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Foreword by Ian Stewart

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Foreword

by Ian Stewart

I know when I first came across the Tower of Hanoi because I still have a copy of the book that I found it in: *Riddles in Mathematics* by Eugene P. Northrop, first published in 1944. My copy, bought in 1960 when I was fourteen years old, was a Penguin reprint. I devoured the book, and copied the ideas that especially intrigued me into a notebook, alongside other mathematical oddities. About a hundred pages further into Northrop's book I found another mathematical oddity: Wacław Sierpiński's example of a curve that crosses itself at every point. That, too, went into the notebook.

It took nearly thirty years for me to become aware that these two curious structures are intimately related, and another year to discover that several others had already spotted the connection. At the time, I was writing the monthly column on mathematical recreations for *Scientific American*, following in the footsteps of the inimitable Martin Gardner. In fact, I was the fourth person to write the column. Gardner had featured the Tower of Hanoi, of course; for instance, it appears in his book *Mathematical Puzzles and Diversions*.

Seeking a topic for the column, I decided to revisit an old favourite, and started rethinking what I knew about the Tower of Hanoi. By then I was aware that the mathematical essence of many puzzles of that general kind—rearranging objects according to fixed rules—can often be understood using the state diagram. This is a network whose nodes represent possible states of the puzzle and whose edges correspond to permissible moves. I wondered what the state diagram of the Tower of Hanoi looked like. I probably should have thought about the structure of the puzzle, which is recursive. To solve it, forget the bottom disc, move the remaining ones to an empty peg (the same puzzle with one disc fewer), move the bottom disc, and put the rest back on top. So the solution for, say, five discs reduces to that for four, which in turn reduces to that for three, then two, then one, then zero. But with no discs at all, the puzzle is trivial.

Instead of thinking, I wrote down all possible states for the Tower of Hanoi with three discs, listed the legal moves, and drew the diagram. It was a bit messy, but after some rearrangement it suddenly took on an elegant shape. In fact, it looked remarkably like one of the stages in the construction of Sierpiński's curve. This couldn't possibly be coincidence, and once I'd noticed this remarkable resemblance, it was then straightforward to work out where it came from: the recursive structure of the puzzle.

Several other people had already noticed this fact independently. But shortly after my rediscovery I was in Kyoto at the International Congress of Mathematicians. Andreas Hinz introduced himself and told me that he had used the connection with the Tower of Hanoi to calculate the average distance between any two points of Sierpiński's curve. It is precisely 466/885 of the diameter. This is an extraordinary result—a rational number, but a fairly complicated one, and far from obvious.

This wonderful calculation is just one of the innumerable treasures in this fascinating book. It starts with the best account I have ever read of the history of the puzzle and its intriguing relatives. It investigates the mathematics of the puzzle and discusses a number of variations on the Tower of Hanoi theme. And to drive home how even the simplest of mathematical concepts can propel us into deep waters, it ends with a list of currently unsolved problems. The authors have done an amazing job, and the world of recreational mathematics has a brilliant new jewel in its crown.

Preface

The British mathematician Ian Stewart pointed out in [307, p. 89] that "Mathematics intrigues people for at least three different reasons: because it is fun, because it is beautiful, or because it is useful." Careful as mathematicians are, he wrote "at least", and we would like to add (at least) one other feature, namely "surprising". The Tower of Hanoi (TH) puzzle is a microcosmos of mathematics. It appears in different forms as a recreational game, thus fulfilling the fun aspect; it shows relations to Indian verses and Italian mosaics via its beautiful pictorial representation as an esthetic graph, it has found practical applications in psychological tests and its theory is linked with technical codes and phenomena in physics.

The authors are in particular amazed by numerous popular and professional (mathematical) books that display the puzzle on their covers. However, most of these books discuss only well-established basic results on the TH with incomplete arguments. On the other hand, in the last decades the TH became an object of numerous—some of them quite deep—investigations in mathematics, computer science, and neuropsychology, to mention just central scientific fields of interest. The authors have acted frequently as reviewers for submitted manuscripts on topics related to the TH and noted a lack of awareness of existing literature and a jumble of notation—we are tempted to talk about a Tower of Babel! We hope that this book can serve as a base for future research using a somewhat unified language.

More serious were the errors or mathematical myths appearing in manuscripts and even published papers (which did *not* go through our hands). Some "obvious assumptions" turned out to be questionable or simply wrong. Here is where many mathematical surprises will show up. Also astonishing are examples of how the mathematical model of a difficult puzzle, like the *Chinese rings*, can turn its solution into a triviality. A central theme of our book, however, is the meanwhile notorious *Frame-Stewart conjecture*, a claim of optimality of a certain solution strategy for what has been called *The Reve's puzzle*. Despite many attempts and even allegations of proofs, this has been an open problem for more than 70 years.

Apart from describing the state of the art of its mathematical theory and applications, we will also present the historical development of the TH from its invention in the 19th century by the French number theorist Édouard Lucas. Although we are not professional historians of science, we nevertheless take historical remarks and comments seriously. During our research we encountered many errors or historical myths in literature, mainly stemming from the authors copying statements from other authors. We therefore looked into original sources whenever we could get hold of them.

Our guideline for citing other authors' papers was to include "the first and the best" (if these were two). The first, of course, means the first to our current state of knowledge, and the best means the best to our (current) taste.

This book is also intended to render homage to Édouard Lucas and one of his favorite themes, namely recreational mathematics in their role in mathematical education. The historical fact that games and puzzles in general and the TH in particular have demonstrated their utility is universally recognized (see, e.g., [295, 123]) more than 100 years after Lucas's highly praised book series started with [209].

Myths

Along the way we deal with numerous myths that have been created since the puzzle appeared on the market in 1883. These myths include mathematical misconceptions which turned out to be quite persistent, despite the fact that with a mathematically adequate approach it is not hard to clarify them entirely. A particular goal of this book is henceforth to act as a myth buster.

Prerequisites

A book of this size can not be fully self-contained. Therefore we assume some basic mathematical skills and do not explain fundamental concepts such as sets, sequences or functions, for which we refer the reader to standard textbooks like [107, 284, 26]. Special technical knowledge of any mathematical field is not necessary, however. Central topics of discrete mathematics, namely combinatorics, graph theory, and algorithmics are covered, for instance, in [197, 36], [336, 41, 72], and [179, 231], respectively. However, we will not follow notational conventions of any of these strictly, but provide some definitions in a glossary at the end of the book. Each term appearing in the glossary is put in **bold face** when it occurs for the first time in the text. This is mostly done in Chapter 0, which serves as a gentle introduction to ideas, concepts and notation of the central themes of the book. This chapter is written rather informally, but the reader should not be discouraged when encountering difficult passages in later chapters, because they will be followed by easier parts throughout the book.

The reader must also not be afraid of mathematical formulas. They shape the language of science, and some statements can only be expressed unambiguously when expressed in symbols. In a book of this size the finiteness of the number of symbols like letters and signs is a real limitation. Even if capitals and lower case, Greek and Roman characters are employed, we eventually run out of them. Therefore, in order to keep the resort to indices moderate, we re-use letters for sometimes quite different objects. Although a number of these are kept rather stable globally, like n for the number of discs in the TH or names of special sequences like Gros's g, many will only denote the same thing locally, e.g., in a section. We hope that this will not cause too much confusion. In case of doubt we refer to the indexes at the end of the book.

