Pseudo-Differential Operators
Theory and Applications
9

Leon Cohen

The Weyl Operator and its Generalization

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To Carol, Valerie, Ken, Livia,
Douglas, Rummy, and Polar

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Preface

The concept of associating ordinary functions with operators has arisen in many areas of science and mathematics and it can be argued that the earliest instance was Leibniz's attempt to define a fractional derivative. Up to the beginning of the twentieth century, many isolated results were obtained and culminated with the remarkable contributions of Heaviside and the efforts to put his methods on a sound mathematical footing. These developments were mostly based on associating a function of one variable with one operator, the operator generally being the differentiation operator.

With the discovery of quantum mechanics in the years 1925-1930, there arose, in a natural way, the issue that one has to associate a function of two variables with a function of two operators that do not commute. This has led to a wonderfully rich mathematical development that has found applications in many fields, including pseudo-differential operators, time-frequency analysis, quantum optics, wave propagation, differential equations, image processing, radar, sonar, chemical physics, and acoustics, among others. The earliest proposal for associating an ordinary function of two variables with an operator was that of Born and Jordan (1925), and subsequently Weyl (1929) and others proposed other rules. There are an infinite number of ways to associate a function of two ordinary variables with a function of two operators because ordinary variables commute while operators generally do not. The rules became known as rules of association, correspondence rules, or ordering rules.

Independently of these developments Wigner, in 1932, and Kirkwood, in 1933 devised a classical-like joint distribution where one can calculate operator averages in the standard probability manner, that is, by phase-space integration. No connection was made between these distributions and correspondence rules until Moyal in 1949 saw things clearly. He *derived* the Wigner distribution using the Weyl correspondence. Subsequently it was realized that for every correspondence rule there is a corresponding phase-space distribution. Now the field of correspondence rules and phase-space distributions are intimately connected. Remarkably, around the same time as Moyal, a similar development occurred in the field of time-varying spectral analysis whose aim was to understand signals with changing frequencies, human speech being the prime historical example. It was realized by Gabor and Ville that one can define time and frequency operators and make a mathematical analogy with the quantum case.

I have aimed to present the basic ideas and results of correspondence rules in a straightforward elementary manner and have included many examples to illustrate the ideas developed. I have strived to make the mathematics accessible to a wide audience and I have avoided delving into advanced formulations such as group theoretical considerations. The level of rigor and terminology are those of the original contributors and standard in mathematical physics and engineering.

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Chapter 1

Introduction and Terminology

This book deals with correspondence rules or rules of association. The fundamental idea is to associate a function of ordinary variables with an operator. While in later chapters we deal with arbitrary operators, in the main portion of the book we deal with the operators X and D where,

$$X = \begin{cases} x & \text{in the } x \text{ representation} \\ i\frac{d}{dp} & \text{in the Fourier representation,} \end{cases} \tag{1.1}$$

$$D = \begin{cases} \frac{1}{i}\frac{d}{dx} & \text{in the } x \text{ representation} \\ p & \text{in the Fourier representation.} \end{cases} \tag{1.2}$$

The fundamental relation between X and D is the commutator

$$[X, D] = XD - DX = i. \tag{1.3}$$

Depending on the field, these operators may be appropriately called position and spatial frequency, or position and momentum, and in time-frequency analysis they correspond to the time and frequency operators.

One of the basic reasons for considering these particular operators is that we can use them to evaluate expectation values of functions in the Fourier or x representation without leaving the representation. In particular, suppose we have a function, $f(x)$, and a "state" function $\varphi(x)$; then

$$\int f(x)\,|\,\varphi(x)|^2\,dx = \int \widehat{\varphi}^*(p) f(X)\,\widehat{\varphi}(p)\,dp \tag{1.4}$$

where $\widehat{\varphi}(p)$ is the Fourier transform of $\varphi(x)$,

$$\widehat{\varphi}(p) \;=\; \frac{1}{\sqrt{2\pi}} \int \varphi(x)\,e^{-ixp}\,dx. \tag{1.5}$$

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Therefore, we say that $f(x)$ in the x domain is associated with or represented by the operator $f(X)$ in the Fourier domain and we write

$$f(X) \leftrightarrow f(x) \tag{1.6}$$

where $\leftrightarrow$ indicates the association. Furthermore, suppose we have a function of p, $g(p)$, in the Fourier domain, then

$$\int g(p) \left|\widehat{\varphi}(p)\right|^2 dp = \int \varphi^*(x) g(D) \varphi(x)\, dx \tag{1.7}$$

and we say that $g(p)$ in the Fourier domain is associated with $g(D)$ in the x domain,

$$g(D) \leftrightarrow g(p). \tag{1.8}$$

The right hand side of Eq. (1.7) is sometimes called *sandwiching* because the operator, $g(D)$, is in between $\varphi^*(x)$ and $\varphi(x)$.

The basic advantage of Eq. (1.7) is that if we want to calculate the expectation of $g(p)$ as defined by the left hand side we do not have to first calculate $\widehat{\varphi}(p)$. We can remain in the x representation and use the right hand side to calculate it.

