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Xiaochun Rong
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Metric and Differential Geometry

The Jeff Cheeger Anniversary Volume

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Part III: Index Theory

Dedicated to

Jeff Cheeger

on his
65th birthday

Preface

The articles in this volume originates from lectures given at the international conference on *Metric and Differential Geometry*, May 11–15, 2009. Held at the Chern Institute of Mathematics, Tianjin and the Capital Normal University, Beijing, the main goal of the conference is to bring together leading experts to survey the recent advances in metric and differential geometry. Major topics covered in the conference include metric spaces, Einstein manifolds, Kähler geometry, geometric flows, index theory and hypoelliptic Laplacian, analytic torsions, and differential K-theory. This is reflected well in the collection in the volume. The conference was also used as an occasion to celebrate the 65th birthday of Jeff Cheeger, whose immense influence can be felt in many of these fields.

We acknowledge the financial support of Chern Institute of Mathematics and Capital Normal University for the realization of the conference and NSF for the preparation of these proceedings. We are very grateful for all the referees for their careful job in reviewing the contributions to this volume and for their thoughtful suggestions for improvements.

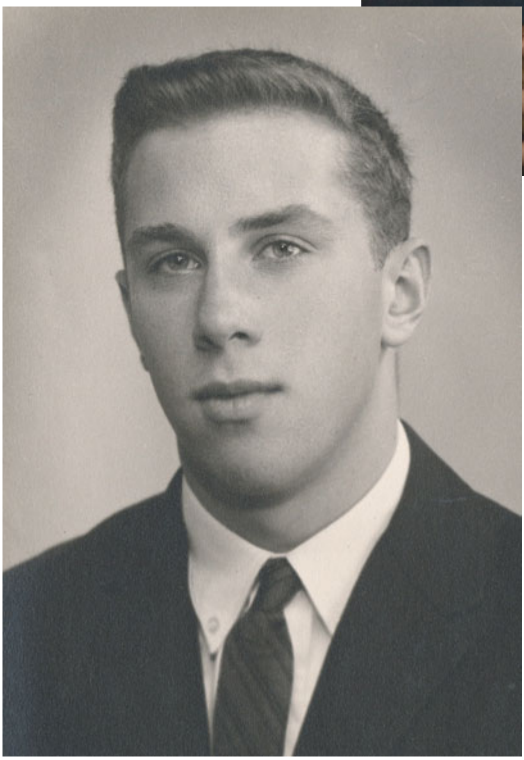
And we thank all the contributors to these proceedings. With submissions that they all could have sent for publication to excellent mathematical journals, they made possible the volume that we present here.

Xianzhe Dai and Xiaochun Rong
January 2012, Santa Barbara

Jeff and college friends, Pompei, Italy, summer after sophomore year in college (1962). From left: Steve, Jeff, Arnie, Jay. (credit: Jeff Cheeger)

On the beaches of Copacabana, Rio de Janeiro (1971), visit to IMPA. (credit: Jeff Cheeger)

Jeff, aged 2 years and
9 months; 1946, Brooklyn,
New York
(credit: Jeff Cheeger)



Jeff, high school graduation picture;
Brooklyn, New York, 1960.
(credit: Jeff Cheeger)

Jeff's parents, Thomas and Pauline Cheeger; Setauket, circa 1981. (credit: Jeff Cheeger)

Jeff and Emily; Setauket, 1981. (credit: Jeff Cheeger)

Calculations on spectral geometry of cones; Geneva 1976. (credit: Jeff Cheeger)

Deer Isle Maine, 1991. (credit: Jeff Cheeger)