Algorithms

The TH has attracted the interest of computer scientists in recent decades, albeit with a widespread lack of rigor. This poses another challenge to the mathematician who was told by Donald Knuth in [178, p. 709] that "It has often been said that a person doesn't really understand something until he teaches it to someone else. Actually a person doesn't really understand something until he can teach it to a computer, i.e. express it as an algorithm." We will therefore provide provably correct algorithms throughout the chapters. Algorithms are also crucial for human problem solvers, differing from those directed to machines by the general human deficiency of a limited memory.

Exercises

Édouard Lucas begins his masterpiece "Théorie des nombres" [213, iii] with a (slightly corrected) citation from a letter of Carl Friedrich Gauss to Sophie Germain dated 30 April 1807 ("jour de ma naissance"): "Le goût pour les sciences abstraites, en général, et surtout pour les mystères des nombres, est fort rare; on ne s'en étonne pas. Les charmes enchanteurs de cette sublime science ne se décèlent dans toute leur beauté qu'à ceux qui ont le courage de l'approfondir."[1] Sad as it is that the first sentence is still true after more than 200 years, the second sentence, as applied to all of mathematics, will always be true. Just as it is impossible to get an authentic impression of what it means to stand on top of a sizeable mountain from reading a book on mountaineering without taking the effort to climb up oneself, a mathematics book has always to be read with paper and pencil in reach. The readers of our book are advised to solve the excercises posed throughout the chapters. They give additional insights into the topic, fill missing details, and challenge our skills. All exercises are addressed in the body of the text. They are of different grades of difficulty, but should be treatable at the place where they are cited. At least, they should then be *read*, because they may also contain new definitions and statements needed in the sequel. We collect hints and solutions to the problems at the end of the book, because we think that the reader has the right to know that the writers were able to solve them.

[1]"The taste for abstract sciences, in general, and in particular for the mysteries of numbers, is very rare; this doesn't come as a surprise. The enchanting charms of that sublime science do not disclose themselves in all their beauty but to those who have the courage to delve into it."

Contents

The book is organized into ten chapters. As already mentioned, Chapter 0 introduces the central themes of the book and describes related historical developments. Chapter 1 is concerned with the Chinese rings puzzle. It is interesting in its own right and leads to a mathematical model that is a prototype for an approach to analyzing the TH. The subsequent chapter studies the classical TH with three pegs. The most general problem solved in this chapter is how to find an optimal sequence of moves to reach an arbitrary regular state from another regular state. An important subproblem solved is whether the largest disc moves once or twice (or not at all). Then, in Chapter 3, we further generalize the task to reach a given regular state from an irregular one. The basic tool for our investigations is a class of graphs that we call *Hanoi graphs*. A variant of these, the so-called Sierpiński graphs, is introduced in Chapter 4 as a new and useful approach to Hanoi problems.

The second part of the book, starting from Chapter 5, can be understood as a study of variants of the TH. We begin with the famous The Reve's puzzle and, more generally, the TH with more than three pegs. The central role is played by the notorious Frame-Stewart conjecture which has been open since 1941. Very recent computer experiments are also described that further indicate the inherent difficulty of the problem. We continue with a chapter in which we formally discuss the meaning of the notion of a variant of the TH. Among the variants treated we point out the *Tower of Antwerpen* and the *Bottleneck TH*. A special chapter is devoted to the *Tower of London*, invented in 1982 by T. Shallice, which has received an astonishing amount of attention in the psychology of problem solving and in neuropsychology, but which also gives rise to some deep mathematical statements about the corresponding *London graphs*. Chapter 8 treats TH type puzzles with oriented disc moves, variants which, together with the more-pegs versions, have received the broadest attention in mathematics literature among all TH variants studied.

In the final chapter we recapitulate open problems and conjectures encountered in the book in order to provide stimulation for those who want to pass their time expediently waiting for some Brahmins to finish a divine task.

Educational aims

With an appropriate selection from the material, the book is suitable as a text for courses at the undergraduate or graduate level. We believe that it is also a convenient accompaniment to mathematical circles. The numerous exercises should be useful for these purposes. Themes from the book have been employed by the authors as a leitmotif for courses in discrete mathematics, specifically by A. M. H. at the LMU Munich and in block courses at the University of Maribor and by S. K. at the University of Ljubljana. The playful nature of the subject lends itself to presentations of the fundamentals of mathematical thinking for a general audience.

The TH was also at the base of numerous research programs for gifted students. The contents of this book should, and we hope will, initiate further activities of this sort.

Feedback

If you find errors or misleading formulations, please send a note to the authors. Errata, sample implementations of algorithms, and other useful information will appear on the *TH-book* website at http://tohbook.info.

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A. M. H. wants to express his appreciation of the hospitality during his numerous visits in Maribor.

Last, but not least, we all thank our families and friends for understanding, patience, and support. We are especially grateful to Maja Klavžar, who, as a librarian, suggested to us that it was about time to write a comprehensive and widely accessible book on the Tower of Hanoi.

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Sandi Klavžar — *Trzin, Slovenia*
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Contents

Chapter 0

The Beginning of the World

The roots of mathematics go far back in history. To present the origins of the protagonist of this book, we even have to return to the Creation.

0.1 The Legend of the Tower of Brahma

"D'après une vieille légende indienne, les brahmes se succèdent depuis bien longtemps, sur les marches de l'autel, dans le Temple de Bénarès, pour exécuter le déplacement de la *Tour Sacrée de* BRAHMA, aux soixante-quatre étages en or fin, garnis de diamants de Golconde. Quand tout sera fini, la Tour et les brahmes tomberont, et ce sera la fin du monde!"

These are the original words of Professor N. Claus (de Siam) of the Collège Li-Sou-Stian, who reported, in 1883, from Tonkin about the legendary origins of a "true annamite head-breaker", a game which he called LA TOUR D'HANOÏ (see [58]). We do not dare to translate this enchanting story written in a charming language and which developed through the pen of Henri de Parville into an even more fantastic fable [256]. W. W. R. Ball called the latter "a sufficiently pretty conceit to deserve repetition" ([23, p. 79]), so we will follow his view and cite Ball's most popular English translation of de Parville's story:

"In the great temple at Benares, beneath the dome which marks the centre of the world, rests a brass-plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disc resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah [sic!]. Day and night unceasingly the priests transfer the discs from one diamond needle to another according to the fixed and immutable laws of Bramah [sic!], which require that the priest must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle on

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which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish."

Ball adds: "Would that English writers were in the habit of inventing equally interesting origins for the puzzles they produce!", a sentence censored by H. S. M. Coxeter in his revised edition of Ball's classic [24, p. 304].

