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A Journey Through 18th to 20th Century Mathematics

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Preface

This book grew out of my research on the history of mathematics over the past 40 years. Time and again, the path of my investigation led me to the consideration of work by Frobenius that played an important role in the historical development I was attempting to understand. I finally decided it would be appropriate to bring these research experiences together into a book on the mathematics of Frobenius, especially since little has been written about him despite the fact that he made many important contributions to present-day mathematics—as suggested by the many theorems and concepts that bear his name.

Initially, the focus of the book was dictated by my earlier research experiences and interests involving Frobenius. These had all involved his work on the theory and application of linear algebra, including the application involved in his creation of the theory of group characters and representations; and so the initial working title was "Frobenius and the History of Linear Algebra." As the reader will see, much of Frobenius' work did indeed involve linear algebra somewhere along the way; but I began to realize that to focus exclusively on this aspect of his work would present a distorted picture of his mathematical activities and their significance, as well as of the sources of his inspiration and the reasons so much of his work has become a part of basic mathematics. His creation of representation theory may have been his most important achievement, but he also did work of lasting significance in many other areas of mathematics. Frobenius was an algebraist at heart, but he looked for interesting problems of an essentially algebraic or formal nature in a broad spectrum of areas of nineteenth- and early twentieth-century mathematics. To do him and his work justice, the scope of the book had to be broadened into a more well-rounded intellectual biography; and that is what I have attempted. Whence the first part of the title of the book: "The Mathematics of Frobenius."

The second part of the title also requires clarification. I have attempted to present the mathematics of Frobenius "in context" in two senses. The first of these involves providing the reader with the historical background necessary to understand why Frobenius undertook to solve a particular problem and to appreciate, by seeing what had been done before, the magnitude of his achievement, as well as what he owed to his predecessors. In addition to the backgrounds peculiar to various particular problems, it was also necessary to say something about Frobenius' educational background, namely his training in the Berlin school of mathematics presided over by Weierstrass (Frobenius' dissertation advisor and principal supporter), Kronecker, and Kummer. Of particular importance is the work done by Weierstrass and Kronecker on the theory of equivalence (respectively, congruence) of families of bilinear (respectively quadratic) forms. As we shall see, from their work Frobenius learned both theorems and concomitant disciplinary ideals that together motivated and informed much of his early work. In addition, from Kummer's groundbreaking work on ideal complex numbers, and Kronecker's interest in extending it, as well as from Dedekind's quite different extension by means of his theory of algebraic numbers and ideals, Frobenius acquired the background that motivated much of his work on the theory of numbers and abstract group theory. Thus considerable attention is given to these arithmetic developments.

I have also attempted to present Frobenius' mathematics "in context" in the sense that I have sought to trace the various ways in which his work was subsequently applied, developed, and ultimately incorporated into present-day mathematics. By presenting the mathematics of Frobenius in context in both these senses, my hope is that the reader will come away not only with an enriched appreciation of Frobenius' work but also with a glimpse of the broad swath of diverse and important strands of eighteenth- to twentieth-century mathematics that results from the contextual approach and that ranges from the work of Lagrange and Laplace on terrestrial and celestial mechanics in the last decades of the eighteenth century, which involved them with the theory of systems of linear differential equations, to the theory of complex abelian varieties in the mid-twentieth century. This is the "Journey through Eighteenth- to Twentieth-Century Mathematics" of the subtitle.

The book has been divided into three parts. Part I is devoted to an overview of Frobenius' entire mathematical career and thus serves as an introduction to the main body of the book. Here, within the context of Frobenius' educational and professional career, his contributions to mathematics and the attendant backgrounds are briefly sketched and their subsequent impact on the development of mathematics indicated. It is my hope that the reader will come away from Part I with a broad sense of Frobenius' many contributions to mathematics, of the institutional and personal connections that affected his work, of the broad scope and progression of his mathematical interests, and of the ways in which his work has been incorporated into present-day mathematics. Of course, in order to gain more than just a vague sense, in order to fully appreciate what Frobenius accomplished, how it grew out of or was motivated by earlier work, and how it has affected present-day mathematics, a reading of the chapters in Parts II and III is necessary. The two chapters that form Part II deal with the development of linear algebra up to and including the work of Weierstrass and Kronecker and are essential background for all that is to follow. The chapters of Part III deal in depth with Frobenius' major works, a subset of the works discussed in Part I. These chapters range over many areas of mathematics and can be read independently of one another, with little loss of continuity thanks to the overview provided by Part I. Thus, for example, a reader particularly interested in Frobenius' arithmetic work could turn next to Chapters 8 and 9, where this work is treated. Readers wishing to know more about his work on group characters and representations could start with Chapter 12. I have provided a detailed table of contents to guide readers to those parts of Frobenius' work of special interest to them.

In addition to a detailed table of contents, I have provided an extensive index that will enable readers to look for a specific topic that may not be included in the table of contents. The index can also be used to find the meaning of unfamiliar terms, such as "Dedekind characters," the "containment theorem of Frobenius," or "winter semester" in German universities. If several pages are given for an entry, the page number containing the explanation of the term is given in boldface. The index is also helpful for tracking down various recurring themes of the book, such as "generic reasoning," "disciplinary ideals of Berlin school" (also found under Kronecker, who articulated them), and "multiple discoveries" involving Frobenius. By the latter term I mean instances in which Frobenius and one or more mathematicians independently made essentially the same discovery or developed similar ideas. As the index shows, Frobenius was involved in many instances of multiple discovery. The entry for Frobenius is particularly extensive and should prove useful in locating various sorts of information about him, such as a listing of all the evaluations Weierstrass made of him and his work, as well as all evaluations Frobenius made of other mathematicians or mathematical theories. In addition, there is a listing of all mathematicians influenced by Frobenius and a listing of all mathematicians who influenced him in the broad sense that includes mathematicians who provided him with useful ideas and results, as well as mathematicians whose work, due to deficiencies, motivated him to develop a theory that removed them.

