Developments in Mathematics

Krishnaswami Alladi
Manjul Bhargava
David Savitt
Pham Huu Tiep *Editors*

Quadratic and Higher Degree Forms

Springer

Developments in Mathematics

VOLUME 31

For further volumes:
http://www.springer.com/series/5834

Krishnaswami Alladi • Manjul Bhargava
David Savitt • Pham Huu Tiep
Editors

Quadratic and Higher Degree Forms

Springer

Editors
Krishnaswami Alladi
Department of Mathematics
University of Florida
Gainesville, FL, USA

Manjul Bhargava
Department of Mathematics
Princeton University
Princeton, NJ, USA

David Savitt
Department of Mathematics
University of Arizona
Tucson, AZ, USA

Pham Huu Tiep
Department of Mathematics
University of Arizona
Tucson, AZ, USA

ISSN 1389-2177
ISBN 978-1-4614-7487-6 ISBN 978-1-4614-7488-3 (eBook)
DOI 10.1007/978-1-4614-7488-3
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013940058

Mathematics Subject Classification (2010): 11E04, 11E16, 11E20, 11E70, 11F03, 05B05, 11H71, 33C70, 33C55, 11F37, 11B65, 05A30, 33E05, 11E12, 11D85, 11E25, 11E76, 11P05, 11N10, 11R52, 11R04

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

There have been dramatic developments in the areas of quadratic and higher degree forms in recent years, and so the time seemed opportune to convene meetings devoted to these topics. During March 2009 there were two major conferences in the area of quadratic forms. One was a research conference at the University of Florida in Gainesville, on “Quadratic forms, sums of squares, and integral lattices” where the latest advances were presented. Immediately after this was the Arizona Winter School on “Quadratic Forms” at the University of Arizona in Tucson, which was an instructional workshop for graduate students with the goal of preparing them for research in this important area. These two conferences were followed by the Conference on Higher Degree Forms at the University of Florida in May 2009.

This volume is an outgrowth of these three conferences, all of which were completely funded by the National Science Foundation. We gratefully acknowledge this support from the NSF. The Tucson conference was the twelfth Arizona Winter School, a longstanding series of NSF-supported workshops on topics in arithmetic geometry. The two Gainesville conferences were in keeping with the tradition there of having annual conferences on various aspects of number theory; they were followed by two Focused Weeks (one on quadratic forms and another on the related topic of integral lattices) at the University of Florida during the Spring of 2010, also fully supported by the NSF. The PIs for the 2009 Florida NSF grant DMS-0753080 were Krishnaswami Alladi and Pham Tiep (then at the University of Florida), with Manjul Bhargava (Princeton) as a consultant. The PIs for the Arizona Winter School NSF grant DMS-0602287 were Matthew Papanikolas, Fernando Rodriguez-Villegas, David Savitt, William Stein, and Dinesh Thakur.

The Arizona Winter School featured instructional lectures by Manjul Bhargava, John Conway, Noam Elkies, Jonathan Hanke, and R. Parimala on various aspects of quadratic forms. The informal (but comprehensive) notes of these lectures are available at the website of the 2009 Arizona Winter School (http://swc.math.arizona.edu). Parimala and Hanke have polished their articles and submitted excellent surveys to this volume.

Even though the Florida conference on quadratic forms was a research conference focusing on the latest developments, there was significant participation by graduate and undergraduate students to help them enter this exciting domain of research. In order to prepare them for the advanced conference lectures, an instructional workshop preceded this conference for which Jonathan Hanke was the main lecturer. Some aspects of his Florida talks are covered in his survey paper in this book.

In his survey, Hanke discusses fundamental connections between the classical theory of quadratic forms over number fields and their rings of integers, and the theory of modular and automorphic forms. In doing so he provides a treatment of theta functions and some aspects of Clifford algebras as well. Hanke's survey is nicely complemented by that of Parimala who provides a lucid introduction to the algebraic theory of quadratic forms, the invariants associated with quadratic forms, and connections with Galois cohomology. She also states some open problems and discusses recent progress. These two surveys are augmented by the survey and research paper of Voight on quaternion algebras and quadratic forms.

The classical theorems of Lagrange that every integer is a sum of four squares and Gauss that every integer is a sum of three triangular numbers motivate the study of "universal forms", namely those that represent all integers, as well as the investigation of ternary forms in general. The papers of Jagy on integral positive ternary quadratic forms, of Berkovich on sums of three squares, and of Chan and Haensch on certain almost universal ternary forms, show that there still are fundamental questions worthy of investigation on very classical topics.

Whereas the study of universal quadratic forms addresses the question of representing all integers, one could consider the question of representing quadratic forms by integral quadratic forms. In 2008 Ellenberg and Venkatesh introduced ergodic theory as a new tool in this study and made dramatic progress going beyond what Eichler and Kneser had achieved using an arithmetic approach. In his survey of such representation problems, Schulze-Pillot sketches three approaches—arithmetic, algebraic and ergodic—and gives a comparative study of them.

The theory of integral lattices has important links with quadratic forms. Bannai and Miezaki discuss a famous conjecture of D. H. Lehmer on the Fourier coefficients of weighted theta series of certain integral lattices and describe recent progress on this classical question. Integral lattices and quadratic forms have links with binary linear codes, and this is investigated by Elkies and Kominers. In doing so, they provide a new structural development of harmonic polynomials on Hamming space analogous to the treatment of harmonic polynomials on Euclidean space, and present several applications.

Finally, the paper of Reznick discusses certain fundamental questions on the length of binary forms of higher degree starting from the seminal work of Sylvester in the mid-nineteenth century. After discussing some current research, he concludes with a list of important open questions.

We hope that this volume, which comprises both introductory survey articles and research papers reporting the latest developments, will be of interest to students and senior mathematicians alike. In conducting the conferences in Florida, we owe a special debt to Frank Garvan as a conference organizer and to Margaret Somers for taking care of all local arrangements. Similarly, we wish to acknowledge Annette

Horn for handling the local arrangements for the 2009 Arizona Winter School. We thank Elizabeth Loew of Springer for her support and interest in including this book in the series Developments in Mathematics.

Gainesville, FL, USA Krishnaswami Alladi
Princeton, NJ, USA Manjul Bhargava
Tucson, AZ, USA David Savitt
Tucson, AZ, USA Pham Huu Tiep

Contents

Toy Models for D. H. Lehmer's Conjecture II

Eiichi Bannai and Tsuyoshi Miezaki*

Abstract In the previous paper under the same title, we showed that the m-th Fourier coefficient of the weighted theta series of the $\mathbb{Z}^2$-lattice and the A_2-lattice does not vanish when the shell of norm m of those lattices is not the empty set. In other words, the spherical 4 (resp. 6)-design does not exist among the nonempty shells in the $\mathbb{Z}^2$-lattice (resp. A_2-lattice). This paper is the sequel to the previous paper. We take 2-dimensional lattices associated to the algebraic integers of imaginary quadratic fields whose class number is either 1 or 2, except for $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, then, show that the m-th Fourier coefficient of the weighted theta series of those lattices does not vanish, when the shell of norm m of those lattices is not the empty set. Equivalently, we show that the corresponding spherical 2-design does not exist among the nonempty shells in those lattices.

Key words and Phrases Weighted theta series • Spherical t-design • Modular forms • Lattices • Hecke operator

Mathematics Subject Classification (2010): Primary 11F03; Secondary 05B30; Tertiary 11R04

---

*The second author was supported by a JSPS Research Fellowship.

E. Bannai (✉)
Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai 200240, China
e-mail: bannai@sjtu.edu.cn

T. Miezaki
Oita National College of Technology, 1666 Oaza-Maki, Oita 870-0152, Japan

Yamagata University, 1-4-12 Kojirakawa, Yamagata 990-8560, Japan
e-mail: miezaki@oita-ct.ac.jp; miezaki@e.yamagata-u.ac.jp

K. Alladi et al. (eds.), *Quadratic and Higher Degree Forms*, Developments in Mathematics 31, DOI 10.1007/978-1-4614-7488-3_1,

1 Introduction

The concept of spherical t-design is due to Delsarte-Goethals-Seidel [7]. For a positive integer t, a finite nonempty subset X of the unit sphere

$$S^{n-1} = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$$

is called a spherical t-design on S^{n-1} if the following condition is satisfied:

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(x)d\sigma(x),$$

for all polynomials $f(x) = f(x_1, x_2, \cdots, x_n)$ of degree not exceeding t. Here, the righthand side means the surface integral on the sphere, and $|S^{n-1}|$ denotes the surface volume of the sphere S^{n-1}. The meaning of spherical t-design is that the average value of the integral of any polynomial of degree up to t on the sphere is replaced by the average value at a finite set on the sphere. A finite subset X in $S^{n-1}(r)$, the sphere of radius r centered at the origin, is also called a spherical t-design if $\frac{1}{r}X$ is a spherical t-design on the unit sphere S^{n-1}.

