SPRINGER BRIEFS IN MATHEMATICS

Kurt Luoto
Stefan Mykytiuk
Stephanie van Willigenburg

An Introduction to Quasisymmetric Schur Functions

Hopf Algebras, Quasisymmetric Functions, and Young Composition Tableaux

Springer

SpringerBriefs in Mathematics

SpringerBriefs in Mathematics showcases expositions in all areas of mathematics and applied mathematics. Manuscripts presenting new results or a single new result in a classical field, new field, or an emerging topic, applications, or bridges between new results and already published works, are encouraged. The series is intended for mathematicians and applied mathematicians.

For further volumes:
http://www.springer.com/series/10030

Kurt Luoto • Stefan Mykytiuk
Stephanie van Willigenburg

An Introduction to Quasisymmetric Schur Functions

Hopf Algebras, Quasisymmetric Functions, and Young Composition Tableaux

Springer

Kurt Luoto
Department of Mathematics
University of British Columbia
Vancouver, British Columbia
Canada

Stefan Mykytiuk
Department of Mathematics and Statistics
York University
Toronto, Ontario
Canada

Stephanie van Willigenburg
Department of Mathematics
University of British Columbia
Vancouver, British Columbia
Canada

ISSN 2191-8198 ISSN 2191-8201 (electronic)
ISBN 978-1-4614-7299-5 ISBN 978-1-4614-7300-8 (eBook)
DOI 10.1007/978-1-4614-7300-8
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013936258

Mathematics Subject Classification (2010): 05A05, 05E05, 05E15, 06A07, 16T05, 16T30

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

For Niall Christie and Madge Luoto

Preface

The history of quasisymmetric functions begins in 1972 with the thesis of Richard Stanley, followed by the formal definition of the Hopf algebra of quasisymmetric functions in 1984 by Ira Gessel. From this definition a whole research area grew and a more detailed, although not exhaustive, history can be found in the introduction.

The history of quasisymmetric Schur functions is far more contemporary. They were discovered in 2007 during the semester on "Recent Advances in Combinatorics" at the Centre de Recherches Mathématiques, and further progress was made at a variety of workshops at the Banff International Research Station and during an Alexander von Humboldt Foundation Fellowship awarded to Steph. The idea for writing this book came from encouragement by Adriano Garsia who suggested we recast quasisymmetric Schur functions using tableaux analogous to Young tableaux. We followed his words of wisdom.

The aim of this monograph is twofold. The first goal is to provide a reference text for the basic theory of Hopf algebras, in particular the Hopf algebras of symmetric, quasisymmetric and noncommutative symmetric functions and connections between them. The second goal is to give a survey of results with respect to an exciting new basis of the Hopf algebra of quasisymmetric functions, whose combinatorics is analogous to that of the renowned Schur functions.

In particular, after introducing the topic in Chapter 1, in Chapter 2 we review pertinent combinatorial concepts such as partially ordered sets, Young and reverse tableaux, and Schensted insertion. In Chapter 3 we give the basic theory of Hopf algebras, illustrating it with the Hopf algebras of symmetric, quasisymmetric and noncommutative symmetric functions, ending with a brief introduction to combinatorial Hopf algebras. The exposition is based on Stefan's thesis, useful personal notes made by Kurt, and a talk Steph gave entitled "Everything you wanted to know about Sym, QSym and NSym but were afraid to ask". Chapter 4 generalizes concepts from Chapter 2 such as Young tableaux and reverse tableaux indexed by partitions, to Young composition tableaux and reverse composition tableaux indexed by compositions. The final chapter then introduces two natural refinements for the Schur functions from Chapter 3: quasisymmetric Schur functions reliant on reverse composition tableaux and Young quasisymmetric Schur functions reliant on Young composition tableaux. This chapter concludes by discussing a number of results for these Schur function refinements and their dual bases. These results are analogous to those found in the theory of Schur functions such as the computation of Kostka numbers, and Pieri and Littlewood–Richardson rules. Throughout parallel construction is used so that analogies may easily be spotted even when browsing.