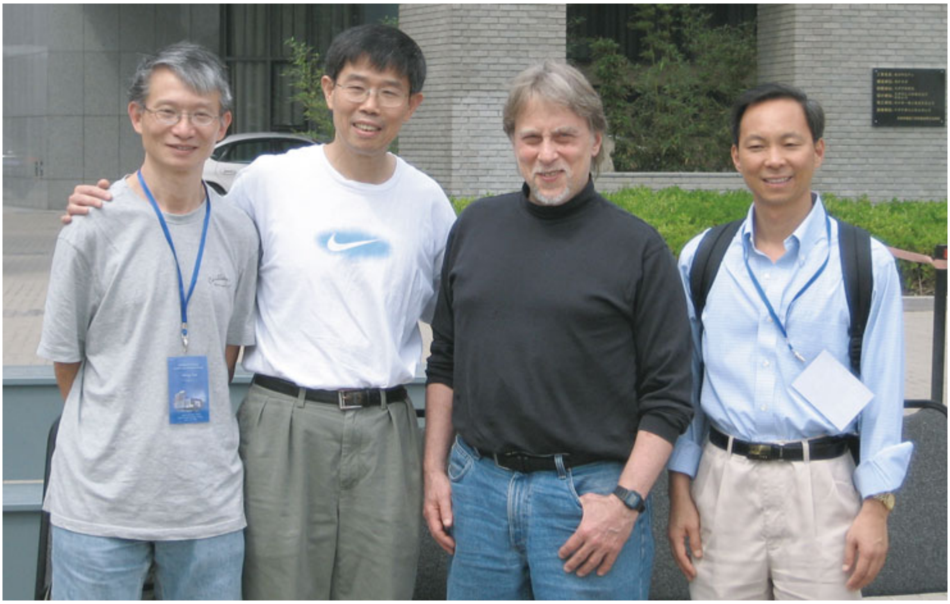
Jeff and Detlef Gromoll, at Jim Simons house, Jim and Marilyn wedding reception, 1977 (credit: Jeff Cheeger)

Jeff, Werner Müller and Robert Schrader in East Berlin, circa 1982 (credit: Jeff Cheeger)

Jeff, Werner and Jean-Michel Bismut in Jerusalem, 2010 (credit: Jeff Cheeger)

Jeff and Misha Gromov, Conference “Differential Geometry, Mathematical Physics, Mathematics and Society” – 60th Birthday Conference Jean Pierre Bourguignon, IHES, 2007. (credit: Gert-Martin Greuel)

Jeff and Blaine Lawson, Conference "Differential Geometry, Mathematical Physics, Mathematics and Society" – 60th Birthday Conference Jean Pierre Bourguignon, IHES, 2007. (credit: Gert-Martin Greuel)



Jeff and students Dagang Yang, Xiaochun Rong and Xianzhe Dai at Chern Institute, 2009 (credit: Xiaochun Rong)

Curriculum Vitae of Jeff Cheeger

Jeff Cheeger, born December 1, 1943, Brooklyn, New York.

Education

- B.A., Harvard College, 1964
- M.S., Princeton University, 1966
- Ph.D., Princeton University, 1967

Career

- Princeton University, Teaching and Research Assistant, 1966–1967
- University of California, Berkeley, N.S.F. Postdoctoral Fellow and Instructor, 1967–1968
- University of Michigan, Assistant Professor, 1968–1969
- SUNY at Stony Brook, Associate Professor, 1969–1971
- SUNY at Stony Brook, Professor, 1971–1985
- SUNY at Stony Brook, Leading Professor, 1985–1992
- SUNY at Stony Brook, Distinguished Professor 1990–1992
- Courant Institute of Mathematical Sciences, NYU, Professor, 1989–
- Courant Institute of Mathematical Sciences, NYU, Silver Professor, 2003–

Honors and Prizes

- Sloan Fellow, 1971–1973
- Invited address, International Congress of Mathematicians, 1974, 1986
- Guggenheim Fellow, 1984–1985
- Max Planck Research Prize, Alexander von Humboldt Society, 1991
- National Academy of Sciences, elected 1997
- Finnish Academy of Science and Letters, foreign member, elected 1998
- Oswald Veblen Prize in Geometry, American Mathematical Society, 2001
- American Academy of Arts and Sciences, 2006