We denote by $\mathrm{Harm}_j(\mathbb{R}^n)$ the set of homogeneous harmonic polynomials of degree j on $\mathbb{R}^n$. It is well known that X is a spherical t-design if and only if the condition

$$\sum_{x\in X} P(x) = 0$$

holds for all $P \in \mathrm{Harm}_j(\mathbb{R}^n)$ with $1 \le j \le t$ [7]. If the set X is antipodal, that is $-X = X$, and j is odd, then the above condition is fulfilled automatically. So we reformulate the condition of spherical t-design on the antipodal set as follows:

Proposition 1.1. *A nonempty finite antipodal subset $X \subset S^{n-1}$ is a spherical $2s+1$-design if the condition*

$$\sum_{x\in X} P(x) = 0$$

holds for all $P \in \mathrm{Harm}_{2j}(\mathbb{R}^n)$ with $2 \le 2j \le 2s$.

It is known [7] that there is a natural lower bound (Fisher type inequality) for the size of a spherical t-design in S^{n-1}. Namely, if X is a spherical t-design in S^{n-1}, then

$$|X| \ge \binom{n-1+[t/2]}{[t/2]} + \binom{n+[t/2]-2}{[t/2]-1}$$

if t is even, and

$$|X| \geq 2\binom{n-1+[t/2]}{[t/2]} \tag{1}$$

if t is odd.

A lattice in $\mathbb{R}^n$ is a subset $\Lambda \subset \mathbb{R}^n$ with the property that there exists a basis $\{v_1, \cdots, v_n\}$ of $\mathbb{R}^n$ such that $\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_n$, i.e., Λ consists of all integral linear combinations of the vectors $v_1, \cdots, v_n$. The dual lattice Λ is the lattice

$$\Lambda^\sharp := \{y \in \mathbb{R}^n \mid (y, x) \in \mathbb{Z}, \text{ for all } x \in \Lambda\},$$

where (x, y) is the standard Euclidean inner product. The lattice Λ is called integral if $(x, y) \in \mathbb{Z}$ for all x, $y \in \Lambda$. An integral lattice is called even if $(x, x) \in 2\mathbb{Z}$ for all $x \in \Lambda$, and it is odd otherwise. An integral lattice is called unimodular if $\Lambda^\sharp = \Lambda$. For a lattice Λ and a positive real number $m > 0$, the shell of norm m of Λ is defined by

$$\Lambda_m := \{x \in \Lambda \mid (x, x) = m\} = \Lambda \cap S^{n-1}(\sqrt{m}).$$

Let $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\,(z) > 0\}$ be the upper half-plane.

Definition 1.1. Let Λ be the lattice of $\mathbb{R}^n$. Then for a polynomial P, the function

$$\Theta_{\Lambda,P}(z) := \sum_{x \in \Lambda} P(x) e^{i\pi z(x,x)}$$

is called the theta series of Λ weighted by P.

Remark 1.1 (See Hecke [9], Schoeneberg [19, 20]).

(i) When $P = 1$, we get the classical theta series

$$\Theta_\Lambda(z) = \Theta_{\Lambda,1}(z) = \sum_{m \geq 0} |\Lambda_m| q^m, \text{ where } q = e^{\pi i z}.$$

(ii) The weighted theta series can be written as

$$\begin{aligned}\Theta_{\Lambda,P}(z) &= \sum_{x \in \Lambda} P(x) e^{i\pi z(x,x)} \\ &= \sum_{m \geq 0} a_m^{(P)} q^m, \text{ where } a_m^{(P)} := \sum_{x \in \Lambda_m} P(x).\end{aligned}$$

These weighted theta series have been used efficiently for the study of spherical designs which are the nonempty shells of Euclidean lattices. (See [5, 6, 16, 23, 24]. See also [2].)

Lemma 1.1 (cf. [23, 24], [16, Lemma 5]). *Let Λ be an integral lattice in $\mathbb{R}^n$. Then, for $m > 0$, the non-empty shell Λ_m is a spherical t-design if and only if*

$$a_m^{(P)} = 0$$

for all $P \in \mathrm{Harm}_{2j}(\mathbb{R}^n)$ *with* $1 \leq 2j \leq t$, *where* $a_m^{(P)}$ *are the Fourier coefficients of the weighted theta series*

$$\Theta_{\Lambda,P}(z) = \sum_{m \geq 0} a_m^{(P)} q^m.$$

We recall the definition of a modular form.

Definition 1.2. Let $\Gamma \subset SL_2(\mathbb{R})$ be a Fuchsian group of the first kind and let χ be a character of Γ. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k for Γ with respect to χ, if the following conditions are satisfied:

(i) $f\left(\frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{\chi(\sigma)}\right)^k f(z)$ for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(ii) $f(z)$ is holomorphic at every cusp of Γ.

If $f(z)$ has period N, then $f(z)$ has a Fourier expansion at infinity, [11]:

$$f(z) = \sum_{m=0}^{\infty} a_m q_N^m,\ q_N = e^{2\pi i z/N}.$$

We remark that for $m < 0$, $a_m = 0$, by the condition (ii). A modular form with constant term $a_0 = 0$, is called a cusp form. We denote by $M_k(\Gamma, \chi)$ (resp. $S_k(\Gamma, \chi)$) the space of modular forms (resp. cusp forms) with respect to Γ with the character χ. When f is the normalized eigenform of Hecke operators, p. 163, [11], the Fourier coefficients satisfy the following relations:

Lemma 1.2 (cf. [11], Proposition 32, 37, 40, Exercise 2, p. 164). *Let* $\alpha \in \mathbb{N}$ *and* $f(z) = \sum_{m \geq 1} a(m) q^m \in S_k(\Gamma, \chi)$. *If* $f(z)$ *is the normalized eigenform of Hecke operators, then the Fourier coefficients of* $f(z)$ *satisfy the following relations*:

$$a(mn) = a(m)a(n)\ if\ (m,n) = 1 \tag{2}$$

$$a(p^{\alpha+1}) = a(p)a(p^{\alpha}) - \chi(p)p^{k-1}a(p^{\alpha-1})\ if\ p\ is\ a\ prime. \tag{3}$$

We set $f(z) = \sum_{m \geq 1} a(m) q^m \in S_k(\Gamma, \chi)$. When $\dim S_k(\Gamma, \chi) = 1$ and $a(1) = 1$, then $f(z)$ is the normalized eigenform of Hecke operators, [11]. So, the coefficients of $f(z)$ have the relations as mentioned in Lemma 1.2. It is known that

$$|a(p)| < 2p^{(k-1)/2} \tag{4}$$

for all primes p, [11, p. 164], [10]. Note that this is the Ramanujan conjecture and its generalization, called the Ramanujan-Petersson conjecture for cusp forms which are eigenforms of the Hecke operators. These conjectures were proved by Deligne as a consequence of his proof of the Weil conjectures, [11, p. 164], [10]. Moreover, for a prime p with $\chi(p) = 1$ the following equation holds, [12].

$$a(p^{\alpha}) = p^{(k-1)\alpha/2}\frac{\sin(\alpha+1)\theta_p}{\sin\theta_p}, \tag{5}$$

where $2\cos\theta_p = a(p)p^{-(k-1)/2}$ and $\alpha \in \mathbb{N}$.

It is well known that the theta series of $\Lambda \subset \mathbb{R}^n$ weighted by harmonic polynomial $P \in \mathrm{Harm}_j(\mathbb{R}^n)$ is a modular form of weight $n/2 + j$ for some subgroup $\Gamma \subset SL_2(\mathbb{R})$ [8]. In particular, when $\deg(P) \geq 1$, the theta series of Λ weighted by P is a cusp form.

For example, we consider the even unimodular lattice Λ. Then the theta series of Λ weighted by harmonic polynomial P, $\Theta_{\Lambda,P}(z)$, is a modular form with respect to $SL_2(\mathbb{Z})$.

Example 1.1. Let Λ be the E_8-lattice. This is an even unimodular lattice of $\mathbb{R}^8$, generated by the E_8 root system. The theta series is as follows:

$$\begin{aligned}\Theta_\Lambda(z) = E_4(z) &= 1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{2m} \\ &= 1 + 240q^2 + 2{,}160q^4 + 6{,}720q^6 + 17{,}520q^8 + \cdots,\end{aligned}$$

where $\sigma_3(m)$ is a divisor function $\sigma_3(m) = \sum_{0<d|m} d^3$.

For $j = 2, 4$ and 6, the theta series of Λ weighted by $P \in \mathrm{Harm}_j(\mathbb{R}^8)$ is a weight $6, 8$ and 10 cusp form with respect to $SL_2(\mathbb{Z})$. However, it is well known that for $k = 6, 8$ and 10, $\dim S_k(SL_2(\mathbb{Z})) = 0$, that is, $\Theta_{\Lambda,P}(z) = 0$. Then by Lemma 1.1, all the nonempty shells of E_8-lattice are spherical 6-design.