1.1 The Fundamental Issue

In the above discussion we had no difficulty in associating a function of one variable $f(x)$ or $g(p)$ with its corresponding operator. But what if we have a function of two variables, for example xp^2, then what will the association be? It could be, for example, XD^2, DXD, or D^2X, among others; all of these associations are proper in the sense that they reduce to xp^2 if we just think of the operators as ordinary variables. However, all these choices are different because X and D do not commute. Formulating such associations for a general function $a(x,p)$ is the fundamental aim. There have been many rules proposed, among them the Weyl, Standard, and Born-Jordan. In the next chapter we study the Weyl rule and subsequently other rules. In Chap. 4 we present a general method that handles all rules in a unified manner.

Furthermore, suppose we did have an association for the function $a(x,p)$ with an operator $A(X,D)$. Then, what is the generalization of Eqs. (1.4) and (1.7)? We will see that the proper generalization is that for a state function $\varphi(x)$ we have to introduce a joint function, $C(x,p)$ called the generalized distribution function which will allow us to write

$$\int \varphi^*(x) A(X,D) \varphi(x)\, dx = \iint a(x,p) C(x,p)\, dx\, dp \tag{1.9}$$

As we will see in Chap. 5 this is the generalization of Eqs. (1.4) and (1.7) and reduces to them when the symbol is a function of x or p only.

1.2 Notation and Terminology in Different Fields

Generally speaking there are five fields where these methods have been used. They are quantum mechanics, pseudo-differential operators, time-frequency analysis, wave propagation, and image processing. We review here some mathematical conventions and contrast the differences in terminology used in various fields.

"Symbol", "classical function", "c-function", and *"ordinary function"* are terms used in different fields to signify the same thing, namely an ordinary function, $a(x, \xi)$, of two variables x and ξ. This is the common notation used in mathematics, while in quantum mechanics it is position and momentum signified by $a(q, p)$, and in time-frequency analysis one generally writes $a(t, \omega)$. In the case of wave propagation and image processing one usually uses $a(x, k)$ where x and k are respectively position and wave-number (spatial-frequency). We will use (x, p).

"Association", "correspondence rule," "rule of association," "quantization," and "ordering rule." These words and phrases all mean the same thing, namely the association of an operator $A(X, D)$ with an ordinary function, $a(x, p)$. The two variables, x and p, are generally conjugate variables. Conjugate generally means that their corresponding operators do not commute. For the Weyl case, the association is symbolized by

$$A(X, D) \leftrightarrow a(x, p). \tag{1.10}$$

We call $A(X, D)$ the Weyl operator. When we study other correspondences we use

$$A^{\mathrm{W}}(X, D) \leftrightarrow a(x, p) \tag{1.11}$$

where the superscript denotes the particular correspondence, in this case the Weyl rule. Similarly for other cases, as, for example

$$A^{\mathrm{BJ}}(X, D) \leftrightarrow a(x, p) \tag{1.12}$$

will denote the Born-Jordan correspondence.

*The state function, signal, and image***.** Generally speaking the state function is an ordinary function upon which the operator operates. In engineering it is called the *signal* and in image processing it is called the image. In acoustics it is the pressure wave and in electrodynamics it may be the electric or magnetic field. The term state function comes from quantum mechanics where it represents the state of the physical system at hand. This is in contrast to the operators which represent physical observables. The state function may be square integrable or not and can be a distribution in the delta function sense.

*Phase-Space and quasi-distributions***.** By a phase-space function we mean a function of the two variables x and p. For example the symbol $a(x, p)$ is a phase-space function. The Wigner and other distributions are also phase-space functions and are called distributions. The terminology "distribution" comes from usage in physics and chemistry and are densities in probability theory. The phase-space distributions we consider in this book are generally not manifestly positive and that is why they are sometimes called quasi-distributions, but we will simply use the term distribution.

*Fourier transform pairs***.** The Fourier transform of the symbol, $a(x,p)$, will be denoted by $\widehat{a}(\theta,\tau)$ and the normalization is taken so that

$$\widehat{a}(\theta,\tau) = \frac{1}{4\pi^2}\iint a(x,p)\, e^{-i\theta x - i\tau p}\, dx\, dp, \tag{1.13}$$

$$a(x,p) = \iint \widehat{a}(\theta,\tau)\, e^{i\theta x + i\tau p}\, d\theta\, d\tau. \tag{1.14}$$

However, due to conventions in different fields we define Fourier transform pairs for the state function by

$$\widehat{\varphi}(p) = \frac{1}{\sqrt{2\pi}}\int \varphi(x)\, e^{-ixp}\, dx, \tag{1.15}$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\int \widehat{\varphi}(p)\, e^{ixp}\, dp, \tag{1.16}$$

and similarly for higher dimensions. The advantage of this convention is that it keeps the symmetry between the two domains in the sense that

$$\int |\varphi(x)|^2\, dx = \int |\widehat{\varphi}(p)|^2\, dp \tag{1.17}$$

and also keeps the symmetry in Eq. (1.4) and (1.7).

Commutator and anti-commutator. The commutator of two operators A and B is standardly denoted by

$$[A,B] = AB - BA \tag{1.18}$$

and the anti-commutator by

$$[A,B]_+ = AB + BA. \tag{1.19}$$

The D operator. We will occasionally use $D_y = \frac{1}{i}\frac{d}{dy}$, etc., otherwise D is understood to be D_x.

*Integrals***.** Integrals without limits imply integration over the reals,

$$\int = \int_{\mathbb{R}} = \int_{-\infty}^{\infty}. \tag{1.20}$$