As with all myths, Claus's legend underwent metamorphoses: the decoration of the discs [étages] with diamonds from Golconda[1] transformed into diamond needles, de Parville put the great temple of Benares[2] to the center of the world and moved "since quite a long time" to "the beginning of the centuries", which Ball interprets as the creation. As if the end of the world would not be dramatic enough, Ball adds a "thunderclap" to it. So the Tower of Brahma became a time-spanning riddle. Apart from his strange spelling of the Hindu god, Ball also re-formulated the rule which de Parville enunciated as "he must not place that disc but on an empty needle or above [au-dessus de] a larger disc". More importantly, while de Parville insists in the task to transport the tower from the first to the third needle, Ball's Bramah did *not* specify the goal needle; we will see that this makes a difference!

In another early account, the Dutch mathematician P. H. Schoute is more precise by insisting to put the disc (or ring in his diction) "on an empty [needle] or on a larger [disc]" [286, p. 275] and specifying the goal by alluding to the Hindu triad of the gods Brahma, Vishnu, and Shiva: Brahma, the creator, placed the discs on the first needle and when they all reach Shiva's, the world will be destroyed; in between, it is sustained by the presence of Vishnu's needle. The latter god will also watch over the observance of what we will call the (cf. [215, p. 55])

divine rule: you must not place a disc on a smaller one.

Let us hope that Sarasvati, Brahma's consort and the goddess of learning, will guide us through the mathematical exploration of the fascinating story of the sacred tower!

Among the many variants of the story, which would fill a book on its own, let us only mention the reference to "an Oriental temple" by F. Schuh [288, p. 95] where 100 alabaster discs were waiting to be transferred by believers from one of two silver pillars to the only golden one. Quite obviously, Schoute's compatriot was more in favor of the decimal system than the Siamese inventor of the puzzle, who, almost by definition, preferred base two.

De Parville, in his short account of what came to be known as the *Tower of Hanoi (TH)*, was also very keen in identifying this man from Indochina. A mandarin, says he, who invents a game based on combinations, will incessantly think about combinations, see and implement them everywhere. As one is never betrayed but by oneself, permuting the letters of the signatory of the TH, *N. Claus (de*

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[1] at the time the most important market for diamonds, located near the modern city of Hyderabad

[2] today's Vārānasī

Siam), mandarin of the collège *Li-Sou-Stian* will reveal *Lucas d'Amiens*, teacher at the lycée *Saint-Louis*.

François Édouard Anatole Lucas (see Figure 0.1) was born on 4 April 1842 in the French city of Amiens and worked the later part of his short life at schools in Paris. Apart from being an emminent number theorist, he published, from 1882,

Figure 0.1: Édouard Lucas, 1842–1891

a series of four volumes of "Récréations mathématiques" [214, 218, 215, 216][3], accomplished posthumously in 1894. They stand in the tradition of J. Ozanam's popular "Récréations mathématiques et physiques" which saw editions from 1694 until well into the 18th century. The fourth volume of this work contains a plate [249, pl. 16 opposite p. 439] showing in its Figure 47 what the author calls "Sigillum Salomonis". It is a mechanical puzzle which Lucas discusses in the first volume of his series under the name of "baguenaudier" (cf. [209, p. 161–186]) and to which his leaflet [58] refers for more details on the TH which only much later enters volume three [215, p. 55–59]. It seems that this more ancient puzzle was the catalyst for Lucas's TH.

0.2 History of the Chinese Rings

The origin of the solitaire game *Chinese rings (CR)*, called jiulianhuan ("Nine linked rings", 九连环) in today's China, seems to be lost in the haze of history.

[3]Throughout we cite editions which were at our disposition.

The legends emerging from this long tradition are mostly frivolous in character, reporting from a Chinese hero who gave the puzzle to his wife when he was leaving for war, for the obvious motive of "entertaining" her during his absence. The present might have looked as in Figure 0.2.

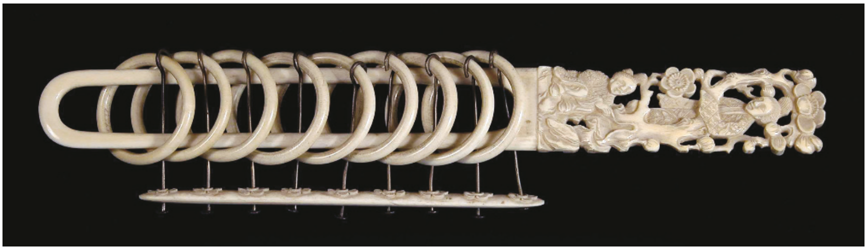


Figure 0.2: The Chinese rings
(courtesy of James Dalgety, http://puzzlemuseum.org)

The most serious attribution has been made by S. Culin in his ethnological work on Eastern games [62, p. 31f]. According to his "Korean informant", the caring husband was Hung Ming, actually a real person (Zhuge Liang, 诸葛亮, 181–234). However, the material currently available is also compatible with a European origin of the game.

The earliest known evidence can be found in Chapter 107 of Luca Pacioli's *De viribus quantitatis* (cf. [250, p. 290–292]) of around 1500, where the physical object, with a certain number of rings, is described and a method to get the rings *onto* the bar is indicated. Pacioli speaks of a "difficult case". A 7-ring version was discussed in the middle of the 16th century in Book 15 of G. Cardano's *De subtilitate libri XXI* as a "useless" instrument ("Verùm nullius vsus est instrumétú ex septem annulis...") which embodies a game of "admirable subtlety" ("miræ subtilitatis") [50, p. 492f]. On the other hand, Cardano claimed that the ingenious mechanism was not that useless at all and therefore employed in locks for chests, a claim supported by Lucas in [214, p. 165, footnote]. The reader may consult [1] for more myths.

Many names have been given to the Chinese rings over the centuries. They have been called *Delay guest instrument* in Korea, *Cardano's rings* or *tiring irons* in England (Dudeney reports in 1917 that "it is said still to be found in obscure English villages (sometimes deposited in strange places such as a church belfry)"; cf. [79, Problem 417].), and *Nürnberger Zankeisen* (quarrel iron) in Germany, but the most puzzling designation is the French *baguenaudier*. In the note [119] of the Lyonnais barrister Louis Gros, almost 5 out of 16 text pages are devoted to the etymology with the conclusion that it should be "baguenodier", deriving from a knot of rings. However, Lucas did not follow Gros's arguments and so the French name of a plant (*Colutea arborescens*) is still attached to the puzzle. But why then has "baguenauder" in French the meaning of strolling around, wasting time?

The puzzle consists of a system of nine rings, bound together in a sophisticated mechanical arrangement, and a bar (or shuttle as in weaving) with a handle at one end. At the beginning, all rings are on the bar. They can be moved off or back onto the bar only at the other end, and the structure allows for just two kinds of individual ring moves, the details of which we will discuss in Chapter 1. The task is to move all rings off the bar. Let us assume for the moment that this and, in fact, all states, i.e. distributions of rings on or off the bar, can be reached. (Lucas [214, p. 177] actually formulates the generalized problem to find a shortest possible sequence of moves to get from an arbitrary initial to an arbitrary goal state.) Then we may view the puzzle as a representation of binary numbers from 0 to $511 = 2^9 - 1$ if we interpret the rings as binary digits, or **bits**, 0 standing for a ring off the bar and 1 for a ring on the bar. This leads us back even further in Chinese history, or rather mythology, namely to the legendary Fu Xi (伏羲), who lived, if at all, some 5000 years ago. To him Leibniz attributes [189, p. 88f] the *ba gua* (八卦), the eight *trigrams* consisting of three bits each, and usually depicted in a circular arrangement as in Figure 0.3, where a broken (*yin*, 阴) line stands for 0, a solid (*yang*, 阳) line for 1, and the least significant bit is the outermost one.