PhD Students

- 1976 Douglas Elerath, SUNY at Stony Brook,
  Dissertation: Nonegatively curved Manifolds Diffeomorphic to Eculidian Space
- 1978 Ping Charng Lue, SUNY at Stony Brook,
  Dissertation: Asymptotic Expansion of the Trace of the Heat Kernel on Generalized Surfaces of Revolution
- 1982 Arthur Chou, SUNY at Stony Brook,
  Dissertation: The Dirac Operator on Singular Spaces
- 1985 Aparna Dar, SUNY at Stony Brook,
  Dissertation: Intersection R-Torsion and Analytic Torsion for Pseudomanifolds
- 1985 Scott Hensley, SUNY at Stony Brook,
  Dissertation: Equivariant Reidemeister Torsion
- 1987 DaGang Yang, SUNY at Stony Brook,
  Dissertation: A Residue Theorem for Secondary Invariants of Collapsed Riemannian Manifolds
- 1989 Xianzhe Dai, SUNY at Stony Brook,
  Dissertation: Adiabatic Limit, Non-multiplicativity and Leray Spectral Sequence
- 1990 Xiaochun Rong, SUNY at Stony Brook,
  Dissertation: Collapsed 3-Manifolds and Rationality of Limiting n-invariants
- 1990 Shunhui Zhu, SUNY at Stony Brook,
  Dissertation: Bounding Topology by Ricci Curvature in Dimension Three
- 1991 Zhong-dong Liu, SUNY at Stony Brook,
  Dissertation: Nonnegative Ricci Curvature Near Infinity and Geometry of Ends
- 1996 Christina Sormani, Courant Institute, NYU,
  Dissertation: Noncompact Manifolds with Lower Ricci Curvature Bounds and Minimal Volume Growth
- 1998 Alireza Ranjbar-Motlagh, Courant Institute, NYU,
  Dissertation: Analysis on Metric-Measure Spaces
- 2001 Yu Ding, Courant Institute, NYU,
  Dissertation: Continuity of Some Analytic Objects under Measured Gromov-Hausdorff Convergence

The Mathematical Work of Jeff Cheeger, a Brief Summary

Xianzhe Dai and Xiaochun Rong

Jeff Cheeger's work, starting from his Ph.D. thesis, has been characterized by deep geometric insights, far-reaching consequences, and widespread influence. Some early work of Jeff Cheeger is exposed in Blaine Lawson's article in this volume, to which we refer for more discussions. Here we list some of the significant contributions of Jeff Cheeger to mathematics.

1. Cheeger-Gromov Compactness/Collapsing Theory. Cheeger's thesis [1] introduced finiteness theorems into Riemannian geometry. Subsequently, in a lecture at the Summer Institute on Global Analysis at Stanford (1973) Cheeger showed that the collection of such Riemannian manifolds is totally bounded in the Lipschitz topology. Hence, in the Lipschitz topology, one can actually take sublimits of sequences of such manifolds. The compactness theory was improved in important work of Gromov. It has had a very big impact on the subject.

Taking as point of departure Gromov's celebrated theorem on almost flat manifolds, Cheeger and Gromov developed their celebrated collapsing theory (F-structures) on the scale of the injectivity radius; [37, 46]). Later, with Fukaya, who had also done fundamental work on collapsing, they gave a theory (N-structures) which holds on a small but fixed scale; [53]. Collapsing theory has been one of the most important developments in Riemannian geometry in the past three decades.

2. Cheeger's Inequality for the Smallest Eigenvalue of the Laplacian. Cheeger proved a lower bound on the first nonzero eigenvalue of the Laplacian, which subsequently became known as "Cheeger's inequality" [6]. The lower bound is in terms of the "Cheeger constant", a certain isoperimetric constant. This had many applications in Riemannian geometry and geometric analysis. An analogous estimate for graphs (also known as Cheeger's inequality) has extraordinarily diverse applications in computer networks, computational complexity, computational geometry, random walks, the theory of error-correcting codes and so on.

3. Cheeger-Gromoll's Soul Theorem and Splitting Theorem. Cheeger's work with Detlef Gromoll [3, 13] on complete manifolds of nonnegative curvature followed soon after work of Gromoll-Meyer on complete noncompact manifolds of strictly positive curvature. The Soul Theorem states that such a manifold is diffeomorphic to the normal bundle of some close totally geodesic (in fact, totally convex) submanifold called a *soul*. In this terminology, Gromoll-Meyer had shown that if the curvature is positive, then the soul is a point. The Soul Theorem has been the subject of many research works; in particular, it is used in Perelman's resolution of the Poincaré conjecture.

