Although the covariant derivative has been introduced by means of locally defined objects, it is possible to show (using the transformation laws of $\{\theta^i\}$ and $\{\theta^i_j\}$ as the coframe changes) that the new tensor field thus obtained is *globally* defined.

Remark. One can verify that the previous definition matches the "canonical" one usually given in terms of the Koszul formalism (see for example [Lee97], [Pet06a]). Indeed, as we will see before long, the operator ∇ coincides precisely with the Levi-Civita connection associated to the metric $\langle\ ,\ \rangle$ of M.

1.1.3 Meaning of the first structure equations

We now want to discuss the geometric meaning of condition (1.5), namely

$$\theta^i_j + \theta^j_i = 0.$$

To this purpose let us compute the covariant derivative of the metric tensor $\langle\ ,\ \rangle$. Using the Leibniz rule, and recalling that for every tangent vector $X_p = X(p) \in T_pM$ (with p in the domain of the o.n. coframe $\{\theta^i\}$) it holds that $\nabla_{X_p}\theta^i = -\theta^i_j(X_p)\theta^j$, we have

$$\begin{aligned}\nabla_{X_p}\langle\ ,\ \rangle &= \nabla_{X_p}(\delta_{ij}\theta^i\theta^j) = \delta_{ij}(\nabla_{X_p}\theta^i \otimes \theta^j + \theta^i \otimes \nabla_{X_p}\theta^j)\\ &= \delta_{ij}(-\theta^i_k(X_p)\theta^k \otimes \theta^j - \theta^j_k(X_p)\theta^i \otimes \theta^k)\\ &= -\theta^i_k(X_p)\theta^k \otimes \theta^i - \theta^i_k(X_p)\theta^i \otimes \theta^k\\ &= -(\theta^i_k + \theta^k_i)(X_p)\theta^k \otimes \theta^i,\end{aligned}$$

and therefore

$$\nabla\langle\ ,\ \rangle = 0 \ \text{ if and only if } \ \theta^i_j + \theta^j_i = 0,$$

i.e., *condition* (1.5) *is equivalent to the parallelism of the metric* (in other words: $\forall\, X, Y, Z \in \mathfrak{X}(M),\ X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$).

On the other hand, the first structure equations (1.4) tell us that the metric is *torsion-free*. Indeed, let X and Y be two vector fields on M and $[X, Y]$ their *Lie bracket*, defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall f \in C^\infty(M). \tag{1.13}$$

We claim that condition

$$[X, Y] = \nabla_X Y - \nabla_Y X \quad \forall\, X, Y \in \mathfrak{X}(M) \tag{1.14}$$

is equivalent to the validity of (1.4). Note that the left-hand side of (1.14) is independent of the choice of a metric on M. Since the *torsion* of a generic connection ∇ on M is the $(0, 2)$ tensor field $\mathrm{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, this justifies the expression “torsion-free” used above. To prove the equivalence, recall that the exterior differential of a 1-form ω is intrinsically defined by

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]);$$

moreover, as a consequence of the definition of covariant derivative,

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y), \tag{1.15}$$

so that

$$X\big(\theta^i(Y)\big) - \theta^i(\nabla_X Y) = (\nabla_X\theta^i)(Y) = -\theta^i_j(X)\theta^j(Y),$$

that is,

$$X\big(\theta^i(Y)\big) + \theta^i_j(X)\theta^j(Y) = \theta^i(\nabla_X Y).$$

Then we compute $\big(d\theta^i + \theta^i_j \wedge \theta^j\big)(X,Y)$, that is

$$\begin{aligned}
&d\theta^i(X,Y) + \theta^i_j \wedge \theta^j(X,Y)\\
&= X(\theta^i(Y)) - Y(\theta^i(X)) - \theta^i([X,Y]) + \theta^i_j(X)\theta^j(Y) - \theta^i_j(Y)\theta^j(X)\\
&= X(\theta^i(Y)) + \theta^i_j(X)\theta^j(Y)) - Y(\theta^i(X)) - \theta^i_j(Y)\theta^j(X)) - \theta^i([X,Y])\\
&= \theta^i(\nabla_X Y - \nabla_Y X - [X,Y]),
\end{aligned}$$

and the claim follows.

Remark. By the fundamental theorem of Riemannian geometry (see for instance [Lee97] or [Pet06b]), we deduce that the connection ∇ coincides, as we said previously, with the Levi-Civita connection of the metric $\langle\ ,\ \rangle$.