My interest in Frobenius began circa 1970 with an attempt to reconstruct the origins of his remarkable theory of group characters [266]. I knew that he had been in correspondence with Dedekind, who had introduced him to group determinants. Important excerpts of Dedekind's side of the correspondence had been published by E. Noether in Dedekind's collected works [119, pp. 414–442], but Frobenius' side of the correspondence seemed lost to posterity until the year after my paper [266] appeared, when Clark Kimberling announced his fortuitous discovery of the Dedekind–Frobenius correspondence [339, §8], which runs to over 300 pages. At my request he kindly provided me with a copy of the correspondence, which showed that my reconstruction of how Frobenius had created his theory of group characters needed to be significantly modified. The result was my paper [268], which quoted extensively (in translation) from the correspondence. Much of that material is incorporated into this book. In addition, the correspondence during 1882 has proved enlightening in discussing Frobenius' work on density theorems, which was done in 1880 but not published until 1896. By the time I investigated Frobenius' work on density theorems, two unpublished transcriptions of the correspondence had been made, the first by the late Walter Kaufmann-Bühler, and the second, building upon the first, by Ralf Haubrich. They kindly sent me copies of drafts of their transcriptions, which greatly facilitated a careful reading of the entire correspondence. The Dedekind–Frobenius correspondence was initially housed at the Clifford Memorial Library of the University of Evansville, and I am grateful to the library for permission to use the correspondence in my publications. In 1995, the correspondence was moved to its present location in the archives of the library of the Technical University at Braunschweig, the institution where Dedekind spent almost all of his mathematical career.[1] All of the citations from Frobenius' letters are from this archival source. The citations from Dedekind's letters that are printed in his collected works are so indicated by footnotes.

Besides the individuals and institutions mentioned above, I am indebted to many others who, at one point or other during the past 40 years, assisted me with some aspect of my work on Frobenius. Although I am sure I have now forgotten some, I do wish to express my gratitude to those I have remembered: Armand Borel, Keith Conrad, Harold Edwards, Walter Feit, Jeremy Gray, Rob Gross, Walter Ledermann, Franz Lemmermeyer, Peter Neuman, Wilfried Parys, Klaus Peters, Peter Roquette, Michael Rosen, David Rowe, Yvonne Schetz, Hans Schneider, Shlomo Sternberg, and Dan Weiner. I am also grateful to the NSF Science and Technology Studies program for providing the financial support that enabled me to initiate my efforts to write a book on Frobenius.[2] My greatest debt of all is to Jean-Pierre Serre. To begin with, shortly before my interests turned toward Frobenius, he took on the burden of editing Frobenius' mathematical works for publication. Frobenius' *Mathematische Abhandlungen* appeared in 1968 [232] and has facilitated my study of his work ever since. In addition, throughout my career as historian of mathematics he has encouraged my efforts and generously given his time to critically evaluate and respond with many helpful suggestions to drafts of various parts of my work. His contributions to the writing of this book in particular have been manifold. Some of these are reflected in the index but many are not. Among the latter, I would mention that the decision to transform my book from "Frobenius and the History of Linear Algebra" into a book that attempts to deal with all of Frobenius' major mathematical contributions was sparked by his remark, on hearing of my plans to write the first sort of book, that I should really look into Frobenius' work on theta functions, since C.L. Siegel had told him that Frobenius had done important work in this area. (That Siegel was right can be seen from Chapter 11.) Initially I dismissed Serre's suggestion of a more inclusive work on the grounds of personal inadequacy, but his suggestion remained in the back of my mind and eventually led to the following book, imperfect as it may prove to be.

Finally, I wish to express my gratitude to David Kramer, whose scrupulous and informed copyediting of the book has resulted in many significant improvements.

Boston, MA, USA Thomas Hawkins

[1] Frobenius' letters to Dedekind are archived under the reference Universitätsarchiv Braunschweig G 98 : 10. G 98. Frobenius' letters are under G 98 : 10.

[2] Through grant SES-0312697.

Contents

Part I
Overview of Frobenius' Career and Mathematics

Chapter 1
A Berlin Education

Ferdinand Georg Frobenius was born in Berlin on 26 October 1849.[1] He was a descendant of a family stemming from Thüringen, a former state in central Germany and later a part of East Germany. Georg Ludwig Frobenius (1566–1645), a prominent Hamburg publisher of scientific works, including those written by himself on philology, mathematics, and astronomy, was one of his ancestors. His father, Christian Ferdinand, was a Lutheran pastor, and his mother, Christiane Elisabeth Friedrich, was the daughter of a master clothmaker.

Frobenius grew up in Berlin and attended high school there at the Joachimstalische Gymnasium, where he distinguished himself as one of its most outstanding students [22, p. 190]. He began his university studies in Göttingen, where he enrolled for the "summer semester" of 1867. In German universities the summer semester runs from about mid-April to mid-July and the winter semester from about mid-October through February. Thus the winter semester corresponds roughly to the fall semester in an American university and the summer semester to the spring semester. Frobenius took two courses in analysis at Göttingen and a course in physics given by the well-known physicist Wilhelm Weber. His primary interest was already in mathematics, and it is likely that his intention from the outset was to do what he ended up doing: enrolling for the winter semester 1867 at the University of Berlin and pursuing his study of mathematics there for six semesters through completion of his doctorate. This would have been a reasonable course of action because at the time, the University of Berlin was the leading center for mathematics in Germany and one of the major centers for mathematics in the world.

[1] The following biographical details about Frobenius are drawn from [4, 22, 553].

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1.1 Student Years: 1867–1870

In his invaluable study of mathematics in Berlin [22], K.-R. Biermann has characterized the period 1855–1892, which he calls the "Kummer–Weierstrass–Kronecker era," as a historical high point for instruction and research in mathematics at the university, what one who was there (Adolf Kneser) described as the "heroic period" [22, p. 75]. Indeed, thanks largely to the concerted efforts of the above three mathematicians, a bona fide school of mathematics emerged during these years. Frobenius became a devoted member of this school, and his choice of research problems and the manner in which he approached them was colored by his experiences at Berlin, which resonated with his own mathematical proclivities. Figure 1.1 shows a youthful Frobenius as he may have looked during his student and postdoctoral years at the University.