By means of his triangle comparision theorem, V. Toponogov proved that if a complete manifold of nonnegative sectional curvature contains a *geodesic line*, then this line splits off as a isometric factor. In [13], by an argument based on Laplacian comparison for distance functions (roughly in the distribution sense) and the maximum principle, Cheeger and Gromoll extended the splitting theorem to the case complete manifolds of nonnegative Ricci curvature. It became a cornerstone of the subject. (As discussed below, much later, Cheeger and Colding extended the splitting theorem to the case of Gromov-Hausdorff limit spaces. This plays a basic role in their structure theory of Gromov-Hausdorff limits of manifolds with a uniform lower Ricci curvature bound.)

4. Cheeger-Simons Differential Characters. Shortly after the famous work of S.S. Chern and James Simons on transgression forms for principle bundles with connection (Chern-Simons invariants) in the early 1970s, Cheeger and Simons developed a corresponding theory of secondary invariants which live in the base space as elements of a new structure called *the ring of differential characters*. (This work was written up in notes distributed at the Stanford Summer Institute (1973) which were published only much later; see [34].) Differential characters are the first geometric model of what is now called differential cohomology, a refinement of ordinary integral cohomology. They have found interesting applications in theoretical physics.

5. The Ray-Singer Conjecture and the Cheeger-Müller Theorem. In [21], Cheeger proved the celebrated Ray-Singer conjecture which asserted that the analytic torsion is equal to the Reidemeister torsion. The proof involved considering two manifolds M_0, M_1 which differ by a surgery and interpolating between them a 1-parameter family M_t of manifolds with boundary which degenerates to M_0 at $t = 0$ and M_1 at $t = 1$. The proof is a true tour de force. It led in particular, to Cheeger's work on analysis on singular spaces. The Ray-Singer conjecture was proved independently and by a different method by Werner Müller. The Cheeger-Müller theorem has been used in Witten's study of two dimensional quantum gauge theories. More recently, it has been used to detect torsion cohomology classes of hyperbolic manifolds by Bergeron-Venkatesh and Müller.

6. Singular Spaces: Poincaré Duality, L_2-cohomology and Spectral Geometry. Cheeger realized that the technique developed in [21] for controlling heat kernels of degenerating families of manifolds with boundary, (he called it the "strong form of the method of separation of variables") gave rise to a functional calculus for the Laplacian on metric cones of arbitrary smooth cross-section. The key idea was to interpret the kernel $K_f(x_1, x_2, r_1, r_2)$ of a function $f(\Delta)$ of the Laplacian on the cone as a canonically associated family (parameterized by the radial variables r_1, r_2) of kernels of the Laplacian $\tilde{\Delta}$ of the cross-section. In this way, the functional calculus for $\tilde{\Delta}$ could be employed to gain control over the eigenfunction expansion of K_f (with respect to x_1, x_2) in regions where the expansion converged badly (or not at all). This led directly to his work on L_2-cohomology and Hodge theory on manifolds with (possibly iterated) conical singularities and to his study of the spectral geometry of piecewise flat pseudomanifolds; [20, 23, 28, 30, 31].

By this route, independently of Goresky and MacPherson, and in an entirely different context, Cheeger made the fundamental discovery that for pseudomanifolds (satisfying a certain condition on the middle dimensional cohomology of links) there exists a cohomology theory, L_2-cohomology, which satisfies Poincaré duality. He told this to Dennis Sullivan, who then informed him of the work of Mark Goresky and Bob MacPherson on intersection homology. Sullivan then conjectured that the two theories must be isomorphic and Cheeger proved that this was the case. Cheeger also introduced the notion of $*$-invariant ideal boundary conditions which can be used to restore Poincaré duality in certain cases in which the above mentioned condition on links fails to hold. On the topological side, this was done independently by Morgan (unpublished). Cheeger-Goresky-MacPherson made important conjectures (now established) asserting that the intersection homology of singular algebraic varieties satisfies properties analogous to those of the ordinary cohomology of smooth Kähler manifolds; [28].

