Ronald L. Graham · Jaroslav Nešetřil
Steve Butler *Editors*

The Mathematics of Paul Erdős II

Second Edition

Springer

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Paul Erdős Multi(media)
by Jiří Načeradský and Jarik Nešetřil.

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Preface to the Second Edition

In 2013 the world mathematical community is celebrating the 100th anniversary of Paul Erdős' birth. His personality is remembered by many of his friends, former disciples, and over 500 coauthors, and his mathematics is as alive and well as if he was still among us. In 1995/1996 we were preparing the two volumes of The Mathematics of Paul Erdős not only as a tribute to the achievements of one of the great mathematicians of the twentieth century but also to display the full scope of his œuvre, the scientific activity which transcends individual disciplines and covers a large part of mathematics as we know it today. We did not want to produce just a "festschrift".

In 1995/1996 this was a reasonable thing to do since most people were aware of the (non-decreasing) Erdős activity only in their own particular area of research. For example, we combinatorialists somehow have a tendency to forget that the main activity of Erdős was number theory.

In the busy preparation of the volumes we did not realize that at the end, when published, our volumes could be regarded as a tribute, as one of many obituaries and personal recollections which flooded the scientific (and even mass) media. It had to be so; the old master left.

Why then do we think that the second edition should be published? Well, we believe that the quality of individual contributions in these volumes is unique, interesting and already partly historical (and irreplaceable—particularly in Part I of the first volume). Thus it should be updated and made available especially in this anniversary year. This we feel as our duty not only to our colleagues and authors but also to students and younger scientists who did not have a chance to meet the wandering scholar personally. We decided to prepare a second edition, asked our authors for updates and in a few instances we solicited new contributions in exciting new areas. The result is then a thoroughly edited volume which differs from the first edition in many places.

On this occasion we would like to thank all our authors for their time and work in preparing their articles and, in many cases, modifying and updating them. We are fortunate that we could add three new contributions: one by

Joel Spencer (in the way of personal introduction), one by Larry Guth in Part IV of the first volume devoted to geometry, and one by Alexander Razborov in Part I of the second volume devoted to extremal and Ramsey problems. We also wish to acknowledge the essential contributions of Steve Butler who assisted us during the preparation of this edition. In fact Steve's contributions were so decisive that we decided to add him as co-editor to these volumes. We also thank Kaitlin Leach (Springer) for her efficiency and support. With her presence at the SIAM Discrete Math. conference in Halifax, the whole project became more realistic.

However, we believe that these volumes deserve a little more contemplative introduction in several respects. The nearly 20 years since the first edition was prepared gives us a chance to see the mathematics of Paul Erdős in perspective. It is easy to say that his mathematics is alive; that may sound cliché. But this is in fact an understatement for it seems that Erdős' mathematics is flourishing. How much it changed since 1995 when the first edition was being prepared. How much it changed in the wealth of results, new directions and open problems. Many new important results have been obtained since then. To name just a few: the distinct distances problem, various bounds for Ramsey numbers, various extremal problems, the empty convex 6-gon problem, packing and covering problems, sum-product phenomena, geometric incidence problems, etc. Many of these are covered by articles of this volumes and many of these results relate directly or indirectly to problems, results and conjectures of Erdős. Perhaps it is not as active a business any more to solve a particular Erdős problem. After all, the remaining unsolved problems from his legacy tend to be the harder ones. However, many papers quote his work and in a broader sense can be traced to him.

There may be more than meets the eye here. More and more we see that the Erdős problems are attacked and sometimes solved by means of tools that are not purely combinatorial or elementary, and which originate in the other areas of mathematics. And not only that, these connections and applications merge to new theories which are investigated on their own and some of which belong to very active areas of contemporary mathematics. As if the hard problems inspire the development of new tools which then became a coherent group of results that may be called theories. This phenomenon is known to most professionals and was nicely described by Tim Gowers as two cultures. [W. T. Gowers, *The two cultures of mathematics*, in Mathematics: Frontiers and Perspectives (Amer. Math. Soc., Providence, RI, 2000), 65–78.] On one side, problem solvers, on the other side, theory builders. Erdős' mathematics seems to be on one side. But perhaps this is misleading. As an example, see the article in the first volume *Unexpected applications of polynomials in combinatorics* by Larry Guth and the article in the second volume *Flag algebras: an interim report* by Alexander Razborov for a wealth of theory and structural richness. Perhaps, on the top level of selecting problems and with persistent activity in solving them, the difference between the two sides becomes less clear. (Good) mathematics presents a whole.

Time will tell. Perhaps one day we shall see Paul Erdős not as a theory builder but as a man whose problems inspired a wealth of theories.