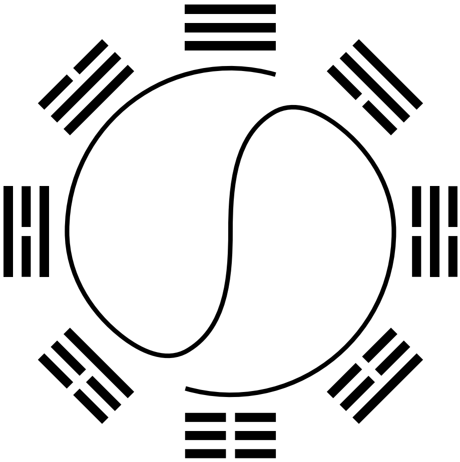


Figure 0.3: Fu Xi's arrangement of the trigrams

Legend has it that Fu Xi saw this arrangement on the back of a tortoise. (Compare also to the Korean flag, the *Taegeukgi.*) Following the yin-yang symbol in the center of the figure, Fu Xi, Leibniz and we recognize the numbers from 0 to 7 in binary representation. Doubling the number of bits will give a supply of 64 *hexagrams* for the I Ching (yi jing, 易经), the famous *Book of Changes*; cf. [352] and [209, p. 149–151]. It seems, however, that Leibniz went astray with the philosophical and religious implications he drew from yin and yang; see [321]. On the other hand, the mathematical implications of binary thinking can not be over-estimated.

0.3 History of the Tower of Hanoi

The sixth chapter in [209] was devoted by Lucas to "The Binary Numeration". Here he describes the advantages of the binary system [209, p. 148f], the Yi Jing [209, p. 149–151], and perfect numbers [209, p. 158–160], before he starts his seventh recreation on the baguenaudier, as mentioned before. We do not find, however, the most famous of Lucas's recreations in this first edition, the TH. This is not surprising though. In the box containing the original game, preserved today in the *Musée des arts et métiers* in Paris, one can find the following inscription, most probably in Lucas's own hand:

> La tour d'Hanoï, —
>
> Jeu de combinaison pour
> appliquer le système de la numération
> binaire, inventé par M. Edouard Lucas
> (novembre 1883) — donné par l'auteur.

So we have a date of birth for the TH. (In [213, p. xxxii], Lucas claims that the puzzle was published in 1882, but there is no evidence for that.) The idea of the game was immediately pilfered around the world with patents approved, e.g., in the United States ($\mathrm{N^o}$ 303, 946 by A. Ohlert, 1884) and the United Kingdom ($\mathrm{N^o}$ 20, 672 by A. Gartner and G. Talcott, 1890). In 1888, Lucas donated the original puzzle (see Figures 0.4 and 0.5), together with a number of mechanical calculating machines to the *Conservatoire national des arts et métiers* (Cnam) in Paris, where he also gave public lectures for which a larger version of the TH was produced; cf. [212].

The cover of the original box shows the fantastic scenery of Figure 0.6. The picture, also published on 19 January 1884 in [59, p. 128], repeats all the allusions to fancy names of places and persons we already found on the leaflet [58] which accompanied the puzzle. Two details deserve to be looked at closer. The man supporting the ten-storied pagoda has a tatoo on his belly: A U—Lucas was "agrégé de l'université", entitling him to teach at higher academic institutions. The crane, a symbol for the Far East, holds a sheet of paper on which is written, with bamboo leaves, the name Fo Hi—the former French transliteration of Fu Xi whom we met before.

But why the name "The Tower of *Hanoï*"? We would not think of the capital of today's Vietnam in connection with Brahmins moving 64 golden discs in the great temple of Benares. However, when Lucas started to market the puzzle in its modest version with only eight wooden discs, French newspapers were full of reports from Tonkin. In fact, Hanoi had been seized by the French in 1882, but during the summer of 1883 was under constant siege by troops from the Chinese province of Yunnan on the authority of the local court of Hué, where on 25 August 1883 the Harmand treaty established the rule of France over Annam and Tonkin

