Ronald L. Graham · Jaroslav Nešetřil
Steve Butler *Editors*

The Mathematics of Paul Erdős I

Second Edition

Springer

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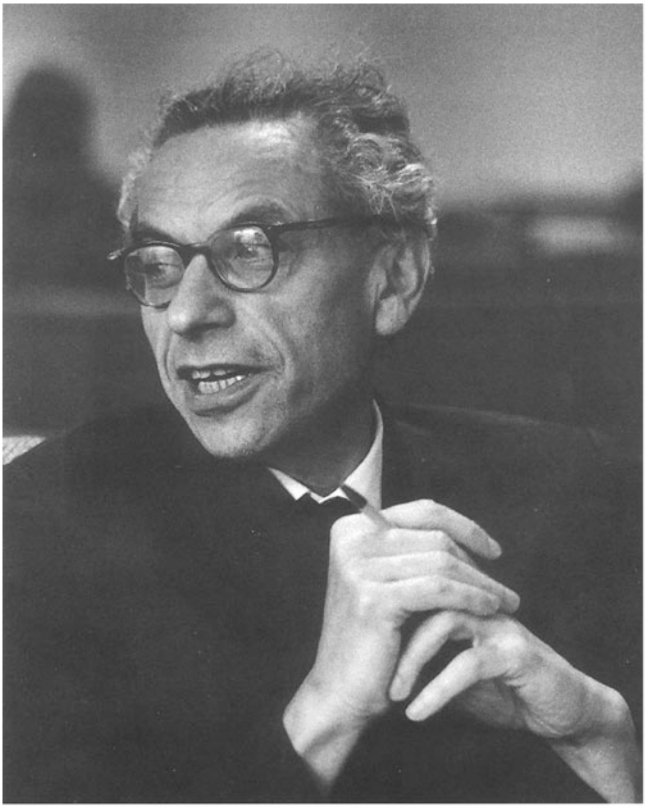

Photo by Adrian Bondy

Paul Erdős p g o m
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Editors

The Mathematics of Paul Erdős I

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ISBN 978-1-4614-7257-5 ISBN 978-1-4614-7258-2 (eBook)
DOI 10.1007/978-1-4614-7258-2
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013940428

Mathematics Subject Classification: 00-02, 01A70, 01A75, 00B15, 00B30, 11-02, 51-02, 52-02, 60-02, 60C99, 68R05, 68R10

Cover design: K. Marx

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface to the Second Edition

In 2013 the world mathematical community is celebrating the 100th anniversary of Paul Erdős' birth. His personality is remembered by many of his friends, former disciples, and over 500 coauthors, and his mathematics is as alive and well as if he was still among us. In 1995/1996 we were preparing the two volumes of The Mathematics of Paul Erdős not only as a tribute to the achievements of one of the great mathematicians of the twentieth century but also to display the full scope of his œuvre, the scientific activity which transcends individual disciplines and covers a large part of mathematics as we know it today. We did not want to produce just a "festschrift".

In 1995/1996 this was a reasonable thing to do since most people were aware of the (non-decreasing) Erdős activity only in their own particular area of research. For example, we combinatorialists somehow have a tendency to forget that the main activity of Erdős was number theory.

In the busy preparation of the volumes we did not realize that at the end, when published, our volumes could be regarded as a tribute, as one of many obituaries and personal recollections which flooded the scientific (and even mass) media. It had to be so; the old master left.

Why then do we think that the second edition should be published? Well, we believe that the quality of individual contributions in these volumes is unique, interesting and already partly historical (and irreplaceable—particularly in Part I of the first volume). Thus it should be updated and made available especially in this anniversary year. This we feel as our duty not only to our colleagues and authors but also to students and younger scientists who did not have a chance to meet the wandering scholar personally. We decided to prepare a second edition, asked our authors for updates and in a few instances we solicited new contributions in exciting new areas. The result is then a thoroughly edited volume which differs from the first edition in many places.

On this occasion we would like to thank all our authors for their time and work in preparing their articles and, in many cases, modifying and updating them. We are fortunate that we could add three new contributions: one by

Joel Spencer (in the way of personal introduction), one by Larry Guth in Part IV of the first volume devoted to geometry, and one by Alexander Razborov in Part I of the second volume devoted to extremal and Ramsey problems. We also wish to acknowledge the essential contributions of Steve Butler who assisted us during the preparation of this edition. In fact Steve's contributions were so decisive that we decided to add him as co-editor to these volumes. We also thank Kaitlin Leach (Springer) for her efficiency and support. With her presence at the SIAM Discrete Math. conference in Halifax, the whole project became more realistic.

