George E. Andrews
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Ramanujan's Lost Notebook

Part IV

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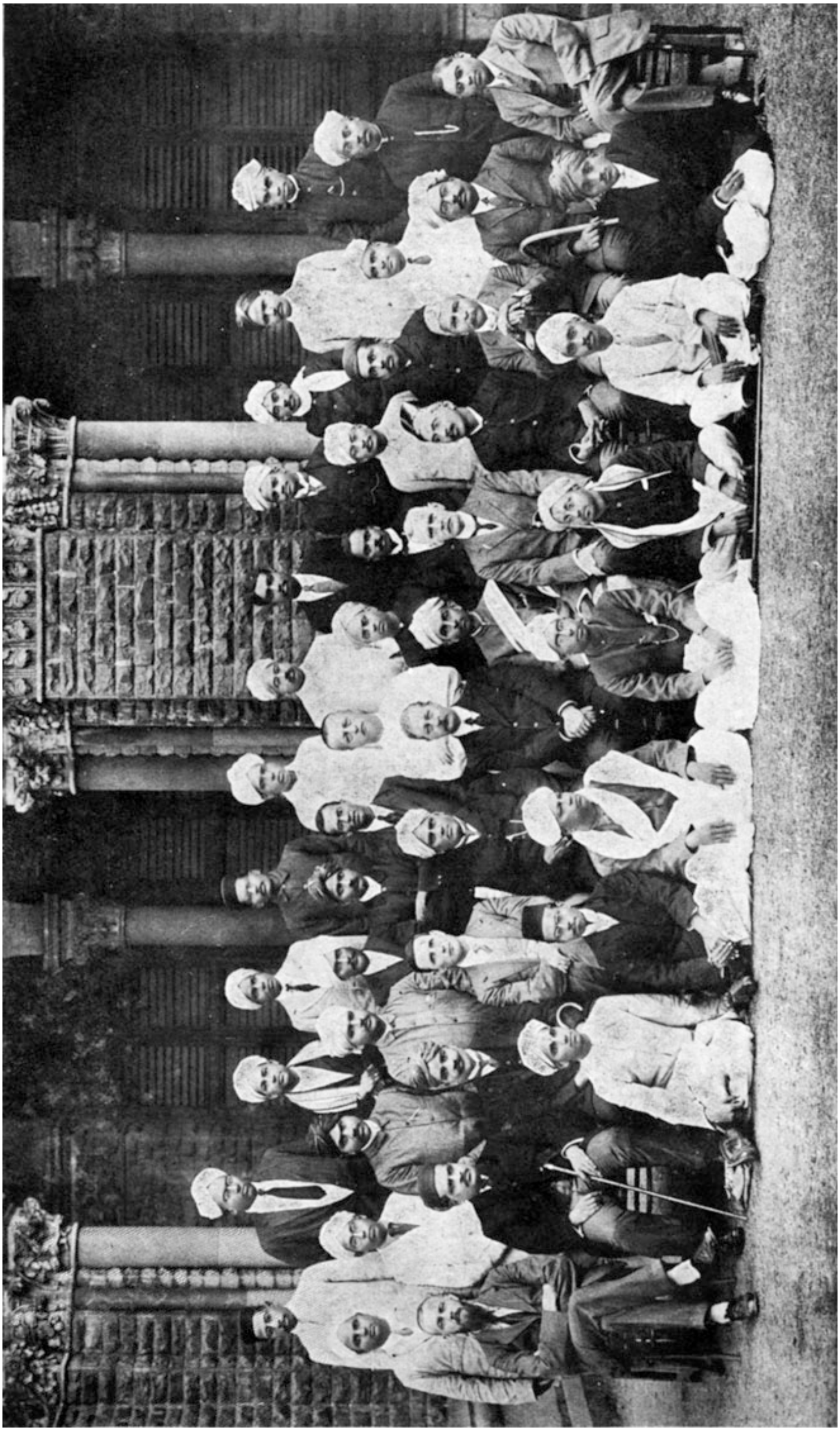
Appearing in this photograph are the participants at the second meeting of the Indian Mathematical Society on 11–13 January 1919 in Bombay. Several people important in the life of Ramanujan are pictured. Sitting on the ground are (third from left) S. Narayana Aiyar, Chief Accountant of the Madras Port Trust Office, and (fourth from left) P.V. Seshu Aiyar, Ramanujan's mathematics instructor at the Government College of Kumbakonam. Sitting in the chairs are (fifth from left) V. Ramaswami Aiyar, the founder of the Indian Mathematical Society, and (third from right) R. Ramachandra Rao, who provided a stipend for Ramanujan for 15 months. Standing in the third row is (second from left) S.R. Ranganathan, who wrote the first book-length biography of Ramanujan in English. Identifications of the remainder of the delegates in