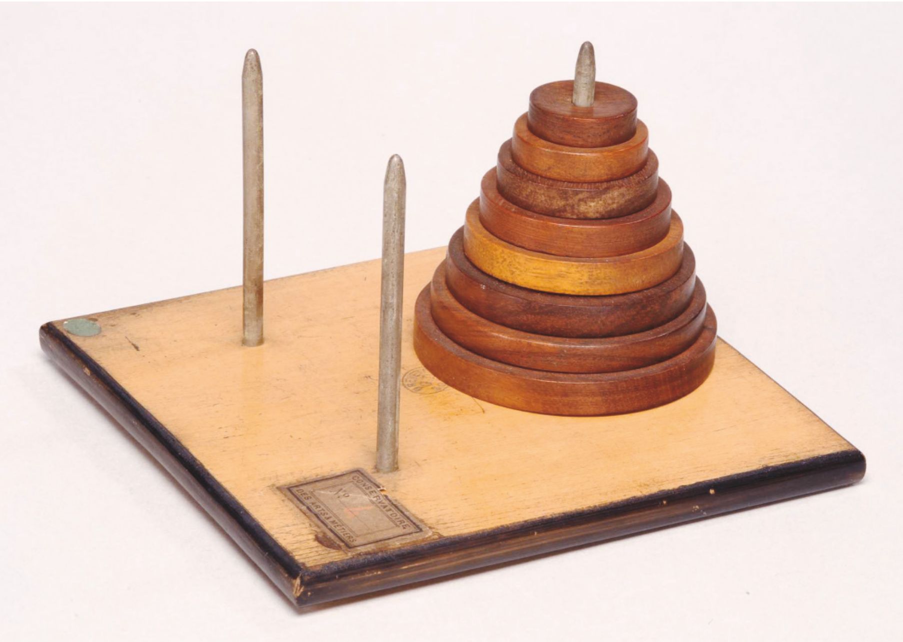


Figure 0.4: The original Tower of Hanoi

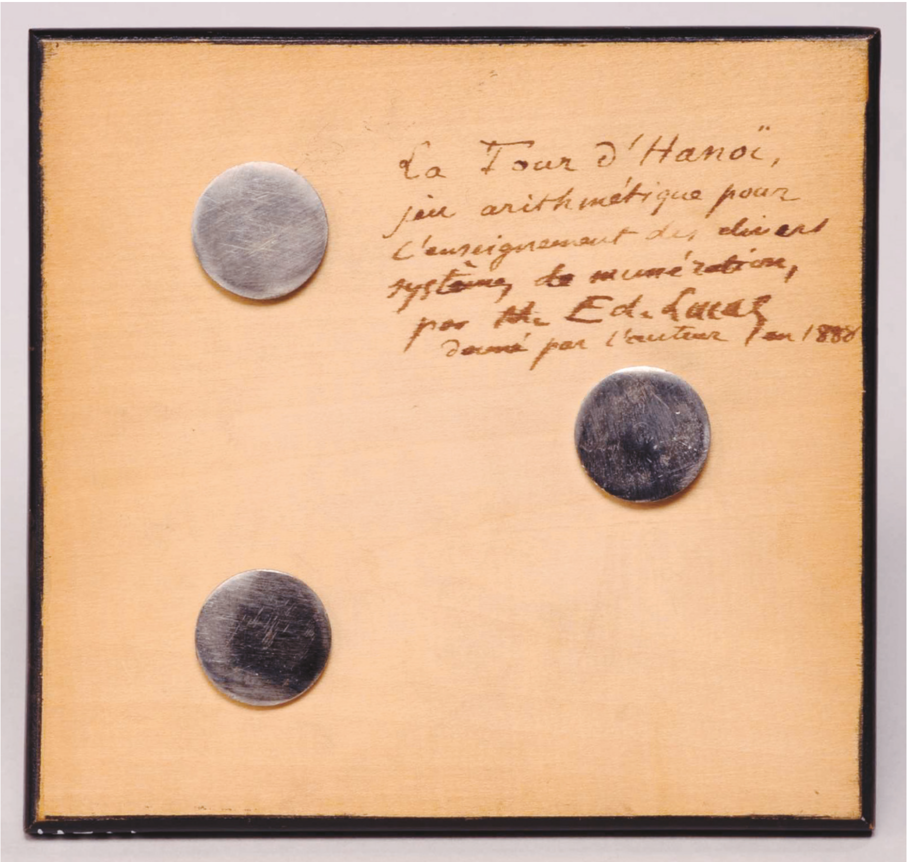
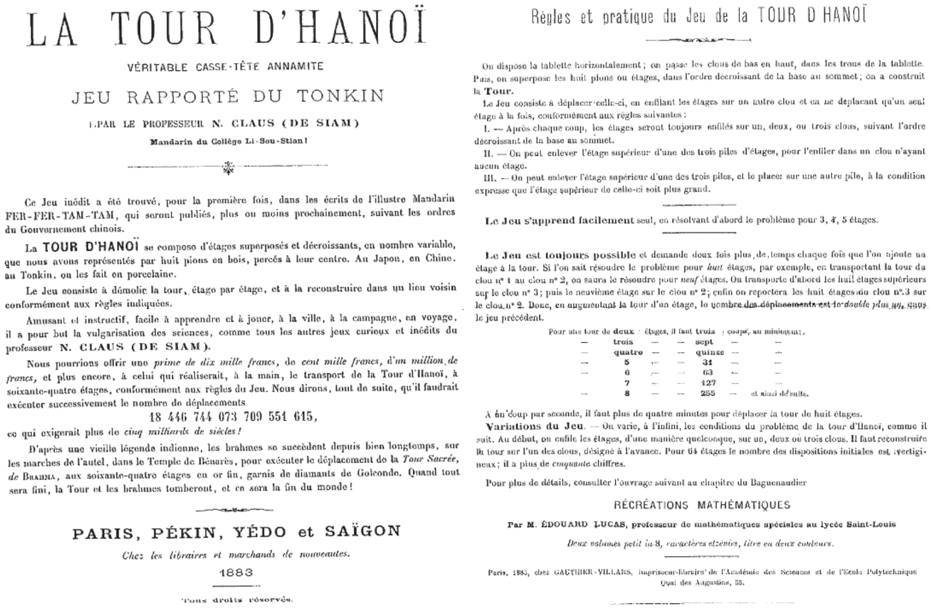


Figure 0.5: Base plate of the original puzzle

Figure 0.6: The cover plate of the Tower of Hanoi

(cf. [252, Section 11]). In [256], de Parville calls a variant of the TH, where discs of increasing diameter were replaced by hollow pyramids of decreasing size, the "Question of Tonkin" and comments the fact that the discs of Claus's TH were made of wood instead of gold as being more prudent because it concerns Tonkin. So Lucas selected the name of Hanoi because it was in the headlines at the time. Most probably, our book today would sell better had we chosen the title "The Tower of *Kabul*"!

Lucas never travelled to Hanoi [210, p. 14]. However, he was a member of the commission which edited the collected works of Pierre de Fermat and was sent on a mission to Rome to search in the famous Boncompagni library for unpublished papers of his illustrious compatriot (cf. [216, p. 91]). Was it on this voyage that Lucas invented the TH? The leaflet, which we reproduce here as Figure 0.7, supports this hypothesis when talking in a typical Lucas style about FER-FER-TAM-TAM, thereby transforming the French government into a Chinese one.



LA TOUR D'HANOÏ

VÉRITABLE CASSE-TÊTE ANNAMITE

JEU RAPPORTÉ DU TONKIN

PAR LE PROFESSEUR **N. CLAUS (DE SIAM)**

Mandarin du Collège Li-Sou-Stian!

Ce Jeu inédit a été trouvé, pour la première fois, dans les écrits de l'illustre Mandarin FER-FER-TAM-TAM, qui seront publiés, plus ou moins prochainement, suivant les ordres du Gouvernement chinois.

La **TOUR D'HANOÏ** se compose d'étages superposés et décroissants, en nombre variable, que nous avons représentés par huit pions en bois, percés à leur centre. Au Japon, en Chine, au Tonkin, on les fait en porcelaine.

Le Jeu consiste à démolir la tour, étage par étage, et à la reconstruire dans un lieu voisin conformément aux règles indiquées.

Amusant et instructif, facile à apprendre et à jouer, à la ville, à la campagne, en voyage, il a pour but la vulgarisation des sciences, comme tous les autres jeux curieux et inédits du professeur **N. CLAUS (DE SIAM)**.

Nous pourrions offrir une *prime de dix mille francs*, de *cent mille francs*, d'*un million de francs*, et plus encore, à celui qui réaliserait, à la main, le transport de la Tour d'Hanoï, à soixante-quatre étages, conformément aux règles du Jeu. Nous dirons, tout de suite, qu'il faudrait exécuter successivement le nombre de déplacements

18 446 744 073 709 551 615,

ce qui exigerait plus de *cinq milliards de siècles!*

D'après une vieille légende indienne, les brahmes se succèdent depuis bien longtemps, sur les marches de l'autel, dans le Temple de Bénarès, pour exécuter le déplacement de la *Tour Sacrée, de* BRAHMA, aux soixante-quatre étages en or fin, garnis de diamants de Golconde. Quand tout sera fini, la Tour et les brahmes tomberont, et ce sera la fin du monde!

PARIS, PÉKIN, YÉDO et SAÏGON

Chez les libraires et marchands de nouveautés.

1883

Tous droits réservés.