We now define the *Lie derivative* of Y in the direction of X to be $\mathcal{L}_X Y = [X,Y]$, so that condition (1.14) can be written in the form

$$\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X. \tag{1.16}$$

Setting also

$$\mathcal{L}_X f = X(f) \tag{1.17}$$

for $f \in C^\infty(M)$, and

$$(\mathcal{L}_X \omega)(Y) = \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X Y), \tag{1.18}$$

if ω is a 1-form, we can extend $\mathcal{L}_X$ to a generic tensor field requiring $\mathbb{R}$-linearity and the validity of the Leibniz rule (see also [Lee03], [Pet06a]). Using (1.15), we compute the Lie derivative of the metric in the direction of X, $\mathcal{L}_X\langle\ ,\ \rangle$ (note that this latter has to be a covariant tensor of order 2, that is, a $(0,2)$-tensor):

$$\begin{aligned}
(\mathcal{L}_X\langle\ ,\ \rangle)(Y,Z) &= ((\mathcal{L}_X\theta^i)\otimes\theta^i + \theta^i\otimes(\mathcal{L}_X\theta^i))(Y,Z)\\
&= \theta^i(Z)(\mathcal{L}_X\theta^i)(Y) + \theta^i(Y)(\mathcal{L}_X\theta^i)(Z)\\
&= \theta^i(Z)\big[\mathcal{L}_X(\theta^i(Y)) - \theta^i(\mathcal{L}_X Y)\big]\\
&\quad + \theta^i(Y)\big[\mathcal{L}_X(\theta^i(Z)) - \theta^i(\mathcal{L}_X Z)\big]\\
&= \theta^i(Z)X(\theta^i(Y)) - \theta^i(Z)\theta^i(\nabla_X Y - \nabla_Y X)\\
&\quad + \theta^i(Y)X(\theta^i(Z)) - \theta^i(Y)\theta^i(\nabla_X Z - \nabla_Z X)\\
&= \theta^i(Z)(\nabla_X\theta^i)(Y) + \theta^i(Y)(\nabla_X\theta^i)(Z)\\
&\quad + \theta^i(Z)\theta^i(\nabla_Y X) + \theta^i(Y)\theta^i(\nabla_Z X)\\
&= (\nabla_X\theta^i\otimes\theta^i + \theta^i\otimes\nabla_X\theta^i)(Y,Z) + \langle\nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle\\
&= (\nabla_X\langle\ ,\ \rangle)(Y,Z) + \langle\nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle\\
&= \langle\nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle\,,
\end{aligned}$$

where in the last equality we have used the fact that the metric is parallel with respect to the Levi-Civita connection. Thus, we have proved the useful identity

$$(\mathcal{L}_X \langle\ ,\ \rangle)(Y, Z) = \langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle \tag{1.19}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, which will be repeatedly used in the sequel. Note that equation (1.19) in components reads as

$$(\mathcal{L}_X \langle\ ,\ \rangle)_{ij} = \langle \nabla_{e_i} X, e_j\rangle + \left\langle e_i, \nabla_{e_j} X\right\rangle = X^j_i + X^i_j. \tag{1.20}$$

It can be proved that the Lie derivative of Y in the direction of X has the following geometric meaning (see e.g. [Lee03]):

$$(\mathcal{L}_X Y)_p = \left.\frac{d}{dt}\right|_{t=0} (\varphi_{-t})_* Y_{\varphi_t(p)} = \lim_{t\to 0} \frac{(\varphi_{-t})_* Y_{\varphi_t(p)} - Y_p}{t},$$

where φ_t is the local flow generated by X and $(\varphi_t)_*$ is the push-forward. The analogous applies to $\mathcal{L}_X h$, with h a generic tensor field (see also Chapter 2 for the special case of $\mathcal{L}_X \langle\ ,\ \rangle$).

1.1.4 Curvature: the second structure equations

We now consider the second structure equations. Let $\{\theta^i\}$ be a local o.n. frame and $\{\theta^i_j\}$ the corresponding Levi-Civita connection forms. The *curvature forms* $\{\Theta^i_j\}$ associated to the coframe are defined through the *second structure equations*,