People outside of mathematics might think of our field as a collection of old tricks. The second edition of mathematics of Paul Erdős is a good opportunity to see how wrong this popular perception of mathematics is.

La Jolla, USA R.L. Graham
Prague, Czech Republic J. Nešetřil

Just a few lines to
remember Szeged restaurant
in Budapest on July 23/96
Paul Erdős

Joachim [illegible]

Laci [illegible]
Walter Deuber
Miklós [illegible]
Noga Alon
Moshe Rosenfeld
[illegible]
[illegible]
NURIT ALON (NOGA'S BOSS)

[illegible]
Rob Tijdeman
Bernie Tijdeman
Vera T. [illegible]
[illegible]
[illegible]
[illegible]
Mária [illegible]

IN MEMORIAM

Paul Erdős

26.3.1913–20.9.1996

The week before these volumes were scheduled to go to press, we learned that Paul Erdős died on September 20, 1996. He was 83. Paul died while attending a conference in Warsaw, on his way to another meeting. In this respect, this is the way he wanted to "leave". In fact, the list of his last month's activities alone inspires envy in much younger people.

Paul was present when the completion of this project was celebrated by an elegant dinner in Budapest for some of the authors, editors and Springer representatives attending the European Mathematical Congress. He was especially pleased to see the first copies of these volumes and was perhaps surprised (as were the editors) by the actual size and impact of the collection (On the opposite page is the collection of signatures from those present at the dinner, taken from the inside cover of the mock-up for these volumes). We hope that these volumes will provide a source of inspiration as well as a last tribute to one of the great mathematicians of our time. And because of the unique lifestyle of Paul Erdős, a style which did not distinguish between life and mathematics, this is perhaps a unique document of our times as well.

R.L. Graham
J. Nešetřil

Preface to the First Edition

In 1992, when Paul Erdős was awarded a Doctor Honoris Causa by Charles University in Prague, a small conference was held, bringing together a distinguished group of researchers with interests spanning a variety of fields related to Erdős' own work. At that gathering, the idea occurred to several of us that it might be quite appropriate at this point in Erdős' career to solicit a collection of articles illustrating various aspects of Erdős' mathematical life and work. The response to our solicitation was immediate and overwhelming, and these volumes are the result.

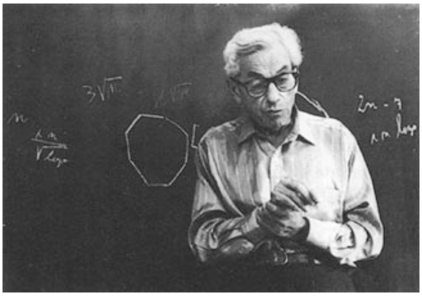
Regarding the organization, we found it convenient to arrange the papers into six chapters, each mirroring Erdős' holistic approach to mathematics. Our goal was not merely a (random) collection of papers but rather a thoroughly edited volume composed in large part by articles explicitly solicited to illustrate interesting aspects of Erdős and his life and work. Each chapter includes an introduction which often presents a sample of related Erdős' problems "in his own words". All these (sometimes lengthy) introductions were written jointly by editors.

We wish to thank the nearly 70 contributors for their outstanding efforts (and their patience). In particular, we are grateful to Béla Bollobás for his extensive documentation of Paul Erdős' early years and mathematical high points; our other authors are acknowledged in their respective chapters. We also want to thank A. Bondy, G. Hahn, I. Ouhel, K. Marx, J. Načeradský and Ché Graham for their help and for the use of their works. At various stages of the project, the book was supported by AT&T Bell Laboratories, GAČR 2167 and GAUK 351. We also are indebted to Dr. Joachim Heinze and Springer Verlag for their encouragement and support. Finally, we would like to record our extreme debt to Susan Pope (at AT&T Bell Laboratories) who somehow (miraculously) managed to convert more than 50 manuscripts of all types into the attractive form they now have.

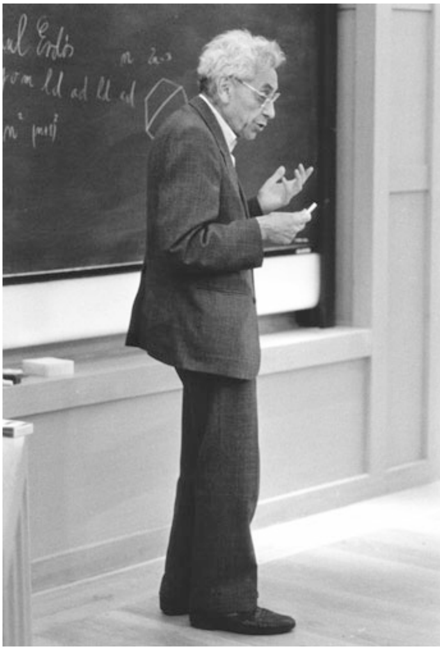
Here then is a unique portrait of a man who has devoted his whole being to "proving and conjecturing" and to the pursuit of mathematical knowledge and understanding. We hope that this will form a lasting tribute to one of the great mathematicians of our time.