For $j = 8$, the theta series of Λ weighted by P is a weight 12 cusp form with respect to $SL_2(\mathbb{Z})$. Such a cusp form is uniquely determined up to constant, i.e., it is Ramanujan's delta function:

$$\Delta(z) = q^2\prod_{m\geq 1}(1-q^{2m})^{24} = \sum_{m\geq 1}\tau(m)q^{2m}.$$

The following proposition is due to Venkov, de la Harpe and Pache [5, 6, 16, 23].

Proposition 1.2 (cf. [16]). *Let the notation be the same as above. Let Λ be the E_8-lattice. Then the following are equivalent*:

(i) $\tau(m) = 0$.
(ii) $(\Lambda)_{2m}$ *is an* 8-*design.*

It is a famous conjecture of Lehmer that $\tau(m) \neq 0$. So, Proposition 1.2 gives a reformulation of Lehmer's conjecture. Lehmer proved in [12] the following theorem.

Theorem 1.1 (cf. [12]). *Let m_0 be the least value of m for which $\tau(m) = 0$. Then m_0 is a prime if it is finite.*

There are many attempts to study Lehmer's conjecture [12, 21], but it is difficult to prove and it is still open.

Recently, however, we showed the "Toy models for D. H. Lehmer's conjecture" [3]. We take the two cases $\mathbb{Z}^2$-lattice and A_2-lattice. Then, we consider the analogue of Lehmer's conjecture corresponding to the theta series weighted by some harmonic polynomial P. Namely, we show that the m-th coefficient of the weighted theta series of $\mathbb{Z}^2$-lattice does not vanish when the shell of norm m of those lattices is not an empty set. Or equivalently, we show the following result.

Theorem 1.2 (cf. [3]). *The nonempty shells in $\mathbb{Z}^2$-lattice (resp. A_2-lattice) are not spherical 4-designs (resp. 6-designs).*

This paper is sequel to the previous paper [3]. In this paper, we take some lattices related to the imaginary quadratic fields. Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and let $\mathcal{O}_K$ be its ring of algebraic integers. Let Cl_K be the ideal classes. In this paper, we only consider the cases $|\mathrm{Cl}_K| = 1$ and $|\mathrm{Cl}_K| = 2$ except for Sect. 6. So, when we consider the cases $|\mathrm{Cl}_K| = 1$ and $|\mathrm{Cl}_K| = 2$, we denote by $\mathfrak{o}$ (resp. $\mathfrak{a}$) the principal (resp. nonprincipal) ideal class.

We denote by d_K the discriminant of K:

$$d_K = \begin{cases} -4d \text{ if } -d \equiv 2,\ 3 & \pmod 4, \\ -d \ \ \text{ if } -d \equiv 1 & \pmod 4. \end{cases}$$

Theorem 1.3 (cf. [25, p. 87]). *Let d be a positive square-free integer, and let $K = \mathbb{Q}(\sqrt{-d})$. Then*

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{-d} & if\ -d \equiv 2,\ 3 \pmod 4, \\ \mathbb{Z} + \mathbb{Z}\dfrac{-1+\sqrt{-d}}{2} & if\ -d \equiv 1 \pmod 4. \end{cases}$$

Therefore, we consider $\mathcal{O}_K$ to be the lattice in $\mathbb{R}^2$ with the basis

$$\begin{cases} (1,0), (1, \sqrt{-d}) & \text{if } -d \equiv 2,\ 3 \pmod 4, \\ (1,0), \left(-\dfrac{1}{2}, \dfrac{\sqrt{-d}}{2}\right) & \text{if } -d \equiv 1 \pmod 4, \end{cases}$$

denoted by $L_{\mathfrak{o}}$.

Generally, it is well-known that there exists one-to-one correspondence between the set of reduced quadratic forms $f(x, y)$ with a fundamental discriminant $d_K < 0$ and the set of fractional ideal classes of the unique quadratic field $\mathbb{Q}(\sqrt{-d})$ [25, p. 94]. Namely, For a fractional ideal $A = \mathbb{Z}\alpha + \mathbb{Z}\beta$, we obtain the quadratic form $ax^2 + bxy + cy^2$, where $a = \alpha\bar{\alpha}/N(A)$, $b = (\alpha\bar{\beta} + \bar{\alpha}\beta)/N(A)$ and $c = \beta\bar{\beta}/N(A)$. Conversely, for a quadratic form $ax^2 + bxy + cy^2$, we obtain the fractional ideal $\mathbb{Z} + \mathbb{Z}(b + \sqrt{d_K})/2a$. We remark that $N(A)$ is a norm of A and $\bar{\alpha}$ is a complex conjugate of α.

Here, we define the automorphism group of $f(x,y)$ as follows:

$$U_f = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; f(\alpha x + \beta y, \gamma x + \delta y) = f(x,y) \right\}.$$

Then, for $n \geq 1$, the number of the nonequivalent solutions of $f(x,y) = n$ under the action of U_f is equal to the number of the integral ideals of norm n [25]. Namely, let $\mathfrak{a}$ be an ideal class and $f_{\mathfrak{a}}(x,y)$ be the reduced quadratic form corresponding to $\mathfrak{a}$. Moreover, let $L_{\mathfrak{a}}$ be the lattice corresponding to $f_{\mathfrak{a}}(x,y)$. Then,

$$\begin{aligned} &\sum_{x \in L_{\mathfrak{a}}} q^{(x,x)} \\ &= 1 + \#U_f \textstyle\sum_{n=1}^{\infty} \#\{A \mid A \text{ is an integral ideal of } \mathfrak{a},\ N(A) = n\}\, q^n, \end{aligned} \tag{6}$$

where $N(A)$ is the norm of an ideal A.

Theorem 1.4 (cf. [25, p. 63]). *Let $f(x,y)$ be the reduced quadratic form with a fundamental discriminant $D < 0$ and U_f be the automorphism group of $f(x,y)$. Then*

$$\#U_f = \begin{cases} 6 & \text{if } D = -3, \\ 4 & \text{if } D = -4, \\ 2 & \text{if } D < -4. \end{cases}$$

These classical results are due to Gauss, Dirichlet, etc.

When $|\mathrm{Cl}_K| = 1$ and 2, we give the generators of $L_{\mathfrak{a}}$ Tables 5 and 6 of Appendix. Here, we remark that when $K = \mathbb{Q}(\sqrt{-1})$ (resp. $K = \mathbb{Q}(\sqrt{-3})$), $L_{\mathfrak{o}}$ is $\mathbb{Z}^2$-lattice (resp. A_2-lattice). We studied the spherical designs of shells of those lattices in the previous paper [3].

In this paper, we take the imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$, with $d \neq 1$ and $d \neq 3$. Then, we consider the analogue of Lehmer's conjecture corresponding to its theta series weighted by some harmonic polynomial P. Here, we consider the following problem of whether the nonempty shells of $L_{\mathfrak{o}}$ and $L_{\mathfrak{a}}$ are spherical 2-designs (hence 3-designs) or not.

In Sect. 4, we study the case that the class number is 1. We show that the m-th coefficient of the weighted theta series of $L_{\mathfrak{o}}$-lattice does not vanish when the shell of norm m of those lattices is not an empty set. Or equivalently, we show the following result:

Theorem 1.5. *Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field whose class number is 1 and $d \neq 1, 3$ i.e., d is in the following set: $\{2, 7, 11, 19, 43, 67, 163\}$. Then, the nonempty shells in $L_{\mathfrak{o}}$ are not spherical 2-designs.*

Similarly, in Sect. 5, we study the case that the class number is 2 and show the following result:

Theorem 1.6. *Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field whose class number is 2 i.e., d is in the following set: $\{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91,$*

115, 123, 187, 235, 267, 403, 427$\}$. *Then, the nonempty shells in* $L_{\mathfrak{o}}$ *and* $L_{\mathfrak{a}}$ *are not spherical* 2*-designs.*

In Sect. 6, we consider the case that the class number is 3 and study the property of Hecke characters. In Sect. 7, we give some concluding remarks and state a conjecture for the future study.

2 Preliminaries

In this section, we review the theory of imaginary quadratic fields.

Theorem 2.1 (cf. [4, p. 104, Proposition 5.16]). *We can classify the prime ideals of a quadratic field as follows*:

1. *If* p *is an odd prime and* $(d_K/p) = 1$ (*resp.* $d_K \equiv 1 \pmod 8$) *then* (p)=$P\overline{P}$ (*resp.* $(2) = P\overline{P}$), *where* P *and* $\overline{P}$ *are prime ideals with* $P \neq \overline{P}$, $N(P)$=$N(\overline{P}) = p$ (*resp.* $N(P) = 2$).
2. *If* p *is an odd prime and* $(d_K/p) = -1$ (*resp.* $d_K \equiv 5 \pmod 8$) *then* (p)=P (*resp.* $(2) = P$), *where* P *is a prime ideal with* $N(P) = p^2$ (*resp.* $N(P)$=4).
3. *If* $p \mid d_k$ *then* $(p) = P^2$, *where* P *is a prime ideal with* $N(P) = p$.