A closer look at the Berlin school is now in order. It will be helpful to begin by indicating the three principal categories of lecturers at a German university. The highest category was that of *ordentlicher Professor*, which I have translated as "full professor." The full professors had substantial salaries and powers within the university, e.g., to recommend new faculty appointments and to direct doctoral dissertations. Then there was the category of "extraordinary professor" (*ausserordentlicher Professor*), which I have translated as assistant professor, since the salary was modest and came with no powers within the university. The lowest category consisted of the "private lecturers" (*Privatdozenten*), which I have translated as "instructors." How someone with a doctorate became an instructor by writing a

Fig. 1.1 Frobenius as he may have looked during the early years of his career, when he absorbed and began applying the teachings of the Berlin school of mathematics. Image courtesy of ETH-Bibliothek Zürich, and located in its Image Archive

Habilitationsschrift is indicated in the next section. There were other lecturers outside these categories as well; an example is provided by Kronecker, as will be seen below.

At Berlin, the two full professors of mathematics at the university in 1867 were Eduard Kummer (1810–1893) and Karl Weierstrass (1815–1897). After impressing Jacobi and Dirichlet with work in analysis, Kummer had achieved more widespread fame for his groundbreaking theory of ideal complex integers associated to the ordinary complex integers of $\mathbb{Z}[\omega_p]$, ω_p a primitive pth root of unity for the prime number p (see Section 9.1.2). He had begun his work on this theory in 1846, but by 1867, when Frobenius arrived, his research interests had shifted to geometry (ray systems). As for Weierstrass, he had gone from the relative obscurity of a high-school mathematics teacher into the mathematical limelight by virtue of his solution, in 1854, to the Jacobi inversion problem for hyperelliptic integrals in any number of variables (see Section 10.1). Kummer's first administrative accomplishment after his appointment to a Berlin professorship in 1855 was to pull the strings necessary to get Weierstrass to Berlin the following year.

Mention should also be made of Carl Borchardt (1817–1880) [22, pp. 61–62]. Borchardt had obtained his doctorate under Jacobi's direction at the University of Königsberg (1843) and subsequently became his friend. Jacobi was in poor health at the time and Borchardt accompanied him to Italy, where (in Rome) they met Jacob Steiner (1796–1863) and P.G. Lejeune Dirichlet (1805–1859), who were then both full professors at the University of Berlin. Borchardt became an instructor at the university in 1848. In 1855, Dirichlet, about to leave for a professorship at Göttingen, persuaded Borchardt, who was independently wealthy, to take over the editorship of "Crelle's Journal"—the *Journal für die reine und angewandte Mathematik*—which had been founded in 1829 by August Crelle, who had edited it until his death in 1855. Borchardt also became a member of the Berlin Academy of Sciences in 1855, at the recommendation of Dirichlet, perhaps in part as a reward for assuming the task of editing *Crelle's Journal*, which was in effect the journal of the Berlin school of mathematics. That is, members of the academy, such as Kronecker, Kummer, and Weierstrass, tended to publish in the proceedings of the academy, but their students aspired to publication in *Crelle's Journal*. For example, from 1871 until 1893, when Frobenius became a Berlin professor and member of the academy, virtually all of his mathematical output was published in that journal.

Borchardt remained editor until his death in 1880, and during his tenure the quality of papers accepted increased. After over a decade as editor he had considered retiring, but when in 1869 Alfred Clebsch, then at the University of Göttingen, founded the rival *Mathematische Annalen*, Borchardt, sensing the competitive challenge, decided to continue as editor. Borchardt also became a close friend to Weierstrass, whom Borchardt had made a point of meeting in 1854, after Weierstrass had solved the Jacobi inversion problem for hyperelliptic integrals. Borchardt was one of the few people whom Weierstrass addressed with the familiar "*Du*" form.

By 1857 Weierstrass had solved the inversion problem for general abelian integrals, but he held back his results because he had discovered that Riemann had already published his quite different solution to the problem that same year.

Fig. 1.2 Weierstrass, through his work, teaching, and support was to exert the greatest influence on Frobenius of any mathematician. Of his many doctoral students, Frobenius was to become the most accomplished. Photo courtesy of Mathematisches Forschungsinstitut, Oberwolfach, Germany

Weierstrass then set for himself the goal of understanding Riemann's results and their relation to his own. Riemann's solution was couched in terms of what are now called Riemann surfaces and were not as rigorously founded as Weierstrass wished. His next goal was to develop his own solution in the light of his understanding of Riemann's results. This he did not do in print but, gradually, through the medium of his lectures on abelian integrals and functions at Berlin. Frobenius probably attended Weierstrass' lectures on this subject as given in the summer semester 1869.[2] Some important aspects of Weierstrass' lectures are discussed in Section 11.1 because of their relevance to Frobenius' work. Part of the foundations for the theory of elliptic and abelian functions was supplied by the general theory of functions of one or more complex variables, which Weierstrass also developed in lectures. Although Weierstrass' primary area of research was complex analysis, he also thought about and occasionally published in other areas. These included the theory of the transformation of quadratic and bilinear forms, to which he devoted two important papers (1858, 1868) that were to prove of great consequence to Frobenius (as indicated below). Weierstrass, who became Frobenius' mentor and dissertation advisor, is pictured in Fig. 1.2.

[2] A listing of the semesters in which Weierstrass lectured on abelian integrals and functions is given in vol. 3 of his *Mathematische Werke*, the volume containing of a version of these lectures.

The third member of the Berlin mathematical triumvirate was Leopold Kronecker (1823–1891). Although Kronecker had studied mathematics at Berlin and obtained a doctorate under the direction of Dirichlet in 1845, he did not pursue a traditional academic career thereafter. Instead he managed the considerable wealth of his family while at the same time indulging his interests in mathematics. These interests were shaped more by Kummer than by Dirichlet. Kummer had known Kronecker since his high-school days at the gymnasium in Liegnitz (now Legnica, in Poland), where he was then teaching, and the bond between them had developed into a friendship based on a mutual interest in mathematics. As Frobenius wrote many years later, "In so far as Kronecker was a student, he was Kummer's student ..." [202, p. 710]. Indeed, one of Kronecker's goals was to extend a version of Kummer's theory of ideal numbers from the context of $\mathbb{Q}(\omega_p)$ to that of far more general fields.[3]