Murray Hill, USA R.L. Graham
Praha, Czech Republic J. Nešetřil

Paul Erdős with Fan Chung.
Photo by George Csicsery.



Paul Erdős lecturing.
Photo by Geňa Hahn.



Paul Erdős lecturing.
Photo by George Csicsery.

Paul Erdős with Ron Graham.

Paul Erdős visiting Spelman college in Spring 1989. Photo by Colm Mulcahy.

Paul Erdős with George Szekeres in 1993.

Paul Erdős with an epsilon.

Paul Erdős with epsilon Jakub, Jarik Nešetřil, and Vojtěch Rödl.

Paul Erdős with Wolfgang Haken.

Paul Erdős around 1921.

Paul Erdős with Ralph Faudree.

Paul Erdős with his mother.

Portrait by Fan Chung (watercolor, 2008).

Portrait by Ivan Ouhel (oil, 1992).

Portrait by Karel Marx (oil, 1993).

Contents

I. Combinatorics and Graph Theory

Introduction

Erdős' work in graph theory started early and arose in connection with D. Kőnig, his teacher in prewar Budapest. The classic paper of Erdős' and Szekeres from 1935 also contains a proof in "graphotheoretic terms." The investigation of the Ramsey function led Erdős to probabilistic methods and seminal papers in 1947, 1958 and 1960. It is perhaps interesting to note that three other very early contributions of Erdős' to graph theory (before 1947) were related to infinite graphs: infinite Eulerian graphs (with Gallai and Vászoni) and a paper with Kakutani on nondenumerable graphs (1943). Although the contributions of Erdős to graph theory are manifold, and he proved (and always liked) beautiful structural results such as the Friendship Theorem (jointly with V. T. Sós and Kövári), and compactness results (jointly with N. G. de Bruijn), his main contributions were in asymptotic analysis, probabilistic methods, bounds and estimates. Erdős was the first who brought to graph theory the experience and rigor of number theory (perhaps being preceded by two papers by V. Jarník, one of his early coauthors). Thus he contributed in an essential way to lifting graph theory up from the "slums of topology."

This part contains a "special" problem paper not by Erdős but by his frequent coauthors from Memphis: R. Faudree, C. C. Rousseau and R. Schelp (well, there is actually an Erdős supplement there as well). We encouraged the authors to write this paper and we are happy to include it in this volume. This part also includes two papers coauthored by Béla Bollobás, who is one of Erdős' principal disciples. Bollobás contributed to much of Erdős' combinatorial activities and wrote important books about them (*Extremal Graph Theory*, *Introduction to Graph Theory*, *Random Graphs*). His contributions to this chapter (coauthored with his two former students G. Brightwell and A. Thomason) deal with graphs (and thus are in this chapter) but they by and large employ random graph methods (and thus they could be also be at home in the other volume). The main questions there may be also considered as extremal graph theory questions (and thus they could fit into the following part). Other contributions to this chapter, which are related to some aspect of Erdős' work or simply pay tribute to him are by N. Alon, Z. Füredi, M. Aigner and E. Triesch, S. Bezrukov and K. Engel, A. Gyárfás, S. Brandt, N. Sauer and H. Wang, H. Fleischner and M. Stiebitz, and D. Beaver, S. Haber and P. Winkler.

In 1995/1996, when the contents of these volumes were already crystallizing, we asked Paul Erdős to isolate a few problems, both recent and old, for each of the eight main parts of this book. To this part on Combinatorics and Graph Theory he contributed the following collection of problems and comments.

Erdős in his own words

Many years ago I proved by the probability method that for every k and r there is a graph of girth $\geq r$ and chromatic number $\geq k$. Lovász when he was still in high school found a fairly difficult constructive proof. My proof still had the advantage that not only was the chromatic number of $G(n)$ large but the largest independent set was of size $< \epsilon n$ for every $\epsilon > 0$ if $n > n_0(\epsilon, r, k)$. Nešetřil and V. Rödl later found a simpler constructive proof.