Lemma 2.1. *Let* $|\mathrm{Cl}_K| = 1$ *and* I *be an integral ideal of* K. *For* $n \in \mathbb{N}$, *if* $N(I) = n$ *and* I *is a principal ideal, namely,* $I \in \mathfrak{o}$ *then there exist* $a, b \in \mathbb{Z}$ *such that for* $-d \equiv 2, 3 \pmod 4$

$$n = a^2 + db^2,$$

for $-d \equiv 1 \pmod 4$

$$n = a^2 + db^2 \quad or \quad n = \frac{a^2 + db^2}{4}.$$

If $|\mathrm{Cl}_K| = 2$, $N(I) = n$ *and* I *is a nonprincipal ideal, namely,* $I \in \mathfrak{a}$ *and assume that* m *is one of the norms of nonprincipal ideals then there exist* $a, b \in \mathbb{Z}$ *such that for* $-d \equiv 2, 3 \pmod 4$

$$mn = a^2 + db^2,$$

for $-d \equiv 1 \pmod 4$

$$mn = a^2 + db^2 \quad or \quad mn = \frac{a^2 + db^2}{4}.$$

Proof. We assume that $|\mathrm{Cl}_K| = 1$. For $-d \equiv 2, 3 \pmod 4$, we can write $I = (a + b\sqrt{-d})$, then $N(I) = a^2+db^2$. For $-d \equiv 1 \pmod 4$, we can write $I = (a+b\sqrt{-d})$ or $I = ((a + b\sqrt{-d})/2)$, then $N(I) = a^2 + db^2$ or $N(I) = (a^2 + db^2)/4$.

Here, we assume that $|\mathrm{Cl}_K| = 2$. Let J be the nonprincipal ideal of K whose norm is m. If I is a nonprincipal ideal then, JI is a principal ideal of K. Therefore, for $-d \equiv 2, 3 \pmod 4$, we can write $JI = (a + b\sqrt{-d})$, then $N(JI) = a^2 + db^2$. Hence, $mn = a^2 + db^2$. for $-d \equiv 1 \pmod 4$, we can write $JI = (a + b\sqrt{-d})$ or $JI = ((a + b\sqrt{-d})/2)$, then $N(JI) = a^2 + db^2$ or $N(JI) = (a^2 + db^2)/4$. Hence, $mn = a^2 + db^2$ or $mn = (a^2 + db^2)/4$. □

Proposition 2.1. *Let $F(m)$ be the number of the integral ideals of norm m of K. Let p be a prime number. Then, if $p \neq 2$*

$$F(p^e) = \begin{cases} e + 1 & if \ \ (d_K/p) = 1, \\ (1 + (-1)^e)/2 & if \ \ (d_K/p) = -1, \\ 1 & if \ p \mid d_K, \end{cases}$$

if $p = 2$

$$F(2^e) = \begin{cases} e + 1 & if \ d_K \equiv 1 \pmod 8, \\ (1 + (-1)^e)/2 & if \ d_K \equiv 5 \pmod 8, \\ 1 & if \ 2 \mid d_K. \end{cases}$$

Proof. When $(d_K/p) = 1$ i.e., $(p) = P\overline{P}$ and $P \neq \overline{P}$, since P and $\overline{P}$ are the only integral ideals of norm p, we have $F(p) = 2$. Moreover, the integral ideals of norm p^e are as follows: P^e, $P^{e-1}\overline{P}$, $\dots$, $(\overline{P})^e$. So, we have $F(p^e) = e + 1$. The other cases can be proved similarly. □

3 Hecke Characters and Theta Series

In this section, we introduce the Hecke character and discuss the relationships between the Hecke character and the weighted theta series of the lattices $L_{\mathfrak{o}}$ and $L_{\mathfrak{a}}$. Then, we show that for $|\mathrm{Cl}_K| = 1$ and $P_1 = (x^2 - y^2)/2$, the weighted theta series $\Theta_{L_{\mathfrak{o}},P_1}$ is a normalized Hecke eigenform. For $|\mathrm{Cl}_K| = 2$ and $P_2 = x^2 - y^2$, a certain sum of the two weighted theta series $c_1\Theta_{L_{\mathfrak{o}},P_2} + c_2\Theta_{L_{\mathfrak{a}},P_2}$ is a normalized Hecke eigenform. Later, we give the explicit values of c_1 and c_2.

For the readers convenience we quote from [15] the notion of the Hecke character (for more information the reader is referred to [15]). A Hecke character ϕ of weight $k \geq 2$ with modulus Λ is defined in the following way. Let Λ be a nontrivial ideal in $\mathcal{O}_K$ and let $I(\Lambda)$ denote the group of fractional ideals prime to Λ. A Hecke character ϕ with modulus Λ is a homomorphism

$$\phi : I(\Lambda) \to \mathbb{C}^\times$$

such that for each $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod \Lambda$ we have

$$\phi(\alpha\mathcal{O}_K) = \alpha^{k-1}. \tag{7}$$

Let ω_ϕ be the Dirichlet character with the property that

$$\omega_\phi(n) := \phi((n))/n^{k-1}$$

for every integer n coprime to Λ.

Theorem 3.1 (cf. [15, p. 9], [14, p. 183]). *Let the notation be the same as above, and define $\Psi_{K,\Lambda}(z)$ by*

$$\Psi_{K,\Lambda}(z) := \sum_A \phi(A) q^{N(A)} = \sum_{n=1}^{\infty} a(n) q^n, \tag{8}$$

where the sum is over the integral ideals A that are prime to Λ and $N(A)$ is the norm of the ideal A. Then $\Psi_{K,\Lambda}(z)$ is a cusp form in $S_k(\Gamma_0(d_K \cdot N(\Lambda)), \left(\frac{-d_K}{\bullet}\right)\omega_\phi)$.

We remark that function (8) is a normalized Hecke eigenform [1, 22]. Moreover, if the class number of K is h then the character as given in (7) will have h extensions to nonprincipal ideals. Namely, the function (8) has h choices, so we denote by $\Psi^{(1)}_{K,\Lambda}(z), \ldots, \Psi^{(h)}_{K,\Lambda}(z)$ these functions (see [17]).

Example 3.1.

(i) $d = 2$.
We calculate $\Psi_{K,\Lambda}(z) = \sum_{m\geq 1} a(m) q^m$, where $\Lambda = (1)$ and the weight of the Hecke character is 3. We remark that $|\mathrm{Cl}_K| = 1$ and ideals are listed in Table 3.
By the definitions (7) and (8), we have $a(1) = 1^2 = 1$, $a(2) = \sqrt{-2}^2 = -2$, $a(3) = (-1+\sqrt{-2})^2 + (-1-\sqrt{-2})^2 = 2$, $a(4) = 2^2, \ldots$. Thus, we obtain

$$\Psi^{(1)}_{K,\Lambda}(z) = q - 2q^2 - 2q^3 + 4q^4 + 4q^6 - 8q^8 - 5q^9 + \cdots.$$

(ii) $d = 5$.
We calculate $\Psi_{K,\Lambda}(z) = \sum_{m\geq 1} a(m) q^m$, where $\Lambda = (1)$ and the weight of the Hecke character is 3. We remark that $|\mathrm{Cl}_K| = 2$ and ideals are listed in Table 4. When A of norm m is a nonprincipal ideal, A^2 is a principal ideal, so, $\phi(A^2)$ is computable by the definition (7). For example, $\phi((2, 1+\sqrt{-5}))^2 = \phi((2)) = 4$, so, we can assume that $\phi((2, 1+\sqrt{-5})) = 2$, i.e., $a(2) = 2$. Then, since $(2, 1+\sqrt{-5})(3, 1+\sqrt{-5}) = (1-\sqrt{-5})$ and $(2, 1+\sqrt{-5})(3, 1-\sqrt{-5}) = (-1-\sqrt{-5})$, we have $a(3) = ((1+\sqrt{-5})^2 + (1-\sqrt{-5})^2)/2 = -4$, $a(4) = 2^2$, ... Thus, we obtain

$$\Psi^{(1)}_{K,\Lambda}(z) = q + 2q^2 - 4q^3 + 4q^4 - 5q^5 - 8q^6 + 4q^7 + 8q^8 + 7q^9 + \cdots.$$

On the other hand, we assume that $\phi((2, 1+\sqrt{-5})) = -2$, i.e., $a(2) = -2$. Then, we have

$$\Psi^{(2)}_{K,\Lambda}(z) = q - 2q^2 + 4q^3 + 4q^4 - 5q^5 - 8q^6 - 4q^7 - 8q^8 + 7q^9 + \cdots.$$