Cheeger's theory of spectral geometry on piecewise flat spaces, which included a treatment of index theory via the heat equation method of McKean-Singer, had several remarkable applications. One was a purely combinatorial formula for the Euler number of a pseudomanifold in terms of certain angle defects which are defined in terms of products of dihedral angles. Another was an (analytically based) canonical local combinatorial formula for L-classes of PL-manifolds in terms of η-invariants of links. More generally, in this way, he defined L-classes (which are ordinary homology classes) for pseudomanifolds. Using intersection homology, a nonlocal topological definition in the spirit of Thom was given independently by Goresky-MacPherson.

Using the functional calculus for the Laplacian on cones, jointly with Michael Taylor, Cheeger gave a highly detailed treatment of diffraction of waves by cones of arbitrary cross section; [26, 27]. It was the first rigorous treatment of this classical problem and gave more information than earlier formal treatments. The first rigorous treatment of a special case of this diffraction problem dealt with diffraction by an edge (which can be viewed as diffraction by the cone on $[0, 2\pi]$ with Neumann boundary conditions). It was done by the physicist Arnold Sommerfeld in 1899 and subsquently rederived by a number of authors by means of different arguments, all involving nontrival summation formulas.

7. Regge Calculus and Lipschitz-Killing Curvatures for Piecewise Flat Spaces. The Lipschitz-Killing curvatures form a sequence of curvature measures (indexed by the degree of the polynomial in curvature). The sequence starts with scalar curvature times the volume measure and ends with the Chern-Gauss-Bonnet density. Cheeger, together with Werner Müller and Robert Schrader, studied the version of these curvatures for piecewise flat spaces which he had derived in his study of the index theory via the heat equation method; [25, 32, 39]. They showed that, for sequences of uniformly fat subdivisions of triangulations of smooth Riemannian manifolds, as the edge length goes to zero, the piecewise flat curvatures converge in the sense of measures to the smooth ones. For the case of the scalar curvature, this gives the first proof of the "classical limit theorem" for Regge calculus, which had long been widely assumed by physicists without rigorous justification.

8. η-Invariants, Families Index for Manifolds with Boundary and L_2-Index Theory. For a bundle with even dimensional fibre whose base space is a circle, Witten conjectured a formula relating the holonomy of an associated determinant line bundle to the adiabatic limit of the η-invariant of the total space. In [40, 41], Cheeger gave a proof of Witten's formula for signature operators. At about the same time, a different proof for general Dirac operators was given by Jean-Michel Bismut and Dan Freed. Subsequently, Bismut and Cheeger generalized the formula to base spaces and fibres of arbitrary dimension [43]. This involved their defining higher η-invariants (Bismut-Cheeger η-forms) which could also be viewed as Chern-Simons forms for certain bundles with infinite dimensional fibre. Cheeger had previously shown that the celebrated index formula of Atiyah-Patodi-Singer comes out naturally from the heat equation approach to the index formula in the conical setting. This is used in an essential way in the work of Bismut-Cheeger on the families index formula for manifolds with boundary [44, 45, 47].

L_2-index theory for infinite covering spaces of compact manifolds (based on the concept of Von Neumann dimension) was introduced by Atiyah and Singer. An extension to complete Riemannian manifolds with bounded curvature and finite volume was used in Cheeger and Gromov's study of integrals of characteristic forms over such manifolds; [35]. In this context, they also defined an L_2 version of η-invariants of compact manifolds and their associated ρ-invariants, which have had extensive applications in knot theory. The "good chopping" theorem, [48], whose generalization played a significant role in the work with Gang Tian on Einstein 4-manifolds ([74]), also arose in this context. (See also [36] for applications L_2-cohomology to group cohomology.)

9. Structure Theory for Limits of Manifolds with Lower Ricci Curvature Bounds and for Einstein Manifolds. In a series of trail blazing works [56, 60, 62, 64, 65], Cheeger and Tobias Colding gave a structure theory of Gromov-Hausdorff limit spaces of manifolds with lower Ricci curvature bounds. The fundamental tools are developed in [60], where some of the most important rigidity results such as the volume cone implies metric cone theorem, Cheng's maximal diameter theorem, and the Cheeger-Gromoll splitting theorem, are generalized quantitatively to almost rigidity theorems. Equivalently, the rigidity theorems are extended to the case of Gromov-Hausdorff limit spaces. An important tool in the discussion is their "segment inequality" which is a refinement of the Poincaré inequality.