Règles et pratique du Jeu de la TOUR D'HANOÏ

On dispose la tablette horizontalement; on passe les clous de bas en haut, dans les trous de la tablette. Puis, on superpose les huit pions ou étages, dans l'ordre décroissant de la base au sommet; on a construit la **Tour.**

Le Jeu consiste à déplacer celle-ci, en enfilant les étages sur un autre clou et en ne déplaçant qu'un seul étage à la fois, conformément aux règles suivantes:

I. — Après chaque coup, les étages seront toujours enfilés sur un, deux, ou trois clous, suivant l'ordre décroissant de la base au sommet.

II. — On peut enlever l'étage supérieur d'une des trois piles d'étages, pour l'enfiler dans un clou n'ayant aucun étage.

III. — On peut enlever l'étage supérieur d'une des trois piles, et le placer sur une autre pile, à la condition expresse que l'étage supérieur de celle-ci soit plus grand.

Le Jeu s'apprend facilement seul, en résolvant d'abord le problème pour 3, 4, 5 étages.

Le Jeu est toujours possible et demande deux fois plus de temps chaque fois que l'on ajoute un étage à la tour. Si l'on sait résoudre le problème pour *huit* étages, par exemple, en transportant la tour du clou nº 1 au clou nº 2, on saura le résoudre pour *neuf* étages. On transporte d'abord les huit étages supérieurs sur le clou nº 3; puis le neuvième étage sur le clou nº 2; enfin on reportera les huit étages du clou nº 3 sur le clou nº 2. Donc, en augmentant la tour d'un étage, le nombre des déplacements est le *double plus un*, dans le jeu précédent.

Pour une tour de **deux** étages, il faut **trois** coups, au minimum;

—	**trois**	—	—	**sept**	—	—	
—	**quatre**	—	—	**quinze**	—	—	
—	**5**	—	—	**31**	—	—	
—	**6**	—	—	**63**	—	—	
—	**7**	—	—	**127**	—	—	
—	**8**	—	—	**255**	—	et ainsi de suite.	

A un coup par seconde, il faut plus de quatre minutes pour déplacer la tour de huit étages.

Variations du Jeu. — On varie, à l'infini, les conditions du problème de la tour d'Hanoï, comme il suit. Au début, on enfile les étages, d'une manière quelconque, sur un, deux ou trois clous. Il faut reconstruire la tour sur l'un des clous, désigné à l'avance. Pour 64 étages le nombre des dispositions initiales est vertigineux; il a plus de *cinquante* chiffres.

Pour plus de détails, consulter l'ouvrage suivant au chapitre du Baguenaudier

RÉCRÉATIONS MATHÉMATIQUES

Par M. ÉDOUARD LUCAS, professeur de mathématiques spéciales au lycée Saint-Louis

Deux volumes petit in-8, caractères elzévirs, titre en deux couleurs.

Paris, 1883, chez GAUTHIER-VILLARS, imprimeur-libraire de l'Académie des Sciences et de l'École Polytechnique Quai des Augustins, 55.

Figure 0.7: Recto and verso of the leaflet accompanying the Tower of Hanoi puzzle

Apart from this, the front page of [58] discloses the motive of professor N. Claus (de Siam) for his game, namely the vulgarization of science. He offers enormous amounts of money for the person who solves, by hand, the TH with 64 discs and reveals the necessary number of displacements, namely

$$18\,446\,744\,073\,709\,551\,615\,, \tag{0.1}$$

together with the claim that it would take more than *five milliard*[4] *centuries* to carry out the task making one move per second. The number in (0.1) is explained, together with the rules of the game, which are imprecise concerning the goal peg and redundant with respect to the divine rule, on the back of [58]. Here one can find the famous *recursive* solution, stated for an arbitrary number of discs, but demonstrated with an example: if one can solve the puzzle for *eight* discs, one can solve it for *nine* by first transferring the upper eight to the spare peg, then moving the ninth disc to the goal peg and finally the smaller ones to that peg too. So by increasing the number of discs by one, the number of moves for the transfer of the tower doubles plus one move of the largest disc. Now the superiority of the binary number system becomes obvious. We write 2^n for an n-fold product of 2s, $n \in \mathbb{N}_0$, e.g., $2^0 = 1$ (by convention, the product of no factors is 1), $2^1 = 2$,

[4]billion, i.e. 10^9

$2^2 = 4$, and so forth. Then every **natural number** $N \geq 1$ can (uniquely) be written as $(N_{K-1} \ \dots \ N_1 \ N_0)_2$, such that $N = \sum_{k=0}^{K-1} N_k \cdot 2^k$ with $K \in \mathbb{N}$ and $N_k \in \{0,1\}$, $N_{K-1} = 1$. (This needs a mathematical proof, but we will not go into it. The index "2" is meant to distinguish this representation of N from the decimal one; the brackets may be omitted.) Clearly, $2^n = 10\dots0_2$ with n bits 0 and by binary arithmetic, $1\dots1_2 = 2^n - 1$ with n bits 1. Now doubling a number and adding 1 means concatenating a bit 1 to the right of the number. The recursive solution therefore needs $2^n - 1$ moves to transfer a tower of n discs. Calculating the 64-fold product of 2 and subtracting 1 in decimal representation, we arrive at the number shown in (0.1).

This number evokes another "Indian" myth which comes in even more versions than the Tower of Brahma. The nicest, but not the first, of these legends is by J. F. Montucla [242, p. 379–381], who tells us, in citing the gorgeous number, that the Indian Sessa, son of Daher (Sissa ben Dahir), invented (a prototype of) the chess game which he presented to the Indian king (Shirham). The latter was so pleased that he offered to Sessa whatever he desires. Contrary to our experience with fairy tales, Sessa did not ask for the daughter of the king, but pronounced a "modest" wish: a grain of wheat (rice in other versions) on the first square of the chess board, two on the second, four on the third and so on up to the last, the 64th one. The king's minister, however, found out that it was impossible to amass such an amount of wheat and we are told by Montucla that the king admired Sessa even more for that subtle request than for the invention of the game. (In less romantic versions, Sessa was beheaded for his impertinence.)

In this "arithmetic bagatelle", as Montucla called it, we recognize immediately, that Sessa asked to concatenate a bit 1 to the left of the binary number of grains when adding a new square of the board, all in all $1\dots1_2 = 2^{64} - 1$ grains.

With the *Mersenne numbers*[5] (for M. Mersenne) $M_k = 2^k - 1$, we have an example of a sequence $(M_k)_{k\in\mathbb{N}_0}$, called *Mersenne sequence*, which fulfills two *recurrences*

$$M_0 = 0, \ \forall\, k \in \mathbb{N}_0 : \ M_{k+1} = 2M_k + 1\,, \tag{0.2}$$

$$M_0 = 0, \ \forall\, k \in \mathbb{N}_0 : \ M_{k+1} = M_k + 2^k\,. \tag{0.3}$$

We will see later that (0.2) and (0.3) are prototypes of a most fundamental type of recurrences, each of them leading to a uniquely determined sequence.