However, we believe that these volumes deserve a little more contemplative introduction in several respects. The nearly 20 years since the first edition was prepared gives us a chance to see the mathematics of Paul Erdős in perspective. It is easy to say that his mathematics is alive; that may sound cliché. But this is in fact an understatement for it seems that Erdős' mathematics is flourishing. How much it changed since 1995 when the first edition was being prepared. How much it changed in the wealth of results, new directions and open problems. Many new important results have been obtained since then. To name just a few: the distinct distances problem, various bounds for Ramsey numbers, various extremal problems, the empty convex 6-gon problem, packing and covering problems, sum-product phenomena, geometric incidence problems, etc. Many of these are covered by articles of this volumes and many of these results relate directly or indirectly to problems, results and conjectures of Erdős. Perhaps it is not as active a business any more to solve a particular Erdős problem. After all, the remaining unsolved problems from his legacy tend to be the harder ones. However, many papers quote his work and in a broader sense can be traced to him.

There may be more than meets the eye here. More and more we see that the Erdős problems are attacked and sometimes solved by means of tools that are not purely combinatorial or elementary, and which originate in the other areas of mathematics. And not only that, these connections and applications merge to new theories which are investigated on their own and some of which belong to very active areas of contemporary mathematics. As if the hard problems inspire the development of new tools which then became a coherent group of results that may be called theories. This phenomenon is known to most professionals and was nicely described by Tim Gowers as two cultures. [W. T. Gowers, *The two cultures of mathematics*, in Mathematics: Frontiers and Perspectives (Amer. Math. Soc., Providence, RI, 2000), 65–78.] On one side, problem solvers, on the other side, theory builders. Erdős' mathematics seems to be on one side. But perhaps this is misleading. As an example, see the article in the first volume *Unexpected applications of polynomials in combinatorics* by Larry Guth and the article in the second volume *Flag algebras: an interim report* by Alexander Razborov for a wealth of theory and structural richness. Perhaps, on the top level of selecting problems and with persistent activity in solving them, the difference between the two sides becomes less clear. (Good) mathematics presents a whole.

Time will tell. Perhaps one day we shall see Paul Erdős not as a theory builder but as a man whose problems inspired a wealth of theories.

People outside of mathematics might think of our field as a collection of old tricks. The second edition of mathematics of Paul Erdős is a good opportunity to see how wrong this popular perception of mathematics is.

La Jolla, USA R.L. Graham
Prague, Czech Republic J. Nešetřil

Just a few lines to
remember Szeged restaurant
in Budapest on July 23/96
Paul Erdős

[illegible]

Laci Lovász

Walter Deuber

[illegible]

Noga Alon

Moshe Rosenfeld

Miki Simonovits

[illegible]

NURIT ALON (NOGA'S BOSS)

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Rob Tijdeman

Berrie Tijdeman

Vera T. Sós

Vesztergombi Kati

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IN MEMORIAM

Paul Erdős

26.3.1913–20.9.1996

The week before these volumes were scheduled to go to press, we learned that Paul Erdős died on September 20, 1996. He was 83. Paul died while attending a conference in Warsaw, on his way to another meeting. In this respect, this is the way he wanted to "leave". In fact, the list of his last month's activities alone inspires envy in much younger people.

Paul was present when the completion of this project was celebrated by an elegant dinner in Budapest for some of the authors, editors and Springer representatives attending the European Mathematical Congress. He was especially pleased to see the first copies of these volumes and was perhaps surprised (as were the editors) by the actual size and impact of the collection (On the opposite page is the collection of signatures from those present at the dinner, taken from the inside cover of the mock-up for these volumes). We hope that these volumes will provide a source of inspiration as well as a last tribute to one of the great mathematicians of our time. And because of the unique lifestyle of Paul Erdős, a style which did not distinguish between life and mathematics, this is perhaps a unique document of our times as well.

R.L. Graham
J. Nešetřil

Preface to the First Edition

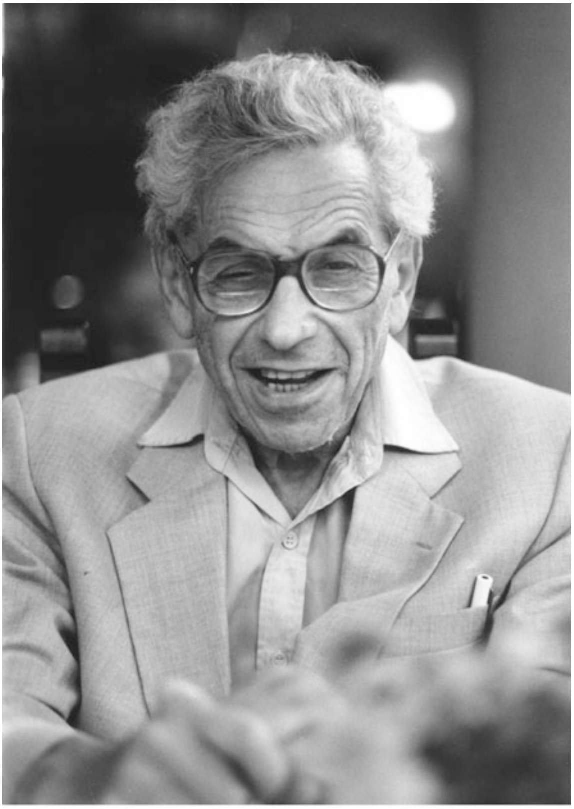
In 1992, when Paul Erdős was awarded a Doctor Honoris Causa by Charles University in Prague, a small conference was held, bringing together a distinguished group of researchers with interests spanning a variety of fields related to Erdős' own work. At that gathering, the idea occurred to several of us that it might be quite appropriate at this point in Erdős' career to solicit a collection of articles illustrating various aspects of Erdős' mathematical life and work. The response to our solicitation was immediate and overwhelming, and these volumes are the result.