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j. \tag{1.21}$$

Obviously the Θ^i_j are 2-forms. Since, according to (1.5), $\theta^i_j + \theta^j_i = 0$, it follows immediately that the Θ^i_j's satisfy the same antisymmetry condition

$$\Theta^i_j + \Theta^j_i = 0. \tag{1.22}$$

Using the basis $\{\theta^i \wedge \theta^j\}$, $1 \le i < j \le m$, of the space of skew-symmetric 2-forms $\bigwedge^2(U)$ on the open set U, we may write

$$\Theta^i_j = \frac{1}{2} R^i_{jkt}\theta^k \wedge \theta^t = \sum_{k<t} R^i_{jkt}\theta^k \wedge \theta^t \tag{1.23}$$

for some coefficients $R^i_{jkt} \in C^\infty(U)$ satisfying

$$R^i_{jkt} + R^i_{jtk} = 0, \tag{1.24}$$

while (1.22) implies that

$$R^i_{jkt} + R^j_{ikt} = 0. \tag{1.25}$$

From (1.24) and (1.25) we thus deduce the symmetries

$$R^i_{jkt} = -R^i_{jtk} = -R^j_{ikt}. \tag{1.26}$$

Differentiating the first structure equation, using the second and the properties of the exterior differential we have

$$\begin{aligned} 0 = d(d\theta^i) = -d(\theta^i_j \wedge \theta^j) &= -d\theta^i_j \wedge \theta^j + \theta^i_j \wedge d\theta^j \\ &= \theta^i_k \wedge \theta^k_j \wedge \theta^j - \Theta^i_j \wedge \theta^j - \theta^i_j \wedge \theta^j_k \wedge \theta^k \\ &= \theta^j \wedge \Theta^i_j, \end{aligned}$$

that is

$$\theta^j \wedge \Theta^i_j = 0. \tag{1.27}$$

These identities go under the name of the *first Bianchi identities*. Using (1.23) we get

$$0 = R^i_{jkt}\theta^j \wedge \theta^k \wedge \theta^t = \sum_{1\leq j<k<t\leq m} (R^i_{jkt} + R^i_{ktj} + R^i_{tjk})\theta^j \wedge \theta^k \wedge \theta^t,$$

and then we deduce the first Bianchi identities in the classical form

$$R^i_{jkt} + R^i_{ktj} + R^i_{tjk} = 0. \tag{1.28}$$

It is possible also to show that another consequence of (1.26) and (1.28) is the last, important symmetry

$$R^i_{jkt} = R^k_{tij}. \tag{1.29}$$

One can verify that the coefficients R^i_{jkt} gives rise to a (global) $(1,3)$-tensor, called the *Riemann curvature tensor*,

$$\mathrm{Riem} = R^i_{jkt}\theta^k \otimes \theta^t \otimes \theta^j \otimes e_i.$$

We warn the reader that there are a number of different conventions for the Riemann curvature tensor (see the discussion in [Lee97]). We will often use the *curvature tensor* of type $(0,4)$ given by

$$R = R_{ijkt}\theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^t,$$

with $R_{ijkt} = R^i_{jkt}$. Note that we have performed the classical operation of lowering indices using the metric tensor, that is, in our chosen orthonormal frame,

$$R_{ijkt} = \delta_{is}R^s_{jkt} = R^i_{jkt}$$

(compare with the discussion about gradient and differential in section 1.1.2). We complete the description of the symmetries of the curvature tensor noting that, from the first Bianchi identities, we can obtain the *second Bianchi identities* involving the covariant derivatives of the components of the curvature tensor, namely:

$$R_{ijkt,l} + R_{ijtl,k} + R_{ijlk,t} = 0. \tag{1.30}$$

Tracing the curvature tensor on its first and third indices (or, equivalently, on its second and fourth) we obtain the *Ricci tensor*

$$\mathrm{Ric} = R(e_i, \cdot, e_i, \cdot) = R(\cdot, e_i, \cdot, e_i,) = R_{ijik}\theta^j \otimes \theta^k = R_{jk}\theta^j \otimes \theta^k,$$

where $\{e_i\}$ is the dual basis of $\{\theta^j\}$. Note that, according to (1.29), Ric is a symmetric $(0,2)$-tensor; in other words, $\forall\, X, Y \in \mathfrak{X}(M)$

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X), \tag{1.31}$$

and in components

$$R_{ij} = R_{ji}. \tag{1.32}$$

The *scalar curvature* S is defined as the trace of Ric, i.e.,

$$S = \mathrm{Ric}(e_i, e_i) = R_{ijij} = R_{ii}.$$

The *sectional curvature of the 2-plane* $\pi \subset T_pM$ spanned by the vectors u and v is defined to be

$$K_p(\pi) = \frac{R(u,v,u,v)}{\langle u,u\rangle \langle v,v\rangle - \langle u,v\rangle^2} \in \mathbb{R}.$$

It is not difficult to verify that the right-hand side of the above formula is in fact independent of the chosen basis of π. Clearly, if u v are an orthonormal basis for π, then

$$K_p(\pi) = R(u,v,u,v).$$

We note that a common notation for the sectional curvature of the plane π spanned by u and v is

$$K_p(\pi) = \mathrm{Sect}(u \wedge v).$$

1.1.5 Einstein manifolds and Schur's Theorem

Definition 1.5. *The manifold* $(M, \langle\ ,\ \rangle)$, $\dim(M) = m \geq 2$, *is said to be* Einstein *if*

$$\mathrm{Ric} = \lambda \langle\ ,\ \rangle, \tag{1.33}$$

for some $\lambda \in \mathbb{R}$.