For noncollapsed limit spaces Y^n, every tangent cone Y_y is shown to be a metric cone. A natural stratification $\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots$ of singular set $\mathcal{S} \subset Y^n$ is defined in terms of the splitting properties of the tangent cones and the Hausdorff dimension bound dim $\mathcal{S}^k \leq k$ is proved. It is also shown that $\mathcal{S} = \mathcal{S}^{n-2}$ and that off a set of codimension 2, the limit space is bi-Hölder equivalent to a smooth connected Riemannian manifold. Moreover, the isometry group is a Lie group. (As had been conjectured by Cheeger-Colding and proved in spectacular recent work of Colding and Aaron Naber, this last statement holds in the collapsed case as well.) Even in the collapsed case, regular points have full measure with respect to any renormalized limit measure and (for such measures) up to a set of measure zero, the limit space is a countable union of measurable subsets, each of which is bi-Lipschitz equivalent to a subset of Euclidean space.

In work of (various subsets of) Cheeger, Colding and Tian, more refined regularity results were obtained in the noncollapsed case, when the members of the sequence are either Einstein, Kähler-Einstein [68] (possibly bounded L_p-norm of the full curvature tensor [69, 70]) or have special holonomy [73, 71].

In [74], the first results on Einstein manifolds whose hypotheses do not require a noncollapsing assumption are obtained. So far, they apply only in dimension 4.

9. Differentiation Theory for Metric Measure Spaces. Rademacher's theorem asserts the almost everywhere differentiability of Lipshitz functions $f : \mathbf{R^n} \to \mathbf{R^m}$. In a highly influential paper, [63], Cheeger formulated and proved a remarkable extension of Rademacher's theorem to spaces which might have fractional Hausdorff dimension and in particular, might not be Euclidean at the infinitesimal level. Let (X, d, μ) denote a doubling metric measure space that satisfies a p-Poincaré inequality for some $p \geq 1$. (In current terminology X is called a PI space.) Then (X, d, μ) has what is now known as a strong measurable differentiable structure (which, by definition, is unique). One way of expressing this is to say that X has a finite dimensional (measurable) cotangent bundle TX^* such that every Lipschitz function has a differential df which is a bounded measurable section of TX^*, such that the map $f \to df$ has the "usual" properties.

Cheeger found the following very generally applicable principle which underlies the whole discussion: If f (in this case a Lipschitz function) has finite energy (in this case, a Dirichlet energy) and the energy is lower semicontinuous then at the infinitesimal level almost everywhere, f will be a minimizer (i.e., harmonic) with constant energy density and hence linear in a generalized sense. Note that in $\mathbf{R^n}$, linear functions are precisely harmonic functions for which the norm of the gradient is a constant function.

One application of the theory is a general bi-Lipschitz nonembedding theorem. If a PI space X admits a bi-Lipschitz embedding into some finite-dimensional

Euclidean space, then at almost every point x, every tangent cone X_x is bi-Lipschitz equivalent to the fibre TX_x^*. (In general dim $X_x \geq TX^*$ and for many X, the inequality is strict.) Cheeger's theorem implies the known non-embedding results both for the Carnot-Carathéodory spaces and for Laakso spaces.

Cheeger and Bruce Kleiner gave an optimal extension of the differentiation theory to Banach space targets having the Radon-Nikodym Property; [79]. In [82, 83] they showed by a different method that the Heisenberg group $\mathbf{H}$ does not admit a bi-Lipschitz embedding into L_1. (The Banach space L_1 does not have the Radon-Nikodym Property; prior to [82, 83] there was no structure theory even for Lipschitz maps $\mathbf{R} \to L_1$.) Besides its significant intrinsic interest as a major advance at the intersection of metric embedding and differentiation theory, this result, when combined with work of Lee and Naor, gave a natural (qualitative) counterexample to the celebrated Goemans-Linial conjecture from theoretical computer science. By means of a highly nontrivial quantitative differentiation argument, this was made quantitative in [84], which provided an exponential improvement of the best previously known counterexample, due to Khot and Vishnoi (2005); for quantitative differentiation, see the discussion below.