Regarding the organization, we found it convenient to arrange the papers into six chapters, each mirroring Erdős' holistic approach to mathematics. Our goal was not merely a (random) collection of papers but rather a thoroughly edited volume composed in large part by articles explicitly solicited to illustrate interesting aspects of Erdős and his life and work. Each chapter includes an introduction which often presents a sample of related Erdős' problems "in his own words". All these (sometimes lengthy) introductions were written jointly by editors.

We wish to thank the nearly 70 contributors for their outstanding efforts (and their patience). In particular, we are grateful to Béla Bollobás for his extensive documentation of Paul Erdős' early years and mathematical high points; our other authors are acknowledged in their respective chapters. We also want to thank A. Bondy, G. Hahn, I. Ouhel, K. Marx, J. Načeradský and Ché Graham for their help and for the use of their works. At various stages of the project, the book was supported by AT&T Bell Laboratories, GAČR 2167 and GAUK 351. We also are indebted to Dr. Joachim Heinze and Springer Verlag for their encouragement and support. Finally, we would like to record our extreme debt to Susan Pope (at AT&T Bell Laboratories) who somehow (miraculously) managed to convert more than 50 manuscripts of all types into the attractive form they now have.

Here then is a unique portrait of a man who has devoted his whole being to "proving and conjecturing" and to the pursuit of mathematical knowledge and understanding. We hope that this will form a lasting tribute to one of the great mathematicians of our time.

Murray Hill, USA R.L. Graham
Praha, Czech Republic J. Nešetřil



Paul Erdős. Photo by George Csicsery.

Paul Erdős on the Queen Elizabeth.
Photo by Ronald Taft.

Paul Erdős in the mountains.

Paul Erdős in 1958.

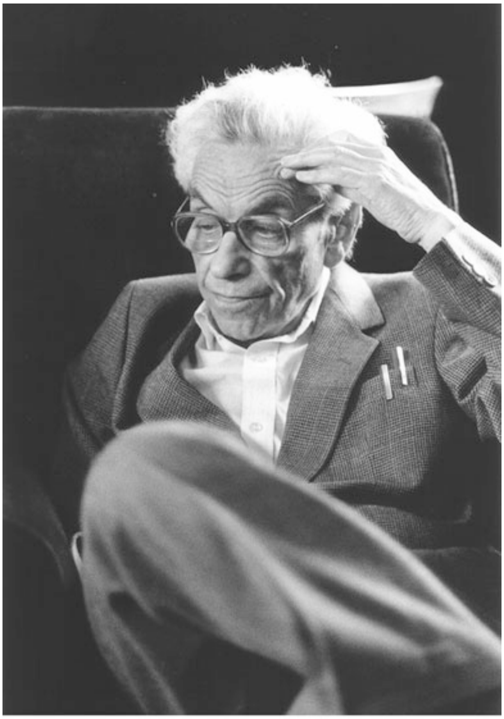
Paul Erdős with Béla Bollobás. Photo by George Csicsery.

Paul Erdős with Vera Sós.

Paul Erdős with Richard Rado.

Paul Erdős with epsilons.

Paul Erdős with Mel Nathanson.



Paul Erdős. Photo by George Csicsery.

Paul Erdős in 1941.

Paul Erdős in the 1950s.

Paul Erdős with Leo Moser.

Paul Erdős receiving a Dr.h.c. degree from Charles University. Photo by Gena Hahn.

Contents

Paul Erdős: Life and Work

Béla Bollobás

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Dipping into the mathematical papers of Paul Erdős is like wandering into *Aladdin's Cave*. The beauty, the variety and the sheer wealth of all that one finds is quite overwhelming. There are fundamental papers on number theory, probability theory, real analysis, approximation theory, geometry, set theory and, especially, combinatorics. These great contributions to mathematics span over six decades; Erdős and his collaborators have left an indelible mark on the mathematics of the twentieth-century. The areas of probabilistic number theory, partition calculus for infinite cardinals, extremal combinatorics, and the theory of random graphs have all practically been created by Erdős, and no-one has done more to develop and promote the use of probabilistic methods throughout mathematics.

Erdős is the mathematician *par excellence*: he thrives on mathematics, living in a state of continuous excitement; he raises, answers and communicates questions, picking up the problems of others and making incisive contributions to them with lightning speed.

Considering what a mild-mannered man he is, it is surprising that everything about Erdős and his mathematics is extreme. He has written over 1,400 papers, more than any mathematician since Euler, and has more than 400 coauthors. If the *Guinness Book of Records* had categories related to mathematical activities, Paul Erdős would hold many of the records by a margin one could not even attempt to estimate, like the thousands of problems posed, the millions of miles travelled, the tens of thousands of mathematical discussions held, the thousands of different beds slept in, the thousands of lectures delivered at different universities, the hundreds of mathematicians helped, and so on.