Table 1 Coefficients, c_1 and c_2

$-d$	-5	-6	-10	-13	-15	-22	-35	-37	-51
c_1	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
c_2	$1/2$	$1/2$	$1/2$	$1/2$	2	$1/2$	3	$1/2$	$1/2$
$-d$	-58	-91	-115	-123	-187	-235	-267	-403	-427
c_1	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
c_2	$1/2$	$5/3$	$1/2$	$1/2$	$7/3$	$1/2$	$1/2$	$11/9$	$1/2$

Here, we discuss the relationships between the Hecke character and the weighted theta series of the lattices $L_{\mathfrak{o}}$ and $L_{\mathfrak{a}}$. First, we quote the following theorem:

Theorem 3.2 (cf. [14, p. 192]). *Let L be a 2-dimensional integral lattice with the Gram matrix A and N be the natural number such that the elements of NA^{-1} are rational integers. Let the character $\chi(d)$ be*

$$\chi(d) = \Big(\frac{(-1)^{(r/2)}\det L}{d}\Big).$$

Then, for $P \in \mathrm{Harm}_2(\mathbb{R}^2)$,

(1) $\Theta_{L,P} \in M_3(\Gamma_0(4N), \chi)$.
(2) *If all the diagonal elements of A are even, then $\Theta_{L,P} \in M_3(\Gamma_0(2N), \chi)$.*
(3) *If all the diagonal elements of A and NA^{-1} are even, then $\Theta_{L,P} \in M_3(\Gamma_0(N), \chi)$.*

Then, we obtain the following lemmas:

Lemma 3.1. *Let K be an imaginary quadratic field whose class number is 1 and $L_{\mathfrak{o}}$ be the lattice corresponding to the principal ideal class $\mathfrak{o}$. Let ϕ be the Hecke character of weight 3 with modulus Λ. Assume that $\Lambda = (1)$ and $P_1 = (x^2-y^2)/2 \in \mathrm{Harm}_2(\mathbb{R}^2)$. Then, $\Psi_{K,\Lambda}(q) = \Theta_{L_{\mathfrak{o}},P_1}(q)$.*

Lemma 3.2. *Let K be an imaginary quadratic field whose class number is 2 and $L_{\mathfrak{o}}$ (resp. $L_{\mathfrak{a}}$) be the lattice corresponding to the principal ideal class $\mathfrak{o}$ (resp. nonprincipal ideal class $\mathfrak{a}$). Let ϕ be the Hecke character of weight 3 with modulus Λ. Assume that $\Lambda = (1)$ and $P_2 = x^2 - y^2 \in \mathrm{Harm}_2(\mathbb{R}^2)$. Then, $\Psi_{K,\Lambda}(q) = c_1\Theta_{L_{\mathfrak{o}},P_2}(q) + c_2\Theta_{L_{\mathfrak{a}},P_2}(q)$, where c_1 and c_2 are given as in Table 1.*

Proof of Lemmas 3.1 and 3.2. First, we assume that the lattices are integral lattices, if not we multiply the Gram matrix of L by 2.

Because of the Theorems 3.1 and 3.2, $\Psi_{K,\Lambda}(q)$, $\Theta_{L_{\mathfrak{o}},P}(q)$ and $\Theta_{L_{\mathfrak{a}},P}(q)$ with $P = P_1$, P_2 are modular forms of the same group Γ. Therefore, we calculate the basis of the space of modular forms of group Γ and check $\Psi_{K,\Lambda}(q) = \Theta_{L_{\mathfrak{o}},P_1}(q)$ and $\Psi_{K,\Lambda}(q) = c_1\Theta_{L_{\mathfrak{o}},P_2}(q) + c_2\Theta_{L_{\mathfrak{a}},P_2}(q)$ explicitly (using "Sage", Mathematics Software [18]). □

Corollary 3.1. *Let the notation be the same as above. If* $|\mathrm{Cl}_K| = 1$ *then* $\Theta_{L_1,P_1}(q)$ *is a normalized Hecke eigenform. If* $|\mathrm{Cl}_K| = 2$ *then* $c_1\Theta_{L_1,P_2}(q) + c_2\Theta_{L_2,P_2}(q)$ *is a normalized Hecke eigenform.*

Proof. The function (8) is a normalized Hecke eigenform [1, 22]. □

Finally, we give the following proposition, which is an analogue of Theorem 1.1 and the crucial part of the proof of Theorems 1.5 and 1.6.

Proposition 3.1. *Assume that* $\sum_{m\geq 1} a(m)q^m$ *is a normalized Hecke eigenform of* $S_3(\Gamma,\chi)$ *and the coefficients* $a(m)$ *are rational integers. Moreover let* p *be the prime such that* $\chi(p) = 1$. *Let* α_0 *be the least value of* α *for which* $a(p^\alpha) = 0$. *If* $a(p) \neq \pm p$ *then* $\alpha_0 = 1$ *if it is finite.*

Proof. Assume the contrary, that is, $\alpha_0 > 1$, so that $a(p) \neq 0$. By the equation (5),

$$a(p^{\alpha_0}) = 0 = p^{\alpha_0}\frac{\sin(\alpha_0+1)\theta_p}{\sin\theta_p}.$$

This shows that θ_p is a real number of the form $\theta_p = \pi k/(1+\alpha_0)$, where k is an integer. Now the number

$$z = 2\cos\theta_p = a(p)p^{-1}, \tag{9}$$

being twice the cosine of a rational multiple of 2π, is an algebraic integer. On the other hand, z is a root of the equation

$$pz - a(p) = 0. \tag{10}$$

Hence z is a rational integer. By (4) and (9), we have $|z| \leq 1$. Therefore $z = \pm 1$ and the equation (10) becomes $a(p) = \pm p$. By assumption, this is a contradiction. □

4 The Case of $|\mathrm{Cl}_K| = 1$

Let $K := \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. If the class number of K is 1 then d is in the following set $\{1, 2, 3, 7, 11, 19, 43, 67, 163\}$. In particular, we only consider the cases where d is in the set: $\{2, 7, 11, 19, 43, 67, 163\}$ since the cases $d = 1$ and $d = 3$ are considered in [3].

In this section, we assume that $a(m)$ and $b(m)$ are the coefficients of the following functions:

$$\Theta_{L_\circ}(q) = \sum_{m\geq 0} a(m)q^m,\ \Theta_{L_\circ,P_1}(q) = \sum_{m\geq 1} b(m)q^m,$$

where $P_1 = (x^2 - y^2)/2 \in \mathrm{Harm}_2(\mathbb{R}^2)$.

Lemma 4.1. *Let p be a prime number. Let d be one of the elements in $\{2, 7, 11, 19, 43, 67, 163\}$. We set $a'(m) = a(m)/2$ for all m. Then,*

$$a'(p^e) = \begin{cases} e+1 & if\ \ (d_K/p) = 1, \\ (1+(-1)^e)/2 & if\ \ (d_K/p) = -1, \\ 1 & if\ p \mid d_K. \end{cases}$$

Proof. Because of the equation (6), $a'(m)$ is the number of integral ideals of K of norm m. Therefore, it can be proved by Proposition 2.1. □

Lemma 4.2. *Let p be a prime number such that $(d_K/p) = 1$. Then, $b(p) \neq 0$. Moreover, if $p \neq d$ then $b(p) \neq \pm p$.*

Proof. We remark that by Lemma 3.1 and Corollary 3.1, $\Theta_{L_{\mathfrak{o}},P_1}(q) = \Psi_{K,\Lambda}(q)$. So, the numbers $b(m)$ are the coefficients of $\Psi_{K,\Lambda}(q)$.

First, we assume that $d \neq 2$, i.e., $-d \equiv 1 \pmod 4$ and $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}(1 + \sqrt{-d})/2$. If $N((a + b\sqrt{-d}))$ is equal to p then by Lemma 2.1

$$p = a^2 + db^2.$$

Because of the definition of $\Psi_{K,\Lambda}(q)$,

$$b(p) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).$$

If $b(p) = 0$ then $a^2 = db^2$. This is a contradiction. Assume that $b(p) = \pm p$. Then,

$$2(a^2 - db^2) = \pm(a^2 + db^2),$$

that is, $a^2 = 3db^2$ or $3a^2 = db^2$. This is a contradiction.

If $N(((a + b\sqrt{-d})/2))$ is equal to p then by Lemma 2.1

$$\frac{a^2 + db^2}{4} = p.$$

Because of the definition of $\Psi_{K,\Lambda}(q)$,

$$b(p) = \left(\frac{a + b\sqrt{-d}}{2}\right)^2 + \left(\frac{a - b\sqrt{-d}}{2}\right)^2 = \frac{a^2 - db^2}{2}.$$

If $b(p) = 0$ then $a^2 = db^2$. This is a contradiction. Assume that $b(p) = \pm p$. Then,

$$\frac{a^2 - db^2}{2} = \pm\frac{a^2 + db^2}{4},$$

that is, $a^2 = 3db^2$ or $3a^2 = db^2$. This is a contradiction.