Kronecker's interest in algebra, and in particular in algebraic equations, had been stimulated by the posthumous publication in 1846 of Évariste Galois' "Memoir on the conditions for the solvability of equations by radicals" [239]. Kronecker, however, did not find Galois' group-theoretic conditions for solvability satisfying [613, pp. 121ff.]. Galois had not described the nature of a polynomial $f(x)$ solvable by radicals; he had only characterized the associated (Galois) group G by the property (stated here in modern terms) of the existence of a chain of subgroups $G \supset G_1 \supset G_2 \supset \cdots \supset G_k = 1$ such that G_i is a normal subgroup of G_{i-1} and the factor group G_{i-1}/G_i is abelian. Kronecker wanted to characterize the equations solvable by radicals in a manner relating more directly to their coefficients and roots. Galois' results showed that it sufficed to do this for abelian equations, i.e., polynomial equations with abelian Galois group. Kronecker obtained many results of considerable interest, including what has become known as the Kronecker–Weber theorem: if $f(x) \in \mathbb{Q}[x]$ is abelian, then all its roots are rational functions of roots of unity.[4]

By the time Frobenius arrived in Berlin, Kronecker, who is pictured in Fig. 1.3, had been residing there for a dozen years and was very much engaged in the mathematical life of the university. He was a member of the Berlin Academy of Sciences, and at Kummer's suggestion, he claimed the right accorded members of presenting lectures at the university. Indeed, along with Kummer and Weierstrass, he was involved in the 2-year lecture cycle that had been devised to provide students, within that time span, with a broad and up-to-date education in what were deemed the main areas of mathematics. Thus, although Kronecker was not a full professor, and so could not officially direct doctoral theses or run the Berlin mathematics

[3] According to Frobenius [202, p. 712], in 1858 Kronecker sent a (now lost) manuscript containing some such extension to Dirichlet and Kummer. In 1882, on the occasion of the fiftieth anniversary of Kummer's doctorate, he presented a sketch of his ideas as they stood then [363], but they were difficult to follow. Edwards [150] has given a possible reconstruction of what Kronecker had in mind.

[4] See Frobenius' discussion of this and related results [202, pp. 712–713].

Fig. 1.3 Through his publications Kronecker was to exert a strong influence on Frobenius, who found in Kronecker's sketchy communications many problems to investigate and resolve. Photo courtesy of Mathematisches Forschungsinsitut, Oberwolfach, Germany

seminar, he played an important role in the educational program of the Berlin school. Reflecting his primary interests in algebra and the theory of numbers, Kronecker's lectures covered such areas as the theory of algebraic equations, number theory, the theory of determinants (including much that would now be classified as linear algebra), and the theory of simple and multiple integrals [22, p. 81]. It was probably through Kronecker that the young Frobenius learned about Galois' memoir [239], which he seems to have carefully studied along with the related works by Abel. Indeed, as we shall see in the following section, Frobenius' first paper beyond his dissertation involved the ideas of Abel and Galois, albeit interpreted within the context of Weierstrassian complex function theory. Kronecker was to exert a major influence on the directions taken by Frobenius' research.

Kummer's lectures covered analytic geometry, mechanics, the theory of surfaces, and number theory. His polished lectures were relatively easy to follow and never ventured beyond well-established theories, unlike both those of Kronecker, whose lectures were very difficult to follow, and those of Weierstrass, whose lectures were challenging but more accessible, the result of an ongoing effort to present the material in a rigorous and appropriate manner. In addition to abelian integrals and functions, Weierstrass' lectures were on such topics as the foundations of the theory of analytic functions, elliptic integrals and functions, the calculus of variations, and applications of elliptic functions to problems in geometry and mechanics. In his lecture cycle Weierstrass strove to present a rigorous development of mathematical analysis that took nothing for granted [22, p. 77]. Although other courses in mathematics were taught by assistant professors, instructors, and other special lecturers, Frobenius managed to take all his mathematics courses from Kummer, Weierstrass, and Kronecker.[5] In addition to courses in mathematics, he took courses in physics and philosophy.

Kronecker's role within the Berlin school was not at all limited to teaching. The Berlin Academy of Sciences afforded him a forum for mathematical discourse with his colleagues. As Frobenius later wrote,

> Nothing gave him more pleasure than to share his views on mathematical works or problems with his colleagues. Insatiable in his craving for scientific discourse, he could hold forth on his ideas until deep into the night with those who listened with intelligence and comprehension; and he who could not be convinced could be certain that already the next morning he would find a write-up of the matter discussed [202, p. 711].

Although in later years, relations between Kronecker and Weierstrass became strained to the breaking point [22, pp. 100ff.], during the period 1868–1875, when Frobenius was in Berlin, the situation was quite different. Even after years of strained relations, Weierstrass retained vivid, fond memories of the early years when he and Kronecker freely exchanged mathematical ideas [202, p. 711]. Several examples of such mutual interaction will be given in the following chapters because they inspired work by Frobenius. By far the most important example of their mutual interaction involved the theory of the transformation of quadratic and bilinear forms, the subject of Part II.

In Part II, I set the stage for, and then present, the work of Weierstrass and Kronecker that informed the Berlin school's approach to what would now be classified as linear algebra. As we shall see, much of Frobenius' work in the early years after his departure from Berlin in 1874 involved, in one way or another, Berlin style linear algebra. In developing and applying linear algebra to diverse mathematical problems drawn from analysis and arithmetic, Frobenius made extensive use of Weierstrass' theory of elementary divisors and the disciplinary ideals implicit in it and made explicit by Kronecker. Weierstrass' theory originated in his interest in the eighteenth-century discussion by Lagrange and Laplace of mechanical problems leading to a system of linear differential equations, which in modern notation would be expressed by $B\ddot{y} + Ay = 0$, where A, B are symmetric matrices, B is positive definite, and $y = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}^t$. Lagrange and Laplace had inherited the method of algebraic analysis that had revolutionized seventeenth-century mathematics. They perfected the method and made brilliant applications of it to terrestrial and celestial mechanics. In their hands, mathematical analysis became both elegant and general in its scope, but it retained the tendency to reason with algebraic symbols as if they had "general" values, a tendency that obscured the possibility of special relations or singularities that need to be considered in order to attain truly general results. In this way they came to believe that the above system of equations would yield stable solutions only if it were assumed, in addition, that the roots of $f(s) = \det(sB - A)$ are real and distinct. Building upon work of Cauchy, Dirichlet, and Jacobi, Weierstrass showed in 1858 that the additional

[5] This is clear from the vita at the end of his doctoral dissertation [171, p. 34], as is the fact that his other courses at Berlin were in physics and philosophy.

assumptions were unnecessary, that by means of rigorous, nongeneric reasoning it could be established that (in the language of matrices introduced later by Frobenius) a real nonsingular matrix P exists such that $P^tBP = I$ and $P^tAP = D$, where D is a diagonal matrix. From this result the correct form for the solutions to $B\ddot{y} + Ay = 0$ then followed, and their stability was established.