Today we live in the age of big mathematical theories, bringing together many sophisticated branches of mathematics. These powerful theories can be very successful in solving down-to-earth problems, as in the case of Andrew Wiles's wonderful proof of *Fermat's Last Theorem*. But no matter how important and valuable these big theories are, they cannot constitute all of mathematics. There are a remarkable number of basic mathematical questions that we would love to answer (nay, we *should* answer!) which seem

R.L. Graham et al. (eds.), *The Mathematics of Paul Erdős I*,
DOI 10.1007/978-1-4614-7258-2_1,

to withstand all our assaults. There is a danger that we turn our backs on such questions, persuading ourselves that they are not interesting when in fact we mean only that we cannot tackle them with our favourite theories. Of course, such an attitude would not be in the proper spirit of science; surely, we should say that we do want to answer these questions, by *whatever means*. And if there are no theories to help us, no bulldozers to move the earth, then we must rely on our bare hands and ingenuity. It is not that we do not want to use big theories to crack our problems, but that the big theories around are unable to say anything deep about our questions. And, with luck, our hands-on approach will tie up with available theories or, better still, will lead to new, more sensitive theories.

Ernst Straus, who as a young man was Einstein's assistant, reported that the reason why Einstein had chosen physics over mathematics was that mathematics was so full of beautiful and attractive questions that one might easily waste one's life working on the "wrong" questions. Einstein was confident that in physics he could identify the "central" questions, and he felt that it was the duty of a scientist to pursue these questions and not let himself be seduced by any problem-no matter how difficult or attractive it might be.

The philosophy of Erdős has been completely different. Throughout his long career, he has been happy to pursue the beautiful problems he encountered, and has raised many others. But this is not an ad hoc process: Erdős has an amazing instinct for discerning beautiful problems that, while appearing innocuous, in fact go right to the heart of the matter. These problems are not chosen indiscriminately; they frequently lead to the discovery of unexpected and exciting phenomena. Like Ramanujan, Erdős uses particular instances of problems to explore an area. Rather than taking whole countries in one sweeping move, he prefers first to occupy some nearby castles, from which he can weigh up the unknown territory before making his next move.

For over 60 years now, Erdős has been the world's most celebrated problem solver and problem poser. Unrivalled, king, nonpareil, ... He has been called an occidental Ramanujan, a modern-day Euler, the Mozart of mathematics. These glowing epithets accurately capture the different facets of Paul Erdős—each is correct in its own way. He has a unique talent to pose penetrating questions. It is easy to ask questions that lead nowhere, questions that are either impossibly hard or too easy. It is a completely different matter to raise, as Erdős does, innocent-looking problems whose solutions shed light on the shape of the mathematical landscape.

An important feature of the problems posed by Erdős is that they carry differing monetary rewards. Needless to say, this is done in jest, but the prizes do indicate Erdős's assessment of the difficulty of the problems. How different this is from the annoying habit of some mathematicians, who casually mention a problem as if they hadn't even thought about it, when in fact they are telling you the central problem they have been working on for a long time!

Two features of his mathematical *œuvre* stand out: his mastery of *elementary* methods and his advocacy of *random* methods. Starting with his very first papers, Erdős championed *elementary* methods in diverse branches of mathematics. He showed, again and again, that elementary methods often succeed against overwhelming odds. In many brilliant proofs he showed that, rather than bringing somewhat foreign machinery to bear on some problems, and thereby trying to fit a square peg into a round hole, one can progress considerably further by facing the complications, going deep into the problem, and tailoring our approach to the intrinsic difficulties of the problem. This philosophy can pay unexpected dividends, as shown by Charles Read's solution of the Invariant Subspace Problem, Miklós Laczkovich's solution of Tarski's problem of "Squaring the Circle", and Tim Gowers' recent solutions of Banach's last unsolved problems, including the Hyperplane Problem.

As to *probabilistic* methods, by now it is widely acknowledged that these can be remarkably effective in tackling main-line questions in diverse areas of mathematics that have nothing to do with probability. It is worth remembering, though, that when Erdős started it all, the idea was very startling indeed. That today we take it in our stride is a sign of the tremendous success of the random method, which is very much his method, still frequently called the *Erdős method*.

Paul Erdős was born on 26th March 1913, in Budapest. His parents were teachers of mathematics and physics; his father translated a book on aircraft design from English into Hungarian. The young Paul did not go to elementary school, but was brought up by his devoted mother, Anna, and, for 3 years, between the ages of 3 and 6, he had a German *Fräulein*. His exceptional talent for mathematics was evident by the time he was 3: his agility at mental arithmetic impressed all comers, and he was not yet 4 when he discovered negative numbers for himself. With the outbreak of the First World War, his father was drafted into the Austro-Hungarian army, and served on the Eastern Front. He was taken prisoner by the Russians, and sent to Siberia to a prisoner of war camp, from which he returned only after about 6 years.