The most ubiquitous of all sequences defined by a recurrence is even older. In his manuscript *Liber abbaci* of 1202/1228, Leonardo Pisano, now commonly called Fibonacci (figlio di Bonaccio), posed the following problem:

Quot paria coniculorum in uno anno ex uno pario germinentur.

The solution of the famous *rabbit problem* invoked the probably most popular integer sequence of all times, which was accordingly named *Fibonacci sequence* by

[5] Some authors use this term only if k is prime.

Édouard Lucas (cf. [213, p. 3]). Its members F_k, the *Fibonacci numbers* (cf. [264]), are given by the recurrence

$$F_0 = 0,\ F_1 = 1,\ \forall\, k \in \mathbb{N}_0 :\ F_{k+2} = F_{k+1} + F_k. \tag{0.4}$$

Lucas employed this sequence in a somewhat more serious context in connection with the distribution law for prime numbers in [207], thereby introducing a variant, namely the sequence (for $k \geq 1$) $L_k := F_{2k}/F_k$ which fulfils the same recurrence relation as in (0.4), but with the *seeds* replaced by $L_1 = 1$ and $L_2 = 3$ (or by $L_0 = 2$, $L_1 = 1$ to start the sequence at $k = 0$). Lucas still named this sequence for Fibonacci as well in [206, p. 935], but it is now called the *Lucas sequence* (and its members *Lucas numbers*). Fibonacci and Lucas numbers can also be calculated explicitly and they exhibit an interesting relation to the famous *Golden section* as shown in Exercise 0.1; cf. [329]. The methods developed in [207] allowed Lucas to decide whether certain numbers are prime or not without reference to a table of primes, and he announced his discovery that $2^{127} - 1$ is a (Mersenne) prime. (Is there a relation to the TH with 127 discs?) With this he was the last human world record holder for the "largest" prime number, to be beaten only by computers 75 years later, albeit using Lucas's method.

Édouard Lucas has never been properly recognized in his home country France, neither in his time, nor today. In 1992, M. Schützenberger writes (to be found in [289]): "...Édouard Lucas, who has no reputation among professional mathematicians, however, because he is schools inspector and does not publish anything else but books on entertaining mathematics." Only in 1998 [69] and with her thesis [70], A.-M. Décaillot put the life and number-theoretical work of Lucas into light. Before that there were just two short biographies in [142, p. 540f] and [339, Section 3.1]. Already in 1907 appeared a collection of biographies and necrologies [219]. As it turns out, Lucas was in the wrong place at the wrong time: with his topics from number theory and his enthusiasm for teaching and popularizing mathematics he put himself outside the infamous main-stream. Being in the wrong place at the wrong time seemed to be Lucas's fate: the story of his death sounds as if it was just a malice of N. Claus (de Siam). Although the source is unknown, we give here a translation of the most moving of all accounts from the journal *La Lanterne* of 6 October 1891 [219, p. 17]:

> The death of this "prince of mathematics", as the young generations of students called him, has been caused by a most vulgar accident. In a banquet at which assisted the members of the [Marseille] congress during an excursion into Provence, a [male] servant, who found himself behind the seat of M. Edouard Lucas, dropped, by unskilfulness, a pile of plates. A broken piece of porcelain came to hit the cheek of M. Lucas and caused him a deep injury from which blood flew in abundance. Forced to suspend his work, he returned to Paris. He took to his bed and soon appeared erysipelas which would take him away.

Édouard Lucas died on 3 October 1891, aged only 49. His tomb, perpetual, but in a deplorable condition, can be found on the Montmartre cemetary of Paris.

Upon his untimely death, Lucas left the second volume of his "Théorie des nombres" [213] unfinished. E. T. Bell writes in 1951 [27, p. 230]: "Some years ago the fantastic price of thirty thousand dollars was being asked for Lucas's manuscripts. In all his life Lucas never had that much money." It is not known where these manuscripts remained. The same fate happened to a collection of six "Jeux scientifiques" (cf. [217, Note III]), for which Lucas earned two medals at the World's Fair in Paris in 1889 (the one for which another tower, namely Eiffel's, was erected). Some of these games are presented in Lucas's article [211]. The puzzle collection, dedicated to his children Paul and Madeleine, was advertised in *Cosmos*, a scientific magazine, of 7 December 1889 as a first series, each single game being sold by "Chambon & Baye" of Paris for the price of 10 francs. Among these was a new version of the TH, this time with five pegs and 16 discs in four colors. He says that the number of problems one can pose about the new TH is uncalculable. Five of the brochures accompanying the puzzles are preserved, the only missing one is about "Les Pavés florentin du père Sébastien", also described in [22, p. 158]. Bell suggested "His widely scattered writings should be collected, and his unpublished manuscripts sifted and edited." Despite some efforts (cf. [129, 18]), this goal is still far from being reached. In particular, it would be desirable to review Lucas's œuvre in recreational mathematics just like Décaillot has done it for his work in number theory.

For now we have to concentrate on Lucas's greatest and most influential invention. At the end of his account on the TH in [215, p. 58f], Lucas writes: "Latterly, the foreign industry has taken possession of the game of our friend [N. Claus (de Siam)] and of his legend; but we can assert that the whole had been imagined, some time ago already [in 1876 according to [210, p. 14]], in n$^{\text{o}}$ 56 of rue Monge, in Paris, in the house built on the site of the one where Pascal died on 19 August 1662."

Indian Verses, Polish Curves, and Italian Pavements

B. Pascal is known, among other things, for the triangle which in his seminal treatise, published posthumously in 1665, he called the *Arithmetical triangle (AT)* (cf. [81]). However, the famous arrangement of numbers was known long before him and can be found implicitly as early as in the tenth century in a commentary on Piṅgala's "Chandaḥśāstra" (ca. -200) by Halāyudha (cf. [161, p. 112f] and [139, Section 3.4]). The *Mount Meru* (Meru-Prastāra) according to this description can be seen in Figure 0.8.

The rule is to start with a 1 in the single square on top and to fill out each subsequent line by writing the sum of the touching squares of the line immediately above. Piṅgala was investigating poetic meters of short (0) and long (1) syllables, a discipline called *prosody*. The problem was to analyze their combinations, and this can be viewed as the birth of *combinatorics*. In how many ways can they be

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Figure 0.8: The Meru-Prastāra

combined to form a word of length k? Quite obviously (for us!), the answer is 2^k, the number of bit strings of length k. But looking closer, how many *combinations* of k syllables are there containing ℓ long and (consequently) $k-\ell$ short ones? The answer is contained in Mount Meru by looking at the ℓth entry from the left in the kth line from the top, both counting from 0. (It is a puzzle why almost all presentations of AT start with the single 1 on top. Although the modern mathematician is used to such abstruse things, how did the Ancients know that there is precisely one way to choose no long syllable (and no short one either for that matter) to form an empty word?) This number will be pronounced "*k choose ℓ*" and is denoted by $\binom{k}{\ell}$. Summing the entries of a single line in Mount Meru immediately leads to the formula

$$\forall\, k \in \mathbb{N}_0 : \sum_{\ell=0}^{k} \binom{k}{\ell} = 2^k .$$

Another question that can be solved by looking at Mount Meru (cf. [161, p. 113f]) is the following: if long syllables take 2 beats (or *morae*), short ones only 1 beat (*mora*), how many words (of varying lengths) can be formed from a fixed number of m morae? Since $m = 2\ell + k - \ell = k + \ell$, this number can be found by summing over all $\binom{k}{\ell} = \binom{m-\ell}{\ell}$, i.e. some mounting diagonal in Mount Meru, with the astonishing result that

$$\forall\, m \in \mathbb{N}_0 : \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-\ell}{\ell} = F_{m+1} , \tag{0.5}$$

where the *floor function* is defined for $x \in \mathbb{R}$ by $\lfloor x \rfloor = \max\{a \in \mathbb{Z} \mid a \le x\}$. (Similarly, the *ceiling function* is characterized by $\lceil x \rceil = \min\{b \in \mathbb{Z} \mid b \ge x\}$.)