Next, we assume that $d = 2$ i.e., $-d \equiv 2 \pmod 4$ and $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$. If $N((a + b\sqrt{-2}))$ is equal to p then by Lemma 2.1

$$p = a^2 + 2b^2.$$

Because of the definition of $\Psi_{K,\Lambda}(q)$,

$$b(p) = (a + b\sqrt{-2})^2 + (a - b\sqrt{-2})^2 = 2(a^2 - 2b^2).$$

If $b(p) = 0$ then $a^2 = 2b^2$. This is a contradiction. Assume that $b(p) = \pm p$. Then,

$$2(a^2 - 2b^2) = \pm(a^2 + 2b^2),$$

that is, $a^2 = 6b^2$ or $3a^2 = 2b^2$. This is a contradiction. □

Proof of Theorem 1.5. We will show that $b(m) \neq 0$ when $(L_{\mathfrak{o}})_m \neq \emptyset$.

By Theorem 3.1, $\Theta_{L_{\mathfrak{o}},P_1}$ is a normalized Hecke eigenform. So, We assume that m is a power of prime, if not we could apply the equation (2). We will divide our considerations into the following three cases.

(i) Case $m = p^\alpha$ and $p \mid d_K$:
By $a(m) = 2$ and the inequality (1), the shells $(L_{\mathfrak{o}})_m$ are not spherical 2-designs. Hence, $b(m) \neq 0$.

(ii) Case $m = p^\alpha$ and $(d_K/p) = -1$:
By Lemma 4.1,

$$a(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

By $a(m) = 2$ and the inequality (1), when n is even, the shells $(L_{\mathfrak{o}})_m$ are not spherical 2-designs. Hence, $b(m) \neq 0$.

(iii) Case $m = p^\alpha$ and $(d_K/p) = 1$:
By Proposition 3.1 and Lemma 4.2, we have $b(m) \neq 0$. This completes the proof of Theorem 1.5. □

5 The Case of $|\mathrm{Cl}_K| = 2$

Let $K := \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. In this section, we assume that the class number of K is 2. So, we consider that d is in the following set: $\{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}$. We denote by $\mathfrak{o}$ (resp. $\mathfrak{a}$) the principal (resp. nonprincipal) ideal class.

In this section, we also assume that $a(m)$ and $b(m)$ are the coefficients of the following functions:

$$\Theta_{L_{\mathfrak{o}}}(q) + \Theta_{L_{\mathfrak{a}}}(q) = \sum_{m \geq 0} a(m) q^m,$$

$$c_1 \Theta_{L_{\mathfrak{o}},P_2}(q) + c_2 \Theta_{L_{\mathfrak{a}},P_2}(q) = \sum_{m \geq 1} b(m) q^m,$$

where c_1 and c_2 are defined in Lemma 3.2.

Table 2 Values of m and $b(m)$

$-d$	-5	-6	-10	-13	-15	-22	-35	-37	-51
m	2	2	2	2	3	2	5	2	3
$b(m)$	2	2	2	2	-3	2	-5	2	3
$-d$	-58	-91	-115	-123	-187	-235	-267	-403	-427
m	2	7	5	3	11	5	3	13	7
$b(m)$	2	-7	-5	3	-11	5	3	-13	7

Lemma 5.1. *Set* $l_1 := \{N(O) \mid O \in \mathfrak{o}\}$ *and* $l_2 := \{N(A) \mid A \in \mathfrak{a}\}$. *Then,* $l_1 \cap l_2 = \emptyset$. *Therefore, the set* $L_{\mathfrak{o}} \cap L_{\mathfrak{a}}$ *consists of the origin.*

Proof. Let p be a prime number such that $(d_K/p) = 1$. Then there exist prime ideals P and P' such that $(p) = PP'$ and $N(P) = N(P') = p$. Since a class number is 2, we have P and $P' \in \mathfrak{o}$ or P and $P' \in \mathfrak{a}$. If P and $P' \in \mathfrak{o}$ we denote by p_i such a prime. If P and $P' \in \mathfrak{a}$ we denote by p_i' such a prime.

Let p be a prime number such that $(d_K/p) = -1$. Then (p) is a prime ideal and $N((p)) = p^2$. We denote by q_i such a prime.

Let p be a prime number such that $p \mid d_K$. Then there exists a prime ideal P such that $(p) = P^2$ and $N(P) = p$. Since a class number is 2, we have $P \in \mathfrak{o}$ or $P \in \mathfrak{a}$. If $P \in \mathfrak{o}$ we denote by r_i such a prime. If $P \in \mathfrak{a}$ we denote by r_i' such a prime.

We take the element $n \in l_1 \cap l_2$ and perform a prime factorization, $n = p_1 \cdots p_1' \cdots q_1 \cdots r_1 \cdots r_1' \cdots$. Then, $p_1 \cdots$, $q_1 \cdots$ and $r_1 \cdots$ correspond to principal ideals. So, if the number of occurrences of each of the primes p' and r' is even then $n \in l_1$ and if the number of occurrences of each of the primes p' and r' is odd then $n \in l_2$. This completes the proof of Lemma 5.1. □

Lemma 5.2. *Let* p *be a prime number. Let* d *be one of the elements in* $\{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}$. *We set* $a'(m) = a(m)/2$ *for all* m. *Then,*

$$a'(p^e) = \begin{cases} e+1 & if\ \ (d_K/p) = 1, \\ (1+(-1)^e)/2 & if\ \ (d_K/p) = -1, \\ 1 & if\ p \mid d_K. \end{cases}$$

Proof. Because of the equation (6), $a'(m)$ is the number of integral ideals of K of norm m. Therefore, it can be proved by Proposition 2.1. □

Lemma 5.3. *Let* p *be a prime number such that* $(d_K/p) = 1$. *Then,* $b(p) \neq 0$. *Moreover, if* $p \neq d$ *then* $b(p) \neq \pm p$.

Proof. We remark that by Lemma 3.2 and Corollary 3.1, $c_1\Theta_{L_{\mathfrak{o}},P_2}(q) + c_2\Theta_{L_{\mathfrak{a}},P_2}(q) = \Psi_{K,\Lambda}(q)$. So, the numbers $b(m)$ are the coefficients of $\Psi_{K,\Lambda}(q)$.

We set $N(J) = p$. When J is a principal ideal, it can be proved by the similar method in Lemma 4.2. So, we assume that J is nonprincipal.

We list the smallest prime number m such that $m \mid d_K$ and $m \in \{N(I) \mid I \in \mathfrak{a}\}$, and the values $b(m)$ are in Table 2.

First, we assume that $-d \equiv 2$ or $3 \pmod 4$. If $N(J)$ is equal to p then by Lemma 2.1

$$mp = a^2 + db^2.$$

Because of the definition of $\Psi_{K,\Lambda}(q)$,

$$b(mp) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).$$

Since $b(mp) = b(m)b(p)$ and the value of $b(m)$ in Table 2, we have $b(p) = a^2 - db^2$. If $b(p) = 0$ then $a^2 = db^2$. This is a contradiction. Assume that $b(p) = \pm p$. Then,

$$a^2 - db^2 = \pm \frac{a^2 + db^2}{2},$$

that is, $a^2 = 3db^2$ or $3a^2 = db^2$. This is a contradiction.

Next, we assume that $-d \equiv 1 \pmod 4$. If $N(J)$ is equal to p then by Lemma 2.1 there exist $a, b \in \mathbb{Z}$ such that

$$mp = a^2 + db^2 \quad or \quad mp = \frac{a^2 + db^2}{4}.$$

Because of the definition of $\Psi_{K,\Lambda}(q)$,

$$b(mp) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).$$

or

$$b(mp) = \Big(\frac{a + b\sqrt{-d}}{2}\Big)^2 + \Big(\frac{a - b\sqrt{-d}}{2}\Big)^2 = \frac{a^2 - db^2}{2}.$$

Since $b(mp) = b(m)b(p)$ and the value of $b(m)$ in Table 2, we have $b(p) = 2/b(m) \times (a^2 - db^2)$ or $b(p) = 1/b(m) \times (a^2 - db^2)/2$. If $b(p) = 0$ then $a^2 = db^2$. This is a contradiction. Assume that $b(p) = \pm p$. Then,

$$\frac{2(a^2 - db^2)}{b(m)} = \pm \frac{a^2 + db^2}{m},$$

or

$$\frac{a^2 - db^2}{2b(m)} = \pm \frac{a^2 + db^2}{4m},$$

that is, $a^2 = 3db^2$ or $3a^2 = db^2$ since $m = \pm b(m)$ for $-d \equiv 1 \pmod 4$. This is a contradiction. □

Proof of Theorem 1.6. Because of Lemma 5.1, it is enough to show that $b(m) \neq 0$ when $(L_{\mathfrak{o}})_m \neq \emptyset$ or $(L_{\mathfrak{a}})_m \neq \emptyset$.