After the unconditional surrender of Hungary at the end of the War, the elected government resigned, as it could not accept the terms of the Allies. These terms left Hungary only the rump of her territory, and in March 1919 the communists took over the country, with the explicit aim of repelling the Allies. The communists formed a *Dictatorship of the Proletariat*, usually referred to as the *Commune*, after its French equivalent in 1871, and set about defending the territory and forcibly reforming the social order.

The Commune could not resist the invasion by the Allies and the Hungarian "white" officers under *Admiral Horlhy*, and it fell after a struggle of 3 months. Unfortunately for the Erdős family, Anna Erdős had a minor post under the Commune, and when Horthy came to power, she lost her job, never to teach again. Later she worked as a technical editor.

He studied elementary school privately with his mother. After that, in 1922, the young Erdős went to Tavaszmező gymnasium, the first year as a private pupil, the second and third years as a normal student, and the fourth year again as a private pupil. After the fourth year he attended St. Stephen's School (Szent István Gimnázium) where his father was a high school teacher. At this time Erdős also received significant instruction from his parents as well. As it happened, my father entered the school just as Erdős left it, so they share many classmates, although they met only many years later.

By the early 1920s the *Mathematical Journal for Secondary Schools* (Középiskolai Matematikai Lapok) was a successful journal, catering for pupils with talent for mathematics. The journal had been founded in 1895 by a visionary young man, Dániel Arany, who hoped to raise the level of mathematics in the whole of Hungary by enticing students to mathematics through beautiful problems. The backbone of the journal was the year-long competition. Every month a number of problems were set for each age group; the readers were invited to submit their solutions, which were marked, and the best published under the names of the authors.

The young Erdős became an ardent reader of this journal, and his love of mathematics was greatly fanned by the intriguing problems in it. In some sense, Erdős's earliest publications date to this time, with the appearance of his solutions in the journal. On one occasion Paul Erdős and Paul Turán were the only ones who managed to solve a particular problem, and their solution was published under their joint names. This was Erdős's first "joint paper" with Turán, whom he had not even met at the time, and who later became one of his closest friends and most important collaborators.

Mathematicians, and especially young mathematicians, learn much from each other. Erdős was very lucky in this respect, for when at the age of 17 he entered the Pázmány Péter Tudományegyetem (the science university of Budapest) he found there an excellent group of about a dozen youngsters devoted to mathematics. Not surprisingly, Erdős became the focal point of this group, but the long mathematical discussions stimulated him greatly.

This little group included Paul Turán, the outstanding number theorist; Tibor Gallai, the excellent combinatorialist; Dezső Lázár, who was later tragically killed by the Nazis; George Szekeres and Esther Klein, who later married and subsequently emigrated to Australia; László Alpár, who became an important member of the Hungarian Mathematical Institute; Márta Svéd, another member of the group who went to live in Australia; and several others. Not only did they discuss mathematics at the university, but also in the afternoons and evenings, when they used to meet at various public places, especially by the *Statue of Anonymous*, commemorating the first chronicler of Hungarian history.

Two of Erdős's professors stand out: Lipót Fejér, the great analyst, and Dénes König, who introduced Erdős to graph theory. The lectures of König led to the first results of Erdős in graph theory: in answer to a question posed in the lectures, in 1931 he extended Menger's theorem to infinite graphs. Erdős never published his proof, but it was reproduced in König's classic, published in 1936.

As an undergraduate, Erdős worked mostly on number theory, obtaining several substantial results. He was not even 20, when in Berlin the great Issai Schur lectured on Erdős's new proof of Bertrand's postulate. He wrote his doctoral dissertation as a second year undergraduate, and it was not long before he got into correspondence with several mathematicians in England, including Louis Mordell, the great number theorist in Manchester, and Richard Rado and Harold Davenport in Cambridge. All three became Erdős's close friends.

When in 1934 Erdős finished university, he accepted Mordell's invitation to Manchester. He left Hungary for England in the autumn of 1934, not knowing that he would never again live in Hungary permanently. On 1st October 1934 he was met at the railway station in Cambridge by Davenport and Rado, who took him to Trinity College, and they immediately embarked on the first of their many long mathematical discussions. Next day Erdős met Hardy and Littlewood, the giants of English mathematics, before hurrying on to Mordell.

Mordell put together an amazing group of mathematicians in Manchester, and Erdős was delighted to join them. First he took up the *Bishop Harvey Goodwin Fellowship*, and was later awarded a *Royal Society Fellowship*. He was free to do research under Mordell's guidance, and he was soon producing papers with astonishing rapidity. In 1937 Davenport left Cambridge to join Mordell and Erdős, and their life-long friendship was soon cemented. I have a special reason to be grateful for the Erdős-Davenport friendship: many years later, I was directed to my present home, Trinity College, Cambridge, only because Davenport was a Fellow here, and he was a good friend of Erdős.