Using the moving frame formalism we now show that, if the dimension of the manifold is greater than or equal to 3 and equation (1.33) holds for some $\lambda \in C^\infty(M)$, where $C^\infty(M)$ is the set of smooth functions defined on M, then

λ is automatically constant. Indeed, first note that, tracing equation (1.33), we immediately obtain

$$\lambda = \frac{S}{m}. \tag{1.34}$$

We now trace the second Bianchi identities (1.30) with respect to the indices i and l to get

$$R_{ijkt,i} + R_{ijti,k} + R_{ijik,t} = 0.$$

Since covariant derivative commutes with contractions, the previous relation yields

$$R_{ijkt,i} = R_{jt,k} - R_{jk,t}, \tag{1.35}$$

whence, contracting again this time with respect to j and k, we get

$$R_{ikkt,i} = R_{kt,k} - R_{kk,t}$$

that is,

$$2R_{kt,\ k} = S_t. \tag{1.36}$$

Now because of (1.33) and (1.34) we have

$$R_{kt} = \frac{S}{m}\delta_{kt},$$

and using again the fact that the metric tensor is parallel we deduce that

$$R_{kt,l} = \frac{1}{m}S_l\delta_{kt}.$$

Now tracing with respect to k and l, we get

$$R_{kt,k} = \frac{1}{m}S_t; \tag{1.37}$$

substituting in (1.36), we obtain

$$\left(\frac{2}{m} - 1\right)S_t = 0,$$

and we conclude that if $m \geq 3$ and M is connected, then the scalar curvature, and therefore λ, are constant. We have thus proved the well-known result of Schur:

Theorem 1.6. *Let $(M, \langle\ ,\ \rangle)$ be a connected Riemannian manifold of dimension $m \geq\ 3$. If*

$$\mathrm{Ric} = \lambda\langle\ ,\ \rangle$$

for some $\lambda \in C^\infty(M)$, then M is Einstein.

Note that, if $m \geq 3$, then $(M, \langle\, , \rangle)$ is Einstein if and only if the *traceless Ricci tensor*

$$T = \mathrm{Ric} - \frac{S}{m} \langle\, , \rangle \tag{1.38}$$

is identically null. Observe also that, in our o.n. coframe,

$$T_{ij} = R_{ij} - \frac{S}{m}\delta_{ij}. \tag{1.39}$$

Using (1.37) (sometimes called *Schur's identities*) we shall obtain a remarkable formula, an infinitesimal version of the *Kazdan-Warner obstruction*, that will be discussed in detail below (see Section 2.1.2).

1.2 Comparison theorems

1.2.1 Ricci identities

We now want to recall that the curvature tensor can be interpreted as an obstruction to the validity of Schwarz's theorem for mixed derivatives of third order and higher. Since this will be useful in the sequel, we consider the case of a function $u : M \to \mathbb{R}$ which we assume to be at least $C^3(M)$. If

$$du = u_i \theta^i \tag{1.40}$$

for some smooth coefficients u_i, the *Hessian* of u is defined as the 2-covariant tensor $\mathrm{Hess}(u) = \nabla du$ of components u_{ij} given by

$$u_{ij}\theta^j = du_i - u_k \theta^k_i, \tag{1.41}$$

that is,

$$\mathrm{Hess}(u) = u_{ij}\theta^j \otimes \theta^i. \tag{1.42}$$

The *Laplacian* of u is the trace of the Hessian, that is,

$$\Delta u = \mathrm{tr}\,(\mathrm{Hess}(u)) = u_{ii}. \tag{1.43}$$

One can verify that the operator Δ defined in (1.43) is the Laplace-Beltrami operator associated to the metric $\langle\, , \rangle$ (see for instance [Pet06b]). Since second derivatives commute we expect $\mathrm{Hess}(u)$ to be a symmetric tensor. This can be verified as follows: we differentiate equation (1.40) and use the structure equations to get

$$\begin{aligned} 0 &= du_i \wedge \theta^i + u_i d\theta^i = (u_{ij}\theta^j + u_k\theta^k_i) \wedge \theta^i - u_i \theta^i_k \wedge \theta^k \\ &= u_{ij}\theta^j \wedge \theta^i \\ &= \frac{1}{2}(u_{ij} - u_{ji})\theta^j \wedge \theta^i, \end{aligned}$$