There is a very great difference between a graph of chromatic number $\aleph_0$ and a graph of chromatic number $\geq \aleph_1$. Hajnal and I in fact proved that if G has chromatic number $\aleph_1$ then G must contain a C_4 and more generally G contains the complete bipartite graph $K(n, \aleph_1)$ for every $n < \aleph_0$. Hajnal, Shelah and I proved that every graph G of chromatic number $\aleph_1$ must contain for some k_0 every odd cycle of size $\geq k_0$ (for even cycles this was of course contained in our result with Hajnal), but we observed that for every k and every m there is a graph of chromatic number m which contains no odd cycle of length $< k$. Walter Taylor has the following very beautiful problem: Let G be any graph of chromatic number $\aleph_1$. Is it true that for every $m > \aleph_1$ there is a graph G_m of chromatic number m all finite subgraphs of which are contained in G? Hajnal and Komjáth have some results in this direction but the general conjecture is still open. If it would have been my problem, I certainly would offer 1,000 dollars for a proof or a disproof. (To avoid financial ruin I have to restrict my offers to my problems.)

Let k be fixed and $n \to \infty$. Is it true that there is an $f(k)$ so that if $G(n)$ has the property that for every m every subgraph of m vertices contains an independent set of size $m/2 - k$ then $G(n)$ is the union of a bipartite graph and a graph of $\leq f(k)$ vertices, i.e., the vertex set of $G(n)$ is the union of three disjoint sets S_1, S_2 and S_3 where S_1 and S_2 are independent and $|S_3| \leq f(k)$. Gyárfás pointed out that even the following special case is perhaps difficult. Let m be even and assume that every m vertices of our $G(n)$ induces an independent set of size at least $m/2$. Is it true then that $G(n)$ is the union of a bipartite graph and a bounded set? Perhaps this will be cleared up before this paper appears, or am I too optimistic?

Hajnal, Szemerédi and I proved that for every $\epsilon > 0$ there is a graph of infinite chromatic number for which every subgraph of m vertices contains an independent set of size $(1-\epsilon)m/2$ and in fact perhaps $(1-\epsilon)m/2$ can be replaced by $m/2 - f(m)$ where $f(m)$ tends to infinity arbitrarily slowly. A result of Folkman implies that if G is such that every subgraph of m vertices contains an independent set of size $m/2 - k$ then the chromatic number of G is at most $2k + 2$.

Many years ago Hajnal and I conjectured that if G is an infinite graph whose chromatic number is infinite, then if $a_1 < a_2 < \ldots$ are the lengths of the odd cycles of G we have

$$\sum_i \frac{1}{a_i} = \infty$$

and perhaps $a_1 < a_2 < \ldots$ has positive upper density. (The lower density can be 0 since there are graphs of arbitrarily large chromatic number and girth.)

We never could get anywhere with this conjecture. About 10 years ago Mihók and I conjectured that G must contain for infinitely many n cycles of length 2^n. More generally it would be of interest to characterize the infinite sequences $A = \{a_1 < a_2 < \ldots\}$ for which every graph of infinite chromatic number must contain infinitely many cycles whose length is in A. In particular, assume that the a_i are all odd.

All these problems are unattackable (at least for us). About 3 years ago Gyárfás and I thought that perhaps every graph whose minimum degree is ≥ 3 must contain a cycle of length 2^k for some $k \geq 2$. We became convinced that the answer almost surely will be negative but we could not find a counterexample. We in fact thought that for every r there must be a G_r every vertex of which has degree $\geq r$ and which contains no cycle of length 2^k for any $k \geq 2$. The problem is wide open.

Gyárfás, Komlós and Szemerédi proved that if k is large and $a_1 < a_2 < \ldots$ are the lengths of the cycles of a $G(n, kn)$, that is, an n-vertex graph with kn edges, then

$$\sum \frac{1}{a_i} > c \log n.$$

The sum is probably minimal for the complete bipartite graphs.

(Erdős-Hajnal) If G has large chromatic number does it contain two (or k if the chromatic number is large) edge-disjoint cycles having the same vertex set? It surely holds if $G(n)$ has chromatic number $> n^\epsilon$ but nothing seems to be known.

Fajtlowicz, Staton and I considered the following problem (the main idea was due to Fajtlowicz). Let $F(n)$ be the largest integer for which every graph of n vertices contains a regular induced subgraph of $\geq F(n)$ vertices. Ramsey's theorem states that $G(n)$ contains a trivial subgraph, i.e., a complete or empty subgraph of $c \log n$ vertices. (The exact value of c is not known but we know $1/2 \leq c \leq 2$.) We conjectured $F(n)/\log n \to \infty$. This is still open. We observed $F(5) = 3$ (since if $G(5)$ contains no trivial subgraph of 3 vertices then it must be a pentagon) . Kohayakawa and I worked out the $F(7) = 4$ but the proof is by an uninteresting case analysis. (We found that this was done earlier by Fajtlowicz, McColgan, Reid and Staton, see Ars Combinatoria vol 39.) It would be very interesting to find the smallest integer n for which $F(n) = 5$, i.e., the smallest n for which every $G(n)$ contains a regular induced subgraph of ≥ 5 vertices. Probably this will be much more difficult than the proof of $F(7) = 4$ since in the latter we could use properties of perfect graphs. Bollobás observed that $F(n) < c\sqrt{n}$ for some $c > 0$.