By Theorem 3.1, $c_1\Theta_{L_{\mathfrak{o}},P_2} + c_2\Theta_{L_{\mathfrak{a}},P_2}$ is a normalized Hecke eigenform. So, We assume that m is a power of prime, if not we could apply the equation (2). We will divide into the three cases.

(i) Case $m = p^\alpha$ and $p \mid d_K$:
By $a(m) = 2$ and (1), the shells $(L)_m$ are not spherical 2-designs. Hence, $b(m) \neq 0$.

(ii) Case $m = p^\alpha$ and $(d_K/p) = -1$:
By Lemma (4.1),

$$a(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

By $a(m) = 2$ and (1), when n is even, the shells $(L)_m$ are not spherical 2-designs. Hence, $b(m) \neq 0$.

(iii) Case $m = p^\alpha$ and $(d_K/p) = 1$:
By Proposition 3.1 and Lemma 5.3, $b(m) \neq 0$. This completes the proof of Theorem 1.6. □

6 The Case of $|\mathrm{Cl}_K| = 3$

In the previous sections, we studied the cases of class number $h = |\mathrm{Cl}_K|$ is either 1 or 2. However, it seems that the situation is somewhat different for the cases of class numbers $h \geq 3$. In this section, we discuss briefly how it is different, by considering the case of $d = 23$ ($h = 3$).

We first remark that one reason of success for the cases $h = 1$ and $h = 2$ is that the coefficients $a(m)$ of the Hecke eigenform $\Psi_{K,\Lambda}$ are all integers. Therefore, by formula (10), we have that $z = a(p)/p$ is a rational number (and since it is an algebraic integer), and so it must be a rational integer. It seems that this situation is no longer true in general for the cases of $h \geq 3$. We will give more detailed information, concentrating the special (and typical) case of $d = 23$.

We denote by $\mathfrak{o}$, $\mathfrak{a}_1$ and $\mathfrak{a}_2$ the ideal classes. The corresponding quadratic forms are $x^2 + xy + 6y^2$, $2x^2 - xy + 3y^2$ and $2x^2 + xy + 3y^2$, namely, $L_{\mathfrak{o}} = \langle(1,0), (1/2, \sqrt{23}/2)\rangle$, $L_{\mathfrak{a}_1} = \langle(2,0), (1/2, \sqrt{23}/2)\rangle$ and $L_{\mathfrak{a}_2} = \langle(2,0), (-1/2, \sqrt{23}/2)\rangle$, respectively. We give the weighted theta series of those ideal lattices. We set $P_1 = x^2 - y^2$ and $P_2 = xy$ in this section.

$\Theta_{L_{\mathfrak{o}}} = 1 + 2q + 2q^4 + 4q^6 + 4q^8 + 2q^9 + 4q^{12} + 2q^{16} + 4q^{18} + 2q^{23} + 4q^{24} + 2q^{25} + 4q^{26} + 4q^{27} + 4q^{32} + 6q^{36} + 4q^{39} + 8q^{48} + 2q^{49} + 4q^{52} + 4q^{54} + 4q^{58} + 4q^{59} + 4q^{62} + 6q^{64} + 8q^{72} + 4q^{78} + 2q^{81} + 4q^{82} + 4q^{87} + 2q^{92} + 4q^{93} + 4q^{94} + 8q^{96} + 2q^{100} + O[q]^{101}$.

$\frac{1}{2} \times \Theta_{L_{\mathfrak{o}},P_1} = q + 4q^4 - 11q^6 - 7q^8 + 9q^9 + q^{12} + 16q^{16} + 13q^{18} - 23q^{23} - 44q^{24} + 25q^{25} + 29q^{26} - 38q^{27} - 28q^{32} + 85q^{36} - 14q^{39} + 77q^{48} + 49q^{49} - 103q^{52} - 99q^{54} - 91q^{58} + 26q^{59} + 101q^{62} - 15q^{64} - 11q^{72} + 133q^{78} + 81q^{81} - 43q^{82} + 82q^{87} - 92q^{92} - 182q^{93} - 19q^{94} - 7q^{96} + 100q^{100} + O[q]^{101}$.

$\Theta_{L_{\mathfrak{o}},P_2} = 0$.

$\Theta_{L_{\mathfrak{a}_1}} = 1+2q^2+2q^3+2q^4+2q^6+2q^8+2q^9+4q^{12}+2q^{13}+4q^{16}+4q^{18}+6q^{24}+2q^{26}+2q^{27}+2q^{29}+2q^{31}+4q^{32}+6q^{36}+2q^{39}+2q^{41}+2q^{46}+2q^{47}+6q^{48}+2q^{50}+4q^{52}+6q^{54}+2q^{58}+2q^{62}+4q^{64}+2q^{69}+2q^{71}+8q^{72}+2q^{73}+2q^{75}+6q^{78}+4q^{81}+2q^{82}+2q^{87}+2q^{92}+2q^{93}+2q^{94}+8q^{96}+2q^{98}+2q^{100}+O[q]^{101}$.
$2\times\Theta_{L_{\mathfrak{a}_1},P_1} = 8q^2-11q^3-7q^4+q^6+32q^8+13q^9-88q^{12}+29q^{13}-56q^{16}+121q^{18}+81q^{24}-103q^{26}-99q^{27}-91q^{29}+101q^{31}+49q^{32}+41q^{36}+133q^{39}-43q^{41}-184q^{46}-19q^{47}-183q^{48}+200q^{50}+232q^{52}-295q^{54}+209q^{58}+41q^{62}-224q^{64}+253q^{69}+77q^{71}+393q^{72}-283q^{73}-275q^{75}-375q^{78}+418q^{81}-247q^{82}-227q^{87}+161q^{92}-203q^{93}+353q^{94}+616q^{96}+392q^{98}-175q^{100}+O[q]^{101}$.
$\frac{4}{\sqrt{23}}\times\Theta_{L_{\mathfrak{a}_1},P_2} = q^3-3q^4+5q^6-7q^9+9q^{13}-11q^{18}+13q^{24}-3q^{26}+9q^{27}-15q^{29}-15q^{31}+21q^{32}-27q^{36}+17q^{39}+33q^{41}-39q^{47}-19q^{48}+45q^{54}+21q^{58}-51q^{62}-23q^{69}+57q^{71}+5q^{72}-15q^{73}+25q^{75}-35q^{78}-38q^{81}+45q^{82}-55q^{87}+69q^{92}+65q^{93}-27q^{94}-75q^{100}+O[q]^{101}$.
$\Theta_{L_{\mathfrak{a}_2}} = 1+2q^2+2q^3+2q^4+2q^6+2q^8+2q^9+4q^{12}+2q^{13}+4q^{16}+4q^{18}+6q^{24}+2q^{26}+2q^{27}+2q^{29}+2q^{31}+4q^{32}+6q^{36}+2q^{39}+2q^{41}+2q^{46}+2q^{47}+6q^{48}+2q^{50}+4q^{52}+6q^{54}+2q^{58}+2q^{62}+4q^{64}+2q^{69}+2q^{71}+8q^{72}+2q^{73}+2q^{75}+6q^{78}+4q^{81}+2q^{82}+2q^{87}+2q^{92}+2q^{93}+2q^{94}+8q^{96}+2q^{98}+2q^{100}+O[q]^{101}$.
$2\times\Theta_{L_{\mathfrak{a}_2},P_1} = 8q^2-11q^3-7q^4+q^6+32q^8+13q^9-88q^{12}+29q^{13}-56q^{16}+121q^{18}+81q^{24}-103q^{26}-99q^{27}-91q^{29}+101q^{31}+49q^{32}+41q^{36}+133q^{39}-43q^{41}-184q^{46}-19q^{47}-183q^{48}+200q^{50}+232q^{52}-295q^{54}+209q^{58}+41q^{62}-224q^{64}+253q^{69}+77q^{71}+393q^{72}-283q^{73}-275q^{75}-375q^{78}+418q^{81}-247q^{82}-227q^{87}+161q^{92}-203q^{93}+353q^{94}+616q^{96}+392q^{98}-175q^{100}+O[q]^{101}$.
$\frac{4}{\sqrt{23}}\times\Theta_{L_{\mathfrak{a}_2},P_2} = -q^3+3q^4-5q^6+7q^9-9q^{13}+11q^{18}-13q^{24}+3q^{26}-9q^{27}+15q^{29}+15q^{31}-21q^{32}+27q^{36}-17q^{39}-33q^{41}+39q^{47}+19q^{48}-45q^{54}-21q^{58}+51q^{62}+23q^{69}-57q^{71}-5q^{72}+15q^{73}-25q^{75}+35q^{78}+38q^{81}-45q^{82}+55q^{87}-69q^{92}-65q^{93}+27q^{94}+75q^{100}+O[q]^{101}$.