Weierstrass presented his results in a paper published by the Berlin Academy in 1858. He had couched his results in the language of the transformation of quadratic forms so as to relate them to the work of Cauchy and Jacobi, which had been motivated by the principal axes theorems of mechanics and the theory of quadric surfaces. Cauchy had been the earliest critic of the generic reasoning of the eighteenth century, whereas Jacobi continued, despite Cauchy's example, to pursue his characteristically elegant form of algebra on the generic level. Jacobi's work on the simultaneous transformation of pairs of quadratic forms into sums of squared terms raised the question as to when, in nongeneric terms, two pairs of quadratic or bilinear forms can be transformed into one another. Weierstrass discovered that the method he had employed in his 1858 paper could be generalized to apply to all nonsingular pairs of bilinear forms so as to give necessary and sufficient conditions for the transformation of one nonsingular pair into another. The result was his theory of elementary divisors, which he presented to the Berlin Academy in a paper of 1868. It will be helpful to briefly explain Weierstrass' theory at this point so as to make the later sections of Part I intelligible to readers unfamiliar with the language of elementary divisors.

Following Frobenius (Section 7.5), I will identify pairs of bilinear forms $\Phi = x^tBy$ and $\Psi = x^tAy$ with their coefficient matrices B,A. Such a pair can be simultaneously transformed into another pair $\tilde{B},\tilde{A}$ if nonsingular matrices P,Q exist such that $PAQ = \tilde{A}$ and $PBQ = \tilde{B}$. In this case, the pairs (B,A) and $(\tilde{B},\tilde{A})$ are said to be *equivalent*. A pair (B,A) is *nonsingular* if $\det B \neq 0$. Weierstrass' main theorem was that two nonsingular pairs are equivalent if and only if they have the same "elementary divisors." To understand what elementary divisors are, observe that pairs (B,A) and $(\tilde{B},\tilde{A})$ are clearly equivalent precisely when the matrix families $sB - A$ and $s\tilde{B} - \tilde{A}$, s a complex variable, are equivalent. By means of determinant-theoretic considerations, Weierstrass introduced a sequence of polynomials $E_n(s),\dots,E_1(s)$ associated to $sB-A$, which, thanks to Frobenius (Chapter 8), can be seen to be the *invariant factors* of $sB-A$ with respect to the polynomial ring $\mathbb{C}[s]$. That is, the Smith normal form of $sB-A$ over $\mathbb{C}[s]$ is the diagonal matrix with $E_n(s),\dots,E_1(s)$ down the diagonal. They satisfy $E_i(s) \mid E_{i+1}(s)$ and $\det(sB-A) = \prod_{i=1}^n E_i(s)$. The $E_i(s)$ of course factor into linear factors over $\mathbb{C}$, and so Weierstrass wrote $E_i(s) = \prod_{j=1}^k (s-a_j)^{m_{ij}}$, where $a_1,\dots,a_k$ denote the distinct roots of $\varphi(s) = \det(sB-A)$ and m_{ij} is a nonnegative integer. The factors $(s-a_i)^{m_{ij}}$ with $m_{ij} > 0$ are Weierstrass' elementary divisors. They are thus the powers of the distinct prime factors of each invariant factor $E_i(s)$ in the polynomial ring $\mathbb{C}[s]$. In order to prove his main theorem, Weierstrass showed that given $sB-A$ (with $\det B \neq 0$), P,Q may be determined such that $W = P(sB-A)Q$ has a simple form from which its elementary divisors can be immediately ascertained. The matrix W is essentially the same as the familiar Jordan canonical form of $sB-A$, which was introduced independently by Camille Jordan at about the same time. The determinant of each "Jordan block" of W equals an elementary divisor $(s-a_i)^{m_{ij}}$ of $sB-A$ with m_{ij} giving the dimension of the block.

For a corollary to his theory, Weierstrass returned to the context of his paper of 1858: pairs (B,A) of symmetric matrices; but he was now able to replace his earlier hypothesis that B is definite with the weaker one that B is nonsingular. His corollary was that two such symmetric pairs, (B,A) and $(\tilde{B},\tilde{A})$, are *congruent* in the sense that a nonsingular P exists such that $P^tBP=\tilde{B}$ and $P^tAP=\tilde{A}$ if and only if $sB-A$ and $s\tilde{B}-\tilde{A}$ have the same elementary divisors. His corollary provided a rigorous counterpoint to Jacobi's generic theorem that a symmetric pair is congruent to a pair of diagonal matrices. It should also be noted (as Frobenius did) that Weierstrass' theory also provided necessary and sufficient conditions that two matrices A and $\tilde{A}$ be *similar* in the sense that $A=S^{-1}\tilde{A}S$ for some nonsingular matrix S. It is only necessary to apply Weierstrass' main theorem to pairs with $B=\tilde{B}=I$.

Kronecker was not a member of the Berlin Academy in 1858 when Weierstrass presented his paper on pairs of quadratic forms, and he was apparently unfamiliar with it when, in 1866, while working with Weierstrass' encouragement on the problem of a viable generalization to abelian functions of the notion of elliptic functions admitting a complex multiplication, he became interested in the problem of the congruent transformation of the special families $sB-B^t$, where B is any $2n\times 2n$ matrix, into a normal form. This problem had emerged from his method of attacking the complex multiplication problem and seems to have overshadowed the original problem in his mind. He dealt with it in the generic spirit of Jacobi, but when he became aware of Weierstrass' paper of 1858 and his subsequent theory of elementary divisors, he abandoned the generic approach and took upon himself the highly nontrivial task of extending Weierstrass' theory to singular pairs (A,B), i.e., the problem of determining a set of invariants for *any* matrix pair (B,A) that would provide necessary and sufficient conditions for two such pairs to be equivalent. (The invariant factors $E_i(s)$ alone are insufficient to this end.)