In 1938 Erdős was offered a fellowship at the Institute for Advanced Study in Princeton, so he soon thereafter sailed for the U.S., where he was to spend the next decade. The war years were rather hard on Erdős, as it was not easy to hear from his parents in Budapest, and when he received news, it was never good. His father died in August 1942, his mother later had to move to the Ghetto in Budapest, and his grandmother died in 1944. Many of his relatives were murdered by the Nazis.

In spite of being cut off from his home, Erdős continued to pour forth wonderful mathematics at a prodigious rate. Having arrived in America, he spent a year and a half at Princeton, before starting on his travels. He visited Philadelphia, Purdue, Notre Dame, Stanford, Syracuse, Johns Hopkins, to mention but a few places, and the pattern was set: like a Wandering Scholar of the Middle Ages, Erdős never stopped again. In addition to the many important papers he wrote by himself, he collaborated more and more with mathematicians from diverse areas, writing outstanding joint papers with Mark Kac, Wintner, Kai Lai Chung, Ivan Niven, Arye Dvoretzky, Shizuo Kakutani, Arthur A. Stone, Leon Alaoglu, Irving Kaplansky, Alfred Tarski, Gabor Szegő, William Feller, Fritz Herzog, George Piranian, and others. Through correspondence, he continued his collaboration with Paul Turán, Harold Davenport, Chao Ko and Tibor Gallai (Grünwald).

In 1954, he left the U.S. to attend the International Congress of Mathematicians in Amsterdam. He had also asked for a reentry permit at that time but his request was denied. So he left without a reentry permit since in his own words, "Neither sam nor joe can restrict my right to travel." Left without a country, Israel came to his aid, offering him employment at the Hebrew University in Jerusalem, and a passport. He arrived in Israel on 30th November 1954, and from then on he has been to Israel practically every year. Before leaving Israel for Europe in July 1955, he applied for a return visa to Israel. When the officials asked him whether he wanted to become an Israeli citizen, he politely refused, saying that he did not believe in citizenship.

After the upheaval following his trip to Amsterdam, he first returned to the U.S. in 1959; the relationship between Erdős and the U.S. Immigration Department was finally normalized in 1963, and since then he has had no problems with them.

In the *Treaty of Yalta*, Hungary was placed within the Soviet sphere of influence; the communists, aided by the Russians, took over the government, and turned Hungary into a *People's Republic*. For ordinary Hungarians, leaving Hungary even for short trips to the West became very difficult. Nevertheless, in 1955 Erdős managed to return to Hungary for a short time, when his good friend, George Alexits, pulled strings and convinced the officials that, if Erdős were to enter the country, he should be allowed to leave.

Later Erdős could return to Hungary at frequent intervals, in order to spend more and more time with his mother, as well as to collaborate with a large number of Hungarian mathematicians, especially Turán and Rényi, In those dark days, Erdős was *the* main link between many Hungarian mathematicians and the West.

As a young pupil, I first heard him lecture during one of his visits: not only did he talk about fascinating problems but he also cut a flamboyant figure, with his suntan, Western suit and casual mention of countries I was sure I could never visit. I got to know him during his next visit: in 1958, having won the National Competition, I was summoned to the elegant hotel he stayed in with his mother. They could not have been kinder: Erdős told to me a host of intriguing questions, and did not talk down to me, while his mother (whom, as most of their friends, I learned to call *Annus Néni* or Aunt Anna) treated me to cakes, ice cream and drinks. Three years later they got to know my parents, and from then on they were frequent visitors to our house, especially for Sunday lunches. My father, who was a physician, looked after both Erdős and Annus Néni.

Seeing them together, there was no doubt that they were very happy in each other's company: these were blissful days for both of them. Erdős thoroughly enjoyed being with his mother, and she was delighted to have her son back for a while. They looked after each other lovingly; each worried whether the other ate well and slept enough or, perhaps, was a little tired.

Annus Néni was fiercely proud of her wonderful son, loved to see the many signs that her son was a great mathematician, and revelled in her role as the *Queen Mother of Mathematics*, surrounded by all the admirers and well-wishers. She was never far from Erdős's mathematics either: she kept Erdős's hundreds of reprints in perfect order, sending people copies on demand.

Annus Néni was not young, having been born in 1880, but her health was good and she was very sharp. To compensate for the many years when they had been kept apart, Annus Néni started to travel with her son in her 80s; their first trip together being to Israel in November 1964. From then on they travelled much together: to England in 1965, many times to other European countries and the U.S., and towards the end of 1968 to Australia and Hawaii. When, tinged with envy, we told her that it must be wonderful to see the world, she replied "*You know that I don't travel because I like it but to be with my son.*" It was a tragedy for Erdős when, in 1971, Annus Néni died during a trip to Calgary. Her death devastated him and for years afterwards he was not quite himself. He still hasn't recovered from the blow, and it is most unlikely that he ever will.

Erdős's brushes with officialdom were not quite over: the communists also managed to upset him. In 1973 there was an international meeting in Hungary, to celebrate his 60th birthday. Erdős's friends from Israel were denied a visa to enter Hungary; this outraged him so much that for 3 years he did not return to Hungary.