Let $G(10n)$ be a graph on $10n$ vertices. Is it true that if every index subgraph of $5n$ vertices of our $G(10n)$ has $\geq 2n^2 + 1$ edges then our $G(10n)$ contains a triangle? It is easy to see that $2n^2$ edges do not suffice. A weaker result has been proved by Faudree, Schelp and myself at the Hakone conference (1992, I believe) see also a paper by Fan Chung and Ron Graham (one of the papers in a volume published by Bollobás dedicated to me).

A related forgotten conjecture of mine states that if our $G(10n)$ has more than $20n^2$ edges and every subgraph of $5n$ vertices has $\geq 2n^2$ edges then our graph must have a triangle. Simonovits noticed that if you replace each vertex of the Petersen graph by n vertices you get a graph of $10n$ vertices, $15n^2$ edges, no triangle and every subgraph of $5n$ vertices contains $\geq 2n^2$ edges.

*****

So much for P. Erdős in 1995. Let us add that since that time some of these problems were solved, some are open and some seem to be dormant. Some were subject of intensive study. The reference to the above Hakone conference is:

P. Erdős, R. J. Faudree, C. C. Rousseau, R. H. Schelp, A local density condition for triangles, Discrete Math. 127, 1–2 (1994), 153–161.
(The conference was The Second Japan Conference on Graph Theory and Combinatorics, Aug 18–22, 1990 in Hakone.)

The mentioned paper by Fan Chung et al. is the following:
F. R. K. Chung and R. L. Graham, On graphs not containing prescribed induced subgraphs, in A Tribute to Paul Erdos, ed. by A. Baker, B. Bollobás and A. Hajnal, Cambridge University Press (1990), 111–120.

One of these problems was quoted by Erdős much earlier. For example the problem of Taylor was mentioned as early as 1975; (W. Taylor: Problem 42. In: Combinatorial Structures and Their Applications, Proc. Calgary Internat. Conf. 1969, Gordon and Breach 1969.)

For more information about Erdős problems on graphs and of their current status see:
F. R. K. Chung, R. L. Graham, *Erdős on Graphs: His Legacy of Unsolved Problems*, A K Peters, Cambridge, MA 1993, xiv+142 pp.

Reconstruction Problems for Digraphs

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Summary. Associate to a finite directed graph $\mathbf{G}(V, E)$ its out-degree resp. in-degree sequences d^+, d^- and the corresponding neighborhood lists N^+, N^- (when $\mathbf{G}$ is a labeled graph). We discuss various problems when sequences resp. lists of sets can be realized as degree sequences resp. neighborhood lists of a directed graph.

1. Introduction

Consider a finite graph $G(V, E)$. Let us associate with G a finite list $P(G)$ of parameters, e.g. the degrees, the list of cliques, the chromatic polynomial, or whatever we like. For any set P of invariants there arise two natural problems:

(R) **Realizability**. Given P, when is $P = P(G)$ for some graph G? We then call P *graphic*, and say that G *realizes* P.

(U) **Uniqueness**. Suppose $P(G) = P(H)$. When does this imply $G \cong H$? In other words, when is P a complete set of invariants?

The best studied questions in this context are probably the reconstruction conjecture for (U), and the degree realization problem for (R). This latter problem was solved in a famous theorem of Erdős-Gallai [4] characterizing graphic sequences. Their theorem reads as follows: Let $d_1 \geq \cdots \geq d_n \geq 0$ be a sequence of integers. Then $(d_1 \geq \cdots \geq d_n)$ can be realized as the degree sequence of a graph if and only if the degree sum is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min(d_j, k) \quad (k = 1, \ldots, n). \tag{1}$$

A variant of this problem concerning neighborhoods was first raised by Sós [13] and studied by Aigner-Triesch [2]. Consider a finite labeled graph $G(V, E)$ and denote by $N(u)$ the neighborhood of $u \in V$. $\mathcal{N}(G) = \{N(u) : u \in V\}$ is called the *neighborhood list* of G. Given a list (multiset) $\mathcal{N} = (N_1, \ldots, N_n)$ of sets. When is $\mathcal{N} = \mathcal{N}(G)$ for some graph G? In contrast to the polynomial

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verification of (1), it was shown in [2] that the neighborhood list problem NL is NP-complete for arbitrary graphs. For bipartite graphs, NL turns out to be polynomially equivalent to the GRAPH ISOMORPHISM problem. A general survey of these questions appears in [3].