Kronecker worked on this problem during 1868–1874 (see Section 5.6), while Frobenius was in Berlin, and he succeeded in solving it, although he only sketched his solution for symmetric pairs (see Theorem 5.13) and then, returning to his work of 1866, he developed from scratch the necessary and sufficient conditions for $sB+B^t$ and $s\tilde{B}+\tilde{B}^t$ to be congruent: $P^t(sB-B^t)P=s\tilde{B}+\tilde{B}^t$, $\det P\neq 0$. It was in these communications to the Berlin Academy that Kronecker—embroiled in a quarrel with Camille Jordan stemming from the latter's criticism of Weierstrass' theory of elementary divisors—explicitly articulated two disciplinary ideals that were implicit in Weierstrass' and his own work on the transformation of forms. Frobenius knew of this work and these ideals, and as we shall see (especially in Chapters 6 and 7), they served to motivate and inform his choice of research problems, the solutions to which involved the creation of mathematics of lasting significance (as indicated in subsequent chapters of Part I).

Weierstrass' paper on elementary divisors appeared in the proceedings of the Berlin Academy during Frobenius' second year at the university, although it is unlikely that he was then aware of it. Weierstrass, however, had already become aware of him. One of the customs at Berlin was for one of the full professors to pose a mathematical prize problem. In 1868, during Frobenius' second semester at Berlin, it was Weierstrass who posed the problem. Seven students submitted solutions, and one of them was Frobenius. He did not win the prize, but he did receive an honorable mention and a prize of 50 thaler [22, p. 88].[6] What had impressed Weierstrass was the uncommon facility for mathematical calculations his solution displayed [22, p. 190], a talent his other teachers had also observed. Whether the 18-year-old also had a talent for independent, original mathematical thought, however, remained to be seen. It was in the Berlin Mathematics Seminar that this talent seems to have first revealed itself. The Berlin Mathematics Seminar, the first seminar in pure mathematics in a German university, had been instituted by Kummer and Weierstrass in order to help students learn to think independently about the mathematics they had been learning [22, p. 72]. The seminar was open only to a limited number of pre- and postdoctoral students who were deemed qualified to benefit from it. No doubt Frobenius' solution to the prize problem was a major factor in his acceptance into the seminar at the beginning of his second year. In fact, he participated in the seminar for four semesters [171, p. 34] and so throughout his final 2 years as a student.

It was in the seminar that Weierstrass realized that Frobenius was much more than a mindless calculator. As he explained in 1872[7]:

> At first he [Frobenius] attracted the attention of his teachers by virtue of his extraordinary facility with mathematical calculations, which enabled him, already as a second-semester student, to solve a prize question posited by the faculty. However, it soon became clear that he possessed to a high degree the mental aptitude and capability necessary for original mathematical research. As a member of the mathematics seminar he produced various works, which would have been worthy of publication and were certainly not inferior in value to many recent publications. In the seminar, when it came to scientific matters, he always proved himself to be an independent thinker, although he was otherwise unassuming and almost childlike in manner. What was dictated to him in lectures he zealously made his own, but rather than being content with that, he always used what he had learned to determine his own scientific endeavors.

Weierstrass had further opportunity to observe Frobenius' talents as the director of his doctoral dissertation.

Incidentally, Frobenius' choice of Weierstrass over Kummer as dissertation advisor is not surprising. During the 9-year period 1867–1874 when Frobenius was in Berlin, 13 doctoral degrees were awarded, and eight of them were done under Weierstrass' direction. During this period, Weierstrass' area of research was no doubt perceived as more in the mainstream. Indicative of this is the fact that Lazarus Fuchs (1833–1902), who had received his doctorate under Kummer's

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[6] In 1875 Frobenius published a paper [176] that was an outgrowth of the work he had done on the prize problem.

[7] The occasion was a proposal to the Prussian minister of culture of a new associate professorship in mathematics, with Frobenius as the choice to fill the new position. The entire document is given by Biermann [22, pp. 189ff.]; the quotation below is from pp. 190–191.

direction in 1858, created quite a stir in Berlin in the period 1865–1868, when, after attending Weierstrass' lectures on abelian integrals and functions in 1863, he had applied Weierstrass' treatment of algebraic differential equations to an important class of linear homogeneous differential equations, thereby displaying the capacity for independent mathematical research needed to be habilitated as an instructor (*Privatdozent*) at the university. Eventually, Frobenius was drawn into the enterprise of developing Fuchs' theory, but not until after he had completed his doctoral dissertation and received his doctorate.

The subject of Frobenius' dissertation appears to have been of his own devising. Its starting point was the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(w)}{w-z}dw.$$

For f analytic in a suitable region including the circle C_ρ defined by $|w| = \rho$, it was well known that the integral formula could be used to derive the Laurent expansion of f by expanding the kernel $\frac{1}{w-z}$ in a geometric series and integrating $\frac{f(w)}{w-z}$ term by term. Frobenius' idea was to consider other expansions of this kernel, suitably chosen so that term-by-term integration of the uniformly convergent expansion would yield series expansions for $f(z)$ of the form $\sum c_n F_n(z)$ with the coefficients c_n given by integrals analogous to those that occur in the Laurent expansion. Frobenius' dissertation revealed a mathematician with a broad knowledge of complex analysis and the related theory of hypergeometric series and differential equations combined with an extraordinary ability to manipulate complicated analytical expressions so as to achieve interesting, original, and conclusive results, although these results were not really part of the mainstream mathematics of his day.[8]

Frobenius officially obtained his doctorate on 28 July 1870, and he had clearly impressed the Berlin faculty.[9] Weierstrass described his dissertation as "a first-rate work," one that was "distinguished by thoroughgoing studies" and "outstanding" in its form of presentation. It left Weierstrass convinced that its author "possesses a definite talent for independent research." Frobenius' oral doctoral examination (on 23 June 1870) was equally impressive. According to the records, Weierstrass asked the candidate questions on the theory of abelian functions and integrals and their theoretical basis in complex function theory—material that was central to Weierstrass' cycle of lectures at the university—and Frobenius showed himself to be "completely familiar with this difficult theory"— and was even able to present with detailed exactitude both complicated proofs and derivations, much to the satisfaction of the examining committee. Kummer then took over the questioning,

[8] A good description of Frobenius' impressive results can be found in Meyer Hamburger's review [259] of the German version of the Latin dissertation that Frobenius published three months later [172].