With the collapse of communism and with the end of the Cold War, Erdős has entered a golden age of travel: not only can he go freely wherever he wants to, but he is even welcomed by officials everywhere.

Having started as a mathematical *prodigy*, by now Erdős is the *doyen* of mathematicians, with more friends in mathematics than the number of people most of us meet in a lifetime. As he likes to put it in his inimitable way, he has progressed from *prodigy to dotigy.* As a Member of the Hungarian Academy of Sciences, Erdős has a permanent position in Budapest. During summer months, he frequently stays in the Guest House of the Academy, two doors away from my mother, visiting Vera Sós, András Hajnal, Miklós Simonovits, András Sárközy, Miklós Laczkovich, and inspiring many others. Another permanent position awaits him in Memphis, where he stays and works with Ralph Faudree, and his other friends, Dick Schelp and Cecil Rousseau. In Israel he visits all the universities, including the Technion in Haifa, Tel Aviv, Jerusalem and the Weizman Institute. But for years now, Erdős has had many other permanent ports of call, including Kalamazoo, where Yousef Alavi looks after him; New Jersey and the New York area, where he stays with Ron Graham and Fan Chung and talks to many others as well, including János Pach, Joel Spencer, Mel Nathanson, Peter Winkler, Endre Szemerédi, Joseph Beck and Herb Wilf; Calgary, mostly because of Eric Milner, Richard Guy and Norbert Sauer; Atlanta, with Dick Duke, Vojtěch Rödl, Ron Gould and Dwight Duffus. And the list could go on and on, with Athens, Baton Rouge, Berlin, Bielefeld, Boca Raton, Bonn, Boston, Cambridge, Chicago, Los Angeles, Lyon, Minneapolis, Paris, Poznań, Prague, Urbana, Warsaw, Waterloo, and many others.

Honours have been heaped upon Erdős, although he could not care less. Every fifth year there is an International Conference in Cambridge on his birthday, and in 1991 Cambridge also awarded him a prestigious Honorary Doctorate, as did the Charles University of Prague a year later, and many other universities since. On the occasion of his 80th birthday, he was honoured at a spate of conferences, not only in Cambridge, but also in Kalamazoo, Boca Raton, Prague and Keszthély.

Nowadays Erdős lectures in more places than ever, interspersing his mathematical problems with stories about mathematicians and his remarks about life. He dislikes cold but, above all, hates old age and stupidity, and so he appreciates the languages in which these evils sound similar. Thus, *old* and *cold* and *alt* and *kalt* go hand in hand in English and German, and in no other language he knows. But Hindi is better still because the two greatest evils sound almost the same: *buddha* is old and *budu* is stupid.

Erdős is fond of paraphrasing poems, especially Hungarian poems, to illustrate various points. The great Hungarian poet at the beginning of this century, Endre Ady, Wrote: *Legyen átkozott aki a helyembe áll! (Let him be cursed who takes my place!)* As a mathematician builds the work of others, so that his immortality depends on those who continue his work, Erdős professes the opposite: *Let him be blessed who takes my place!*

But Erdős does not wait for posterity to find people to continue his work: his extraordinary number of collaborators ensures that many people carryon his work all around the world. The collaborators who particularly stand out are Paul Turán, Harold Davenport, Richard Rado, Mark Kac, Alfréd Rényi, András Hajnal, András Sarközy, Vera Sós and Ron Graham: they have all done much major work with Erdős. In a moment we shall see a brief account of some of this work. Needless to say, our review of Erdős's mathematics will be woefully brief and inadequate, and will also reflect the taste of the reviewer.

Erdős wrote his first paper as a first-year undergraduate, on Bertrand's postulate that, *for every* $n \geq 1$*, there is a prime* p *satisfying* $n < p \leq 2n$. Bertrand's postulate was first proved by Chebyshev, but the original proof was rather involved, and in 1919 Ramanujan gave a considerably simpler proof of it. In his fundamental book, *Vorlesungen über Zahlentheorie*, published in Leipzig in 1927, Landau gave a rather simple proof of the assertion that for some $q > 1$ and every $n \geq 1$, there is a prime between n and qn. However, Landau's q could not be taken to be 2. In his first paper, Erdős sharpened Landau's argument, and by studying the prime factors of the binomial coefficient $\binom{2a}{a}$, gave a simple and elementary proof of Bertrand's postulate.

Erdős was quick to develop further the ideas in his first paper. In 1932, Breusch made use of L-functions to generalize Bertrand's postulate to the arithmetic progressions $3n+1$, $3n+2$, $4n+1$ and $4n+3$: for every $m \geq 7$ there are primes of the form $3n+1$, $3n+2$, $4n+1$ and $4n+3$ between m and $2m$. By constructing expressions containing, as factors, all terms of the arithmetic progression at hand, and rather few other factors, Erdős managed to give an elementary proof of Breusch's theorem, together with various extensions of it to other arithmetic progressions. These results constituted the Ph.D. thesis Erdős wrote as a second-year undergraduate, and published in Sárospatak in 1934.