We calculate the Hecke character of weight 3 and modulus (1), i.e, we calculate $\Psi_{K,\Lambda} = \sum_{m\geq 1} a(m)q^m$, where $\Lambda = (1)$ and $k = 3$. When A of norm m is a nonprincipal ideal, A^3 is a principal ideal. Then we set $\phi(A)^3 = \phi(A^3)$. For example, $(2, -1/2+\sqrt{-23}/2)^3 = (-3/2-\sqrt{-23}/2)$. Because of

$$\phi\Big(\Big(\frac{-3-\sqrt{-23}}{2}\Big)\Big) = \Big(\frac{-3-\sqrt{-23}}{2}\Big)^2 = \frac{-7+3\sqrt{-23}}{2},$$

$\phi((2, -1/2+\sqrt{-23}/2))$ is one of the roots of

$$x^3 - \Big(\frac{-7+3\sqrt{-23}}{2}\Big) = 0. \tag{11}$$

We denote by α_1, α_2 and α_3 the roots of equation (11), namely, $\alpha_1 \sim -1.86272+0.728188i$, $\alpha_2 \sim 0.300733-1.97726i$ and $\alpha_3 \sim 1.56199+1.24907i$, respectively. Then, $\phi((2, -1/2+\sqrt{-23}/2))$ is one of α_1, α_2 or α_3. (Actually there are three different Hecke characters in this case.) First let us set $\phi((2, -1/2+\sqrt{-23}/2)) = \alpha_1$. By the equation $(2, -1/2+\sqrt{-23}/2)\times(2, 1/2+\sqrt{-23}/2) = (2)$,

$$\phi\Big(\Big(2, \frac{-1+\sqrt{-23}}{2}\Big)\Big) \times \phi\Big(\Big(2, \frac{1+\sqrt{-23}}{2}\Big)\Big) = \phi((2)).$$

We get

$$\alpha_1 \times \phi\Big(\Big(2, \frac{1+\sqrt{-23}}{2}\Big)\Big) = 4,$$

hence, $\phi((2, 1/2 + \sqrt{-23}/2)) = 4/\alpha_1$. So,

$$a(2) = \phi\Big(\Big(2, \frac{-1+\sqrt{-23}}{2}\Big)\Big) + \phi\Big(\Big(2, \frac{1+\sqrt{-23}}{2}\Big)\Big) = \alpha_1 + 4/\alpha_1.$$

By the equation $(2, -1/2+\sqrt{-23}/2) \times (3, 1/2-\sqrt{-23}/2) = (1/2-\sqrt{-23}/2)$,

$$\phi\Big(\Big(2, \frac{-1+\sqrt{-23}}{2}\Big)\Big) \times \phi\Big(\Big(3, \frac{1-\sqrt{-23}}{2}\Big)\Big) = \phi\Big(\Big(\frac{1-\sqrt{-23}}{2}\Big)\Big).$$

We get

$$\alpha_1 \times \phi\Big(\Big(3, \frac{1-\sqrt{-23}}{2}\Big)\Big) = \Big(\frac{1-\sqrt{-23}}{2}\Big)^2 = \frac{-11-\sqrt{-23}}{2},$$

hence, $\phi((3, 1/2-\sqrt{-23}/2)) = (-11-\sqrt{-23})/2 \times 1/\alpha_1$. Similarly, $\phi((3, -1/2-\sqrt{-23}/2)) = (-11+\sqrt{-23})/2 \times \alpha_1/(\alpha_1^2+4)$. So,

$$\begin{aligned} a(3) &= \phi\Big(\Big(3, \frac{1-\sqrt{-23}}{2}\Big)\Big) + \phi\Big(\Big(3, \frac{-1-\sqrt{-23}}{2}\Big)\Big) \\ &= \frac{-11-\sqrt{-23}}{2} \times \frac{1}{\alpha_1} + \frac{-11+\sqrt{-23}}{2} \times \frac{\alpha_1}{\alpha_1^2+4}. \end{aligned}$$

We recall $\alpha_1 \sim -1.86272 + 0.728188i$. Then, we obtain

$$\Psi^{(1)}_{K,\Lambda} = q - 3.72545q^2 + 4.24943q^3 + \cdots.$$

Actually, it is possible to continue this calculation, but we need the information on the basis of all the ideals, which is rather complicated. So, we determine the Hecke eigenforms $\Psi^{(i)}_{K,\Lambda}$ by a different method. By computer calculation (using "Sage" [18]), we know that the space of the modular forms of weight 3 where $\Psi_{K,\Lambda}$ belongs is of dimension 3. We can calculate the basis of this modular form explicitly, and the three basis elements are of the form:

$$\begin{aligned} &q + 4q^4 - 11q^6 - 7q^8 + 9q^9 + \cdots, \\ &q^2 - 5q^4 + 7q^6 + 4q^8 - 8q^9 + \cdots, \\ &q^3 - 3q^4 + 5q^6 - 7q^9 + \cdots. \end{aligned}$$

On the other hand, because of Theorems 3.1 and 3.2, $\Theta_{L_{\mathfrak{o}},P_1}$, $\Theta_{L_{\mathfrak{a}_1},P_1}$ and $\Theta_{L_{\mathfrak{a}_2},P_2}$ are in the same space of Hecke eigenforms $\Psi^{(i)}_{K,\Lambda}$. Therefore, comparing the first three coefficients of the following equation:

$$\Psi^{(1)}_{K,\Lambda}(q) = \frac{1}{2}\Theta_{L_{\mathfrak{o}},P}(q) + a2\Theta_{L_{\mathfrak{a}_1},P}(q) + b\frac{4}{\sqrt{23}}\Theta_{L_{\mathfrak{a}_2},P}(q),$$

we can find numbers a and b as follows:

$$(a,b) = \begin{cases} (A_1, B_2), \\ (A_2, B_1), \\ (A_3, B_3), \end{cases}$$

where A_1, A_2 and A_3 are the elements defined by

$$\{x \mid 512x^3 - 96x + 7 = 0\}$$
$$= \{A_1 = -0.465681, A_2 = 0.0751832, A_2 = 0.390498\},$$

respectively, and B_1, B_2 and B_3 are the elements defined by

$$\{x \mid 512x^3 - 2208x + 1587 = 0\}$$
$$= \{B_1 = -2.37065, B_2 = 0.873067, B_3 = 1.49759\},$$

respectively.

In this way, we can calculate the Hecke eigenforms $\Psi^{(i)}_{K,\Lambda}$. Namely,
$\Psi^{(1)}_{K,\Lambda} = q - 3.72545q^2 + 4.24943q^3 + 9.87897q^4 - 15.831q^6 - 21.9018q^8 + 9.05761q^9 + 41.9799q^{12} - 21.3624q^{13} + 42.0781q^{16} - 33.7437q^{18} - 23q^{23} - 93.07q^{24} + 25q^{25} + 79.5844q^{26} + 0.244826q^{27} + 55.473q^{29} - 33.9378q^{31} - 69.1528q^{32} + 89.4799q^{36} - 90.7777q^{39} - 8.78692q^{41} + 85.6853q^{46} + 42.8975q^{47} + 178.808q^{48} + 49q^{49} - 93.1362q^{50} + O[q]^{51}$.

$\Psi^{(2)}_{K,\Lambda} = q + 0.601466q^2 + 1.54364q^3 - 3.63824q^4 + 0.928445q^6 - 4.59414q^8 - 6.61718q^9 - 5.61612q^{12} + 23.5162q^{13} + 11.7897q^{16} - 3.98001q^{18} - 23q^{23} - 7.09168q^{24} + 25q^{25} + 14.1442q^{26} - 24.1073q^{27} - 42.4015q^{29} - 27.9663q^{31} + 25.4677q^{32} + 24.0749q^{36} + 36.3005q^{39} + 74.9986q^{41} - 13.8337q^{46} - 93.8839q^{47} + 18.1991q^{48} + 49q^{49} + 15.0366q^{50} + O[q]^{51}$.

$\Psi^{(3)}_{K,\Lambda} = q + 3.12398q^2 - 5.79306q^3 + 5.75927q^4 - 18.0974q^6 + 5.49593q^8 + 24.5596q^9 - 33.3638q^{12} - 2.15383q^{13} - 5.86788q^{16} + 76.7237q^{18} - 23q^{23} - 31.8383q^{24} + 25q^{25} - 6.72853q^{26} - 90.1376q^{27} - 13.0715q^{29} + 61.9041q^{31} - 40.3149q^{32} + 141.445q^{36} + 12.4773q^{39} - 66.2117q^{41} - 71.8516q^{46} + 50.9864q^{47} + 33.993q^{48} + 49q^{49} + 78.0996q^{50} + O[q]^{51}$.

The coefficients $a(m)$ for this case are far from integers. In fact they are not elements in a cyclotomic number field in general. So, it seems difficult to use the