[9] See [22, p. 85], where the quotations below about Frobenius' dissertation and doctoral examination are given in the original German.

which covered questions about the application of the theory of elliptic functions in number theory and mechanics and ended with some questions about geometric problems. Here too the committee found the candidate well informed throughout. Unlike a typical mathematics doctoral candidate nowadays, Frobenius was also questioned in philosophy (by Harms) and physics (by Dove). He was judged to have a "deep and thorough" understanding of Kant's *Critique of Pure Reason* and to have satisfactorily answered Dove's questions relating to phenomena in the theory of heat. His overall performance earned him a pass summa cum laude.

A doctoral candidate had to submit three theses, one of which he would be asked to defend at his oral doctoral examination. Frobenius' three theses [171, p. 34] were (in English translation from the Latin[10]): "(1) Kant did not support his thesis concerning time and space by sufficiently weighty arguments; (2) The theory of definite integrals ought to precede the treatment of differential calculus; (3) It is better for the elements of higher analysis to be taught in the schools than the elements of the more recent synthetic geometry." Judging by the committee's remarks about Frobenius' understanding of Kant's *Critique of Pure Reason*, it would seem that the committee chose the first thesis. The other two theses are of interest because they were essentially pedagogical in nature. They indicate Frobenius' early concern for the teaching of mathematics—what should be taught and how it should be presented. As we shall see, Frobenius proved to be an excellent teacher, and his creative mathematical work was always characterized by a concern for a clear and rigorous presentation in a form he deemed the most appropriate. These pedagogical tendencies of his mathematical output were probably reinforced by his exposure to Weierstrass' lectures. I believe they were one of the reasons why Frobenius' work proved to be influential.

1.2 Postdoctoral Years: 1870–1874

After obtaining his doctorate, as was the custom, Frobenius took, and passed, the examination required to become a secondary school teacher.[11] He also received an invitation from the University of Freiburg to habilitate there so as to become an instructor. To become an instructor, a further published proof of independent original mathematical work—called a *Habilitationsschrift*—was required. Undoubtedly on the basis of a glowing report from Weierstrass, Freiburg was offering him a generous remuneration and the promise of rapid advancement; but Frobenius declined the offer due to family matters. Instead, he spent a probationary year teaching at the Joachimstalische Gymnasium in Berlin, where he had himself studied, and he proved to be an excellent high-school teacher. An experienced schoolmaster who carefully observed him during that year reported that he possessed an unmistakable

[10] I am grateful to my colleague Dan Weiner for supplying the translations.

[11] The information in this paragraph is drawn from a document published by Biermann [22, p. 191].

inborn pedagogical talent. Having thus passed his probationary year with great success, he was given a regular teaching position at another high school in Berlin (the *Sophienrealschule*).

Despite his teaching duties, Frobenius sought to pursue a career as a mathematician. As a first step, within three months of passing his doctoral examination, he submitted a German language version of his dissertation to *Crelle's Journal* [172]. That it was accepted for publication was certainly due to the influence of Weierstrass, who praised the dissertation for its "many new ideas and results" [22, p. 191]. Still, the dissertation topic was not part of mainstream mathematics; it was something of a mathematical dead end. Frobenius seemed to realize this, for he never returned to the subject. Instead, he turned to a subject that had recently become of considerable interest in Berlin due to the work of Lazarus Fuchs (1833–1902) mentioned above: the application of the new complex function theory to the study of linear differential equations.

Stimulated by the landmark papers of Gauss (1812) and Riemann [495] on the hypergeometric differential equation, Fuchs combined Weierstrassian power series techniques and Weierstrass' theory of algebraic differential equations with the monodromy method introduced by Riemann to study, in groundbreaking papers of 1865–1868, linear homogeneous differential equations.[12] Fuchs had initially published his results in 1865, in the proceedings of the *Gewerbeschule* [236] (later to become the Berlin Technical Institute), where he was teaching. Weierstrass was impressed with Fuchs' results, seeing in them proof of the latter's capability for independent, original mathematical work [22, p. 94]. As a result, in 1866, Fuchs was appointed as an instructor at the university, his 1865 paper [236], with a version published in *Crelle's Journal* in 1866 [237], serving as *Habilitationsschrift*. He remained in that position until 1868 (the year after Frobenius arrived in Berlin), when he left for a professorship at the University of Greifswald. The instructorship vacated by Fuchs was filled by L. Wilhelm Thomé (1841–1910), who had received his doctorate under Weierstrass' direction in 1865.[13] Thomé was regarded an "an indispensable replacement for Fuchs" [22, p. 95], and in 1870, he was made an assistant professor. Thomé was indeed a replacement for Fuchs, since beginning in 1872, he began publishing papers related to Fuchs' theory and its generalization. Frobenius was familiar with the work of Fuchs and Thomé, and it was to Fuchs' theory that he decided to turn for a new direction in research.

Fuchs had studied linear differential equations of the form

$$L(y) = y^{(n)} + q_1(z)y^{(n-1)} + \cdots + q_n(z)y = 0, \tag{1.1}$$

[12] In discussing Fuchs' work, as well as the related work of Frobenius, I have drawn upon Gray's more definitive account [255, Chs. II–III].

[13] Thomé should not be confused with Carl Johannes Thomae (1840–1921), who also worked in complex function theory but had received his doctorate from Göttingen in 1864 and then spent two semesters attending Weierstrass' lectures in Berlin before becoming an instructor in Göttingen in 1866. In 1874 he became a full professor at the University of Freiburg, where he spent the rest of his career.

where $y^{(k)} = d^k y/dz^k$, and the coefficient functions $q_i(z)$ are meromorphic in a simply connected region of the complex plane and have at most a finite number of poles there. Thus the total number of singular points of the coefficients is finite. As Fuchs showed, the singular points of the coefficient functions are the only possible points of singularity of the solutions. One of his main achievements was to characterize those equations (1.1) with the property that in a neighborhood of a singular point $z = a$ of the coefficients, all solutions, when multiplied by $(z-a)^\rho$, for some complex number ρ, remain bounded. These later became known as *linear differential equations of the Fuchsian class*. Fuchs was able to establish a fundamental set of solutions in a neighborhood of such a singular point.