The formula in (0.5) can be proved by recourse to (0.4), which is even more surprising because it apparently had been found 50 years before Fibonacci!

During its long history, the AT reappeared in many disguises. For instance, evaluating the kth power of the *binomial* term $a+b$, we have to sum up products of k factors containing either an a or a b from each of the k individual factors of the power. Identifying a with short and b with long syllables from prosody, we see that the product $a^{k-\ell}b^{\ell}$ shows up precisely $\binom{k}{\ell}$ times, i.e. we arrive at

$$\forall\, k \in \mathbb{N}_0 : \ (a+b)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} a^{k-\ell} b^{\ell} \,.$$

This is the famous *binomial theorem*, and the $\binom{k}{\ell}$ are now commonly known as *binomial coefficients*. However, this denomination addresses only one particular aspect of these numbers and it is a breach of the historical facts; we therefore prefer to call them *combinatorial numbers*.

In more modern terms, the combination of short and long syllables in a word of length k can also be interpreted as choosing a subset from a set K with $|K| = k$ elements. Here 0 means that an element of K does not belong to the subset chosen, while 1 means that it is an element of the subset. In other words, there are exactly $\binom{k}{\ell}$ subsets of K which have precisely ℓ elements. Writing $\binom{K}{\ell}$ for the set of all subsets of K of order ℓ, we get

$$\forall\, \ell \in \{0,\ldots,|K|\} : \ \left|\binom{K}{\ell}\right| = \binom{|K|}{\ell} .$$

A sound mathematical description of the AT, based on this formula, and some important notation can be found in Exercise 0.2.

The direct application of arguments based on choices is the starting point for proving many interrelations between combinatorial numbers. For instance, if you want to select ℓ balls from a collection of b black and w white balls, you may successively pick λ black ones and then $\ell-\lambda$ white balls. There are $\binom{b}{\lambda}\binom{w}{\ell-\lambda}$ such combinations, whence

$$\forall\, b, w, \ell \in \mathbb{N}_0 : \ \binom{b+w}{\ell} = \sum_{\lambda=0}^{\ell} \binom{b}{\lambda}\binom{w}{\ell-\lambda} . \tag{0.6}$$

This can be viewed as an extension of the recursive formula in Exercise 0.2 c), which is case $b = 1$ of (0.6) and could therefore be called the *black sheep formula*.

Combinatorics is also the historical starting point for *probability theory*, which has its early roots in discussions about games of chance. One of these was treated mathematically by Euler in 1751 (see [97]) under the name of "Game of encounter" (Jeu de rencontre). It had been studied earlier by Pierre Rémond de Montmort in the first work entirely devoted to probability, his "Essay d'Analyse sur les Jeux de Hazard", whose second edition appeared in 1713, but Euler's presentation is, as usual, more clearly written.

Two players, A and B, hold identical packs of cards which are shuffled. They start to compare card by card from their respective decks. If during this procedure two cards coincide, then A wins. If all cards have been used up without such an encounter having occurred, then B is the winner. The question is: what is the probability of A (or B for that matter) to win. (In Montmort's version, the number of cards in each deck was 13, and the game was accordingly called "Jeu du Treize".)

Euler solved this question in the following way. Let us assume that a deck has $k \in \mathbb{N}_0$ cards and that the deck of A is naturally ordered from 1 to k; this has to be compared with the $k!$ possible decks B can hold. Some easy cases are done first: for $k = 0$, B wins; for $k = 1$, A is the winner; for $k = 2$, the chances are $1:1$. The most instructive case is $k = 3$, where $2! = 2$ decks of B, namely $(1,2,3)$ and $(1,3,2)$, will make A the winner in the first round. In the second round, the same number of decks ($(1,2,3)$ and $(3,2,1)$), would be favorable for A, however, one obviously has to delete deck $(1,2,3)$, from this list. This is because it would have made A the winner in the first round of a game with only 2 cards, namely without card 2 present. This is a prototype application of the **inclusion-exclusion principle**: an element must not be counted twice in the union of two (or more) finite sets. Of the remaining 3 decks, A will win the third round with deck $(2,1,3)$ in B's possession only, so all in all in 4 out of $3! = 6$ cases. The general rule which Euler found for the number $f_{k,\ell}$ of captures of A in round $\ell \in [k]$ is

$$\forall\, k \in \mathbb{N}:\ f_{k,1} = (k-1)!\ \wedge\ \forall\, \ell \in [k-1]:\ f_{k,\ell+1} = f_{k,\ell} - f_{k-1,\ell}\,.$$

From this, Euler deduces, by what we would call a double induction on k and ℓ today,

$$\forall\, k \in \mathbb{N}\ \forall\, \ell \in [k]:\ f_{k,\ell} = \sum_{m=0}^{\ell-1} (-1)^m \binom{\ell-1}{m} (k-1-m)!\,. \tag{0.7}$$

To obtain the number f_k of decks of B favorable for A to win, we have to sum equation (0.7) over ℓ. Euler used an ingenious trick based on an observation about the AT: if one adds the entries in a diagonal parallel to the left side of the triangle down to a certain entry, one obtains the entry immediately to the right of it in the next line, i.e.

$$\forall\, k \in \mathbb{N},\ m \in [k]_0:\ \sum_{\ell=m+1}^{k} \binom{\ell-1}{m} = \binom{k}{m+1};$$

this formula can also easily be proved by induction. We now get for $k \in \mathbb{N}$:

$$\begin{aligned} f_k = \sum_{\ell=1}^{k} f_{k,\ell} &= \sum_{\ell=1}^{k} \sum_{m=0}^{\ell-1} (-1)^m \binom{\ell-1}{m} (k-1-m)! \\ &= \sum_{m=0}^{k-1} \sum_{\ell=m+1}^{k} (-1)^m \binom{\ell-1}{m} (k-1-m)! \\ &= \sum_{m=0}^{k-1} (-1)^m (k-1-m)! \sum_{\ell=m+1}^{k} \binom{\ell-1}{m} \\ &= \sum_{m=0}^{k-1} (-1)^m \binom{k}{m+1} (k-1-m)! \quad = \quad k! \sum_{m=0}^{k-1} \frac{(-1)^m}{(m+1)!}\,. \end{aligned}$$