In the present paper we consider directed graphs $\mathbf{G}(V,E)$ on n vertices with or without loops and discuss the corresponding realizability problems for the degree resp. neighborhood sequences. We assume throughout that there is at most one directed edge (u,v) for any $u,v \in V$. To every $u \in V$ we associate its *out-neighborhood* $N^+(u) = \{v \in V : (u,v) \in E\}$ and its *in-neighborhood* $N^-(u) = \{v \in V : (v,u) \in E\}$ with $d^+(u) = |N^+(u)|$ and $d^-(u) = |N^-(u)|$ being the *out-degree* resp. *in-degree* of u.

For both the degree realization problem and the neighborhood problem we have three versions in the directed case:

($\mathcal{D}^+$) Given a sequence $d^+ = (d_1^+, \dots, d_n^+)$ of non-negative integers. When is d^+ realizable as the out-degree sequence of a directed graph?

Obviously, ($\mathcal{D}^-$) is the same problem.

($\mathcal{D}_-^+$) Given a sequence of pairs $d_-^+ = ((d_1^+, d_1^-), \dots, (d_n^+, d_n^-))$. When is there a graph $\mathbf{G}$ with $d^+(u_i) = d_i^+$, $d^-(u_i) = d_i^-$ for all i?

($\mathcal{D}^+, \mathcal{D}^-$) Given two sequences $d^+ = (d_1^+, \dots d_n^+)$, $d^- = (d_1^-, \dots, d_n^-)$. When is there a directed graph such that d^+ is the out-degree sequence (in some order) and d^- the in-degree sequence?

In an analogous way, we may consider the realization problems ($\mathcal{N}^+$), ($\mathcal{N}_-^+$), ($\mathcal{N}^+, \mathcal{N}^-$) for neighborhood lists.

In Sect. 2 we consider the degree problems and in Sect. 3 the neighborhood problems. Section 4 is devoted to simple directed graphs when there is at most one edge between any two vertices and no loops.

2. Degree Sequences

Depending on whether we allow loops or not there are six different reconstruction problems whose solutions are summarized in the following diagram:

Degrees	($\mathcal{D}^+$)	($\mathcal{D}_-^+$)	($\mathcal{D}^+, \mathcal{D}^-$)
With loops	Trivial	Gale-Ryser	Gale-Ryser
Without loops	Trivial	Fulkerson	Fulkerson

The problems ($\mathcal{D}^+$) have the following trivial solutions: (d^+) is realizable with loops if and only if $d_i^+ \leq n$ for all i, and without loops if and only if $d_i^+ \leq n-1$ for all i.

Consider ($\mathcal{D}_-^+$) with loops. We represent $\mathbf{G}(V,E)$ as usual by its adjacency matrix M where the rows and columns are indexed by the vertices $u_1, \dots, u_n$

with $m_{ij} = 1$ if $(u_i, u_j) \in E$ and 0 otherwise. To realize a given sequence (d^{+}_{-}) is therefore equivalent to constructing a $0,1$-matrix with given row-sums d_i^+ and column-sums d_i^- which is precisely the content of the Gale-Ryser Theorem [7, 11]. In fact, the Gale-Ryser Theorem applies to the situation $(\mathcal{D}^+, \mathcal{D}^-)$ as well by permuting the columns. If we do not allow loops, then the realization problem (D^{+}_{-}) is settled by an analogous theorem of Fulkerson [5, 6]. He reduces the problem of constructing a $0,1$-matrix with zero trace and given row and column sums to a network flow problem, an approach which can also be used in the case of the Gale-Ryser theorem thus showing that both problems are polynomially decidable. Finally, we remark that the case $(\mathcal{D}^+, \mathcal{D}^-)$ can be reduced to the case (D^{+}_{-}) in view of the following proposition.