11. Quantitative Differentiation. The simplest instance of "quantitative differentiation" arises in work of Peter Jones (1988) (who did not use this terminology). It makes the following assertion about functions $f : [0,1] \to \mathbf{R}$ with $|f'| \leq 1$. There is a natural measure on the collection of subintervals $J \subset [0,1]$ whose mass is infinite. However, for all $\epsilon > 0$, the measure of the collection of subintervals J such that $f \mid J$ fails to be ϵ-close to a linear function (in the appropriately scaled sense) is finite and has a bound in terms of ϵ (independent of the particular function f).

Through the work [63, 84] Cheeger discovered a surprisingly general, versatile, and powerful version of "quantitative differentiation"; see Section 14 of [84] and for a more precise formulation [88]. It can be viewed as the quantitative version of the above-mentioned general principle underlying [63]. When coupled with ϵ-regularity theorems, there are numerous applications to geometric analysis and partial differential equations. The technology for these applications (in which a key role is played by "cone structure") has been developed in (ongoing) work of Cheeger and Aaron Naber; see [86, 87]. A result from [86], which deals with non-collapsed Gromov-Hausdorff limits of Einstein manifolds, serves to illustrate the theme. Namely, the bound (from [68, 70]) on the $(n-4)$-dimensional Hausdorff measure of the singular set $\mathcal{S}$ is very significantly strengthened to the assertion that the volume of the set of points at which the C^2-harmonic radius is $\leq r$ is bounded by $c(n)r^4$.

12. Expository books and articles. Cheeger has written several influential expository works; see [17, 49, 67, 81, 71, 88]. In particular, [17], his book with David Ebin, "Comparison theorems in Riemannian Geometry" is a classic.

List of Publications of Jeff Cheeger

[1] Comparison and finiteness theorems for Riemannian manifolds, PhD Thesis, Princeton University, 1967.

[2] The relation between the diameter and the smallest eigenvalue of the Laplacian for manifolds of nonnegative curvature, Archiv. der Mathematik (1968) 558–560.

[3] The structure of complete manifolds of nonnegative curvature (with D. Gromoll) Bull. Amer. Math. Soc. (1968) 1147–1150.

[4] Pinching theorems for a certain class of Riemannian manifolds, Amer. J. Math., XCI, No. 3 (July 1969) 807–834.

[5] Infinitesimal isometries and Pontrjagin numbers (with P. Baum) Topology (1969) 173–193.

[6] A lower bound for the smallest eigenvalue of the Laplacian, Proc. of Princeton Conf. in Honor of Prof. S. Bochner (1969) 195–199.

[7] A combinatorial formula for Stiefel-Whitney classes, Proc. of Georgia Topology Conference (1969) 470–471.

[8] Counting topological manifolds, (with J. Kister) Proc. of Georgia Topology Conference (1969).

[9] Compact manifolds of nonnegative curvature, Proc. of Oberwolfach Conference on Differential Geometry (1969) 25–41.

[10] Finiteness theorems for Riemannian manifolds, Amer. J. Math. XCII, No. 1 (1970) 61–74.

[11] Homeomorphism types of topological manifolds (with J. Kister) Topology 9, (1970) 149–151.

[12] The splitting theorem for manifolds of nonnegative Ricci curvature (with D. Gromoll) J. Diff. Geom. 6, No. 1 (1971) 119–128.

[13] On the structure of complete manifolds of nonnegative curvature (with D. Gromoll) Ann. of Math. 96, No. 3. (1972) 413–443.

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[27] On the diffraction of waves by canonical singularities II (with M. Taylor) Com. Pur. App. Math. XXXV (1982) 487–529.

[28] L^2-cohomology and intersection homology of algebraic varieties (with R. MacPherson and M. Goresky) Sem. Diff. Geom., S.T. Yau Ed. (Ann. Math. Study 102) Princeton Univ. Press (1982) 303–340.

[29] Finite propagation speed, kernel estimates for functions of the Laplace operator, and the asymptotic geometry of complete Riemannian manifolds (with M. Gromov and M. Taylor) J. Diff. Geom. 17 (1982) 15–53.

[30] Spectral geometry of singular Riemannian spaces, J. Diff. Geom. 18 (1983) 575–657.

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