None of this would be possible without the support of a number of people, whom we would now like to thank. Firstly, Adriano Garsia has our sincere thanks for his ardent support of pursuing quasisymmetric Schur functions. We are also grateful to our advisors and mentors who introduced us to, and fuelled our enthusiasm for, quasisymmetric functions: Nantel Bergeron, Lou Billera, Sara Billey, Isabella Novik and Frank Sottile. This enthusiasm was sustained by our coauthors on our papers involving quasisymmetric Schur functions: Christine Bessenrodt, Jim Haglund and Sarah Mason, with whom it was such a pleasure to do research. We are also fortunate to have visited a variety of stimulating institutes to conduct our research and our thanks go to Francois Bergeron and a host of enthusiastic colleagues who arranged the aforementioned semester. Plus we are most grateful for the opportunities at Banff afforded to us by the director of BIRS, Nassif Ghoussoub, his team, the organizers of each of the meetings we attended and the participants all of whom gave us a stimulating and supportive atmosphere for us to pursue our goals. We are also grateful to the reviewers of this book, to Ole Warnaar, and to Moss Sweedler for their advice.

Our various universities, York University, the University of Washington and the University of British Columbia, are thanked for their support, in particular the A+E Combinatorics reading group at the latter: Omer Angel, Caleb Cheek, Andrew Rechnitzer, Tom Wong, and especially Ed Richmond and Vasu Tewari, both of whom kindly agreed to proofread this manuscript. No research is possible without funding, and we are grateful to be supported by the Natural Sciences and Engineering Research Council of Canada and the Alexander von Humboldt Foundation.

We would also like to thank Razia Amzad at Springer US for her help and support during the preparation of this manuscript, and our families and friends for all their love and support. Lastly, we would like to thank you, the reader, who we hope finds our book a rewarding read.

Vancouver, BC, Canada — Kurt Luoto
Toronto, ON, Canada — Stefan Mykytiuk
Vancouver, BC, Canada — Stephanie van Willigenburg

Notation

α	Composition
$\tilde{\alpha}$	Underlying partition of α
α^c	Complement of α
α^r	Reversal of α
α^t	Transpose of α
$\alpha/\!\!/_{\check{c}}\beta$	Skew shape
$\alpha/\!\!/_{\hat{c}}\beta$	Skew shape
col	Column sequence of a tableau
comp	Composition corresponding to a subset, or to a descent set of a tableau
cont	Content of a tableau
χ	Forgetful map
d	Descent set of a permutation
D	Descent set of a chain
Des	Descent set of a tableau
δ_{ij}	1 if $i = j$ and 0 otherwise
Δ	Coalgebra coproduct
e_λ	Elementary symmetric function
$\mathbf{e}_\alpha$	Elementary noncommutative symmetric function
F_α	Fundamental quasisymmetric function
$\mathscr{H}$	Hopf algebra
$\mathscr{H}^*$	Dual Hopf algebra
h_λ	Complete homogeneous symmetric function
$\mathbf{h}_\alpha$	Complete homogeneous noncommutative symmetric function
ℓ	Length of a composition or partition
$\mathscr{L}$	Set of linear extensions of a poset

$\mathscr{L}_{\check{c}}$	Reverse composition poset
$\mathscr{L}_{\hat{c}}$	Young composition poset
$\mathscr{L}_Y$	Young's lattice
λ	Partition
λ^t	Transpose of λ
λ/μ	Skew shape
m_λ	Monomial symmetric function
M_α	Monomial quasisymmetric function
$[n]$	The set $\{1,2,\ldots,n\}$
NSym	Hopf algebra of noncommutative symmetric functions
P	Poset
P^*	Dual poset
P	P-tableau of a list
QSym	Hopf algebra of quasisymmetric functions
r_α	Ribbon Schur function
$\mathbf{r}_\alpha$	Noncommutative ribbon Schur function
$\check{\rho}_\alpha$	Bijection between SSRCT and SSRT
$\hat{\rho}_\alpha$	Bijection between SSYCT and SSYT
S	Antipode
set	Set corresponding to a composition
sh	Shape of a tableau
s_λ	Schur function
$\check{\mathscr{S}}_\alpha$	Quasisymmetric Schur function
$\hat{\mathscr{S}}_\alpha$	Young quasisymmetric Schur function
$\check{\mathbf{s}}_\alpha$	Noncommutative Schur function
$\hat{\mathbf{s}}_\alpha$	Young noncommutative Schur function
SRT	Standard reverse tableau
$SRCT$	Standard reverse composition tableau
$SSRT$	Semistandard reverse tableau
$SSRCT$	Semistandard reverse composition tableau
Sym	Hopf algebra of symmetric functions
SYT	Standard Young tableau
$SYCT$	Standard Young composition tableau
$SSYT$	Semistandard Young tableau
$SSYCT$	Semistandard Young composition tableau
$\mathfrak{S}_n$	Symmetric group
$\check{T}$	Reverse tableau
T	Young tableau