Proposition 1. *Suppose two sequences $d^+ = (d_1^+, \dots, d_n^+)$, $d^- = (d_1^-, \dots, d_n^-)$ are given. Denote by $\overline{d}^+$ (resp. $\overline{d}^-$) a non-increasing (resp. non-decreasing) rearrangement of d^+ (resp. d^-). If there exists a $0,1$-matrix $M = (m_{ij})$ with $\sum_{j=1}^n m_{jj} = 0$, $\sum_{j=1}^n m_{ij} = d_i^+$, $\sum_{j=1}^n m_{ji} = d_i^-$, $1 \le i \le n$, then there exists a $0,1$-matrix $\overline{M} = (\overline{m}_{ij})$ satisfying $\sum_j \overline{m}_{jj} = 0$, $\sum_j \overline{m}_{ij} = \overline{d}_i^+$, $\sum_i \overline{m}_{ji} = \overline{d}_i^-$, $1 \le i \le n$.*

Proof. Suppose M is given as above. By permuting the rows and columns of M by the same permutation (which does not change the trace) we may assume that $d^+ = \overline{d}^+$. □

Now suppose that for some indices $i < j$, $d_i^- > d_j^-$. We will show that M can be transformed into a matrix $\hat{M}$ with zero trace, row sum vector d^+ and column sum vector $\hat{d}^-$, where $\hat{d}^-$ arises from d^- by exchanging d_i^- and d_j^-. Since each Permutation is generated by transpositions, this will obviously complete the proof. To keep notation simple, we give the argument only for the case $i = n-1$, $j = n$ but it is immediately clear how the general case works. Suppose

$$M = \left(\begin{array}{c|c|c} \qquad\qquad & a & b \\ \hline & 0 & y \\ \hline & x & 0 \end{array}\right).$$

(i) If $x \le y$, then we exchange (a_i, b_i) when $a_i = 1$ and $b_i = 0$ for $d_{n-1}^- - d_n^-$ indices $i \le n-2$.

(ii) If $x = 1$, $y = 0$, then since $d_{n-1}^+ \ge d_n^+$ there exists some $\ell < n-1$ such that

$$\begin{pmatrix} m_{n-1,\ell} \\ m_{n,\ell} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now let

$$M = \left(\begin{array}{c|c|c} & b & a \\ \hline 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array} \right)$$

3. Neighborhood Lists

As in the undirected case the neighborhood problems are more involved. Again, we summarize our findings in a diagram and then discuss the proofs.

Neighbors	$(\mathcal{N}^+)$	$(\mathcal{N}^+_-)$	$(\mathcal{N}^+, \mathcal{N}^-)$
With loops	Trivial	$\geq$ GRAPH ISOM.	$\approx$ GRAPH ISOM.
Without loops	Bipartite matching	NP-complete	NP-complete

Let us consider $(\mathcal{N}^+)$ first. Allowing loops, any list (N_i^+) can be realized. In the absence of loops, (N_i^+) can be realized if and only if $({N_1^+}^c, \ldots, {N_n^+}^c)$ has a transversal, where N^c is the complement of N. So, this problem is equivalent to the bipartite matching problem and, in particular, polynomially decidable.

Let us treat next the problem $(\mathcal{N}^+_-)$ without loops. The special case $N_i^+ = N_i^-$ for all i clearly reduces to the (undirected) neighborhood list problem NL which as mentioned is NP-complete. Accordingly, $(\mathcal{N}^+_-)$ is NP-complete as well.

We show next that the decision problem $(\mathcal{N}^+_-)$ with loops is polynomially equivalent to the matrix symmetry problem MS defined as follows:

The input is an $n \times n$-matrix A with $0,1$-entries, with the question: Does there exist a permutation matrix P such that $(PA)^T = PA$ holds?

Let us represent $\mathbf{G}(V, E)$ again by its adjacency matrix. Then, clearly, MS is the special case of $(\mathcal{N}^+_-)$ where $N_i^+ = N_i^-$ for all i. To see the converse denote by x^i (resp. y^i) the incidence vectors of N_i^+ (resp. N_i^-) as row vectors, and set

$$X = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \quad Y = \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}.$$

The problem $(\mathcal{N}^+_-)$ is thus equivalent to the following decision problem: Does there exist a permutation matrix P such that

$$PX = (PY)^T?$$

Note that $PX = (PY)^T \iff (PX)^T = PY$. Now consider the $4n \times 4n$-matrix

$$\Gamma = \begin{array}{|c|c|c|c|} \hline O & I & X & U \\ \hline I & O & I & W \\ \hline Y & I & O & Z \\ \hline U^T & W^T & Z^T & O \\ \hline \end{array}$$

where O, I are the zero-matrix and identity matrix, respectively, and U, W, Z are matrices with identical rows each which ensure that a permutation matrix R satisfying $(R\Gamma)^T = R\Gamma$ must be of the form

$$R = \begin{array}{|c|c|c|c|} \hline R_1 & O & O & O \\ \hline O & R_2 & O & O \\ \hline O & O & R_3 & O \\ \hline O & O & O & R_4 \\ \hline \end{array}\,.$$

Clearly, such matrices U, W, Z exist. Now $(R\Gamma)^T = R\Gamma$ if and only if

$$(R_1X)^T = R_3Y, R_2^T = R_1, R_3 = R_2^T = R_1,$$

i.e. if and only if $(R_1X)^T = R_1Y$, and the result follows.

It was mentioned in [2] that MS is at least as hard as GRAPH ISOMORPHISM, but we do not know whether they are polynomially equivalent.

Let us, finally, turn to the problems $(\mathcal{N}^+, N^-)$. We treat $(\mathcal{N}^+, \mathcal{N}^-)$ with loops, the other version is settled by a matrix argument as above. Using again the notation x^i, y^i, X, Y, our problem is equivalent to the following matrix problem: Do there exist permutation matrices P and Q such that

$$PX = Y^TQ, \text{ i.e. } PXQ^T = Y^T?$$

This latter problem is obviously polynomially equivalent to HYPERGRAPH ISOMORPHISM which is known to be equivalent to GRAPH ISOMORPHISM (see [2]).

4. Simple Directed Graphs and Tournaments

Let us now consider simple directed graphs and, in particular, tournaments. In contrast to the non-simple case, where $(\mathcal{D}^+)$ and $(\mathcal{N}^+)$ are trivial resp. polynomially solvable (bipartite matching), the problems now become more involved.

As for $(\mathcal{D}^+)$, a necessary and sufficient condition for (d_i^+) to be realizable as an out-degree sequence of a *tournament* was given by Landau [9]. Assume $d_1^+ \geq \cdots \geq d_n^+$, then (d_i^+) is realizable if and only if

$$\sum_{i=1}^{k} d_i^+ \leq (n-1) + \ldots + (n-k) \quad (K = 1, \ldots, n)$$

with equality for $k = n$.

Deleting the last condition yields the corresponding result for arbitrary simple digraphs.

Theorem 1. *A sequence $(d_1^+ \geq \cdots \geq d_n^+)$ is realizable as out-degree sequence of a simple directed graph if and only if*

$$\sum_{i=1}^{k} d_i^+ \leq \sum_{i=1}^{k} (n-i) \quad (K = 1, \ldots, n). \tag{2}$$

Proof. The condition is obviously necessary. For the converse we make use of the dominance order of sequences $p = (p_1 \geq \cdots \geq p_n)$, $q = (q_1 \geq \cdots \geq q_n)$:

$$p \leq q \Longleftrightarrow \sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \quad (k = 1, \ldots, n). \tag{3}$$

Suppose $m = \sum_{i=1}^{n} d_i^+$, and denote by $L(m)$ the lattice of all sequences $p = (p_1 \geq \cdots \geq p_n)$, $\sum_{i=1}^{n} p_i = m$, ordered by (3), see [1] for a survey on the uses of $L(m)$. Let $S(m) \subseteq L(m)$ be the set of sequences which are realizable as out-degree sequences of a simple digraph with m edges.

Claim 1. *$S(m)$ is a down-set, i.e. $p \in S(m), q \leq p \Longrightarrow q \in S(m)$.*

It is well-known that the order $\leq$ in $L(m)$ is transitively generated by successive "pushing down boxes", i.e. it suffices to prove the claim for $q \leq p$ with $q_r = p_r - 1$, $q_s = p_s + 1$ for some $p_r \geq p_s + 2$, and $q_i = p_i$ for $i \neq r, s$. Now suppose $\mathbf{G}$ realizes the sequence p, with $d^+(u_i) = p_i$. Since $d^+(u_r) \geq d^+(u_s) + 2$, there must be a vertex v_t with $(u_r, u_t) \in E, (u_s, u_t) \notin E$. If $(u_t, u_s) \in E$, replace $(u_r, u_t), (u_t, u_s)$ by $(u_t, u_r), (u_s, u_t)$, and if $(u_t, u_s) \notin E$, replace (u_r, u_t) by (u_s, u_t). In either case, we obtain a simple directed graph $\mathbf{G}'$ with q as out-degree sequence, and the claim is proved.

Claim 2. *Suppose $m = (n-1) + \ldots + (n-\ell+1) + r$ with $r \leq n - \ell$, then $\overline{p}_m = (n-1, \ldots, n-\ell+1, r)$ is the only maximal element of $S(m)$ in $L(m)$.*

By the definition (3), the sequence $\overline{p}_m$ clearly dominates any sequence in $S(m)$, and since $\overline{p}_m$ can obviously be realized as an out-degree sequence of a simple digraph, it is the unique maximum of $S(m)$.

Taking Claims 1 and 2 together yields the characterization

$$d \in S(m) \Longleftrightarrow d \leq \overline{p}_m$$

and this latter condition is plainly equivalent to (1). $\square$