Fuchs made an observation in his paper [237, §6] that seems to have piqued Frobenius' interest in his theory. Fuchs observed that due to earlier work of Puiseux, it followed that the class of linear homogeneous differential equations all of whose solutions are algebraic was contained within the class of differential equations he had studied. Although Fuchs did not mention it, his observation suggested the problem of characterizing those differential equations of the Fuchsian class for which all solutions are algebraic functions. By 1871, Frobenius was thinking about this problem [173, p. 65]. One reason may have been due to work of Kummer's former student H.A. Schwarz (Ph.D., 1864), who since 1869 was a professor at the Zurich Polytechnic (now the Eidgenössische Technische Hochschule Zürich). In August 1871, Schwarz announced that he had solved the problem of determining the hypergeometric differential equations for which all solutions are algebraic functions. The solution involved some beautiful mathematics, and Weierstrass and his circle in Berlin were no doubt discussing Schwarz's work.[14] The hypergeometric equations were special second-order equations of the Fuchsian class.

Schwarz's work may have encouraged Frobenius to think about the analogous but far more formidable problem for nth-order equations of Fuchsian type. I believe this general problem appealed to him because he found it analogous to the problem solved by Galois in his "Memoir on the conditions for solvability of equations by radicals" [239]: characterize those polynomial equations $f(x) = 0$ that can be solved "algebraically," i.e., by means of radicals. The problem implied by Fuchs' paper was to characterize those linear differential equations $L(y) = y^{(n)} + q_1(z)y^{(n-1)} + \cdots + q_n(z)y = 0$ that can be *integrated algebraically* (as Frobenius later put it [177]) in the sense that all solutions are algebraic functions. Such differential equations even resembled polynomials with the kth powers x^k of the unknown being replaced by a kth derivative $y^{(k)}$ of the unknown function y. And the problem of characterizing the algebraically integrable ones seems analogous to the problem solved by Galois, namely to characterize those polynomial equations that are algebraically solvable in the sense of solvable by radicals.

It was natural for Frobenius, an algebraist at heart, but a student of Weierstrass as well, to look to Galois' theory for function-theoretic analogues of what he regarded as the most important elements of Galois work—the theory of the Galois group

[14]For an account of Schwarz's work see [255, pp. 70–77].

associated to a polynomial and the concept of the irreducibility of a polynomial (with respect to a field of known coefficients) [172, p. 65]—in order to deal with the problem of characterizing algebraically integrable equations $L(y) = 0$. Frobenius and—independently and quite differently—Sophus Lie were among the earliest mathematicians to consider how to apply Galois' ideas to differential equations.[15]

The starting point of the theory of groups in Galois' work was his construction of what later became known as a *Galois resolvent* V associated to a polynomial $f(x)$ of degree n with "known" coefficients and no multiple roots. Expressed using modern terminology, the assumption is that $f(x) \in \mathbb{K}[x]$, where $\mathbb{K} \supset \mathbb{Q}$ is the field of "known quantities." Let $a_1, \ldots, a_n$ denote the roots of $f(x)$ and $\mathbb{L} = \mathbb{K}(a_1, \ldots, a_n)$ the associated splitting field. Galois began by sketching a proof that constants $c_1, \ldots, c_n$ can be chosen from $\mathbb{K}$ in various ways, including as integers, such that $V = c_1a_1 + \cdots + c_na_n$ takes on $n!$ distinct numerical values when the roots $a_1, \ldots, a_n$ are subjected to all $n!$ possible permutations. Using this property of V, he was able to show that every root of $f(x)$ is a rational function of V with known coefficients, i.e., as we would put it, $\mathbb{L} = \mathbb{K}(V)$. Galois used V to define a set G of permutations of the roots $a_1, \ldots, a_n$ defining what he called the "group of the equation $f(x) = 0$." This group corresponds to the Galois group in the modern sense, although Galois' definition of it was complicated.[16]

Getting back to Frobenius, he realized that Galois' construction of the Galois resolvent V could be extended to the context of the above-mentioned problem suggested by Fuchs' paper. That is, if $L(y) = 0$ is a differential equation of the Fuchsian class with singular point $z = a \in \mathbb{C}$ and if all its solutions are algebraic functions of z, then, Frobenius realized, this means that a polynomial in y,

$$f(y,z) = a_m(z)y^m + \cdots + a_1(z)y + a_0(z), \tag{1.2}$$

of degree $m \geq n$, n the order of $L(y) = 0$, exists with rational functions $a_k(z)$ as coefficients such that in accordance with Weierstrass' theory, the m roots $y_k(z)$ defined locally by power series in $z - a$ and satisfying $f(y_k(z), z) = 0$ in a neighborhood of $z = a$ are all solutions to $L(y) = 0$; furthermore, n of these

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[15] Regarding Lie, see [276, Ch. 1]. For an accessible exposition of the modern approach to applying Galois' ideas to the theory of differential equations of the Fuchsian class, see [377].

[16] Let $g(x) \in \mathbb{K}[x]$ denote the minimal polynomial of V with $m = \deg g(x)$. Then since $\mathbb{K}(a_1, \ldots, a_n) = \mathbb{K}[V]$, each root a_i of $f(x)$ is uniquely expressible as a polynomial in V, $a_i = \phi_i(V)$, where $\phi(x) \in \mathbb{K}[x]$ has degree at most $m-1$. If $V', V'', \ldots, V^{(m-1)}$ are the other roots of the minimal polynomial $g(x)$, then G consists of m permutations $\sigma_1, \ldots, \sigma_m$ of $a_1, \ldots, a_n$ with σ_k the mapping that takes the root $a_i = \phi_i(V)$ to $\phi_i(V^{(k)})$, which is also a root $a_{i'}$ of $f(x)$, so $\sigma_k : a_i \to a_{i'}$ for $i = 1, \ldots, n$ and $k = 0, \ldots, m-1$. In the nineteenth century, permutations in the sense of mappings of a finite set of symbols were called "substitutions." In the example at hand, σ_k substituted the arrangement (or permutation) $a_{1'}, \ldots, a_{n'}$ for the original arrangement $a_1, \ldots, a_n$. Readers interested in a more detailed and historically accurate portrayal of Galois' ideas, including a detailed working out of Galois' sketchy remarks about the construction and properties of V, should consult Edwards' lucid exposition of Galois' memoir [148], which includes as appendix an annotated English translation of the memoir.