Schur, who had been Breusch's supervisor in Berlin, was quick to recognize the genius of the author of the beautiful elementary proof of Breusch's theorem. When, a little later, Erdős proved a conjecture of Schur on abundant numbers, and solved another problem of Schur, Erdős became "der Zauberer von Budapest" ("the magician of Budapest")—no small praise from the great German for a young man of 20.

Abundant numbers figured prominently among the early problems tackled by Erdős. In his lectures on number theory, Schur conjectured that the abundant numbers have positive density: $\lim_{x\to\infty} A(x)/x$ exists, where $A(x)$ is the number of abundant numbers not exceeding x. (A natural number n is *abundant* if $\sigma(n)$, the sum of its positive divisors, is at least $2n$.) The beautiful elementary proof Erdős gave of this conjecture led him straight to other problems concerning the distribution of the values of real-valued additive arithmetical functions $f(n)$, that is functions $f : \mathbb{N} \to \mathbb{R}$ satisfying $f(ab) = f(a) + f(b)$ whenever $(a,b) = 1$.

These problems were first investigated by Hardy and Ramanujan in 1917, but were more or less forgotten for over a decade. As eventually proved by Erdős and Wintner in 1939, a real-valued additive arithmetical function $f(n)$ behaves rather well if the following three series are convergent:

$$\sum_{|f(p)|>1} 1/p, \qquad \sum_{|f(p)|\le 1} f(p)/p \quad \text{and} \quad \sum_{|f(p)|\le 1} f(p)^2/p,$$

with the summations over primes p. To be precise, the three series above are convergent if and only if $\lim_{x\to\infty} A(x)/x$ exists for every real c, where $A_c(x)$ stands for the number of natural numbers $n \le x$ with $f(n) < c$.

In 1934, Turán gave a marvelous proof of an extension of the Hardy-Ramanujan theorem on the "typical number of divisors" of a natural number. Writing $\nu(n)$ for the number of *distinct* prime factors of n (so that $\nu(12) = 2$), Turán proved that

$$\sum_{n=1}^{N} \{\nu(n) - \log\log n\}^2 = N \log\log N + o(N \log\log N).$$

It is a little disappointing that Hardy, one of the greatest mathematicians alive, failed to recognize the immense significance of this new proof. Erdős, on the other hand, not only saw the significance of the paper, but was quick to make use of the probabilistic approach and so became instrumental in the birth of a very fruitful new branch of mathematics, *probabilistic number theory*. In a ground-breaking joint paper he wrote with Kac in 1939, Erdős proved that if a bounded real-valued arithmetical function $f(n)$ satisfies $\sum_p f(p)^2/p = \infty$ then, for every $x \in \mathbb{R}$,

$$\lim_{m\to\infty} A_x(m)/m = \Phi(x),$$

where $A_x(m)$ is the number of positive integers $n \le m$ satisfying

$$f(n) < \sum_{p\le m} f(p)/p + x\Bigl(\sum_{p\le m} f(p)^2/p\Bigr)^{1/2},$$

and, as usual

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$$

is the *standard normal distribution*. In other words, the arithmetical function $f(n)$ satisfies the Gaussian law of error! It took the mathematical community quite a while to appreciate the significance and potential of results of this type.

Note that for $\nu(n)$, the number of prime factors of n, the Erdős-Kac theorem says that if $x \in \mathbb{R}$ is fixed then

$$\lim_{m\to\infty} \frac{1}{m}\bigl|\{n \le m \text{ and } \nu(n) \le \log\log m + x(\log\log m)^{1/2}\}\bigr| = \Phi(x).$$

Starting with his very first papers, Erdős championed "*elementary*" methods in number theory. That the number theorists in the 1930s appreciated elementary methods was due, to some extent, to Shnirelman's great success in studying integer sequences, with a view of attacking, perhaps, the Goldbach conjecture. To study integer sequences, Shnirelman introduced a density, now bearing his name: an integer sequence $0 \le a_1, < a_2 < \dots$ is said to have *Shnirelman density* α if

$$\inf_{x\ge 1} \frac{1}{x}\sum_{a_n\le x} 1 = \alpha.$$

Thus if $a_1 > 1$ then the Shnirelman density of the sequence $(a_n)_{n=1}^{\infty}$ is 0.

Khintchine discovered the rather surprising fact that if $(a_n)_{n=1}^{\infty}$ is an integer sequence of Shnirelman density α with $0 < \alpha < 1$, and $(b_n)_{n=1}^{\infty}$ is the sequence of squares $0^2, 1^2,\ 2^2, \dots$, then the "sum-sequence" $(a_n + b_m)$ has Shnirelman density *strictly greater* than α. The original proof of this result, although elementary, was rather involved.

When Landau lectured on Khintchine's theorem in 1935 in Cambridge, he presented a somewhat simplified proof he had found with Buchstab. Nevertheless, talking to Landau after his lecture, Erdős expressed his view that the proof should be considerably simpler and, to Landau's astonishment, as soon as the next day he came up with a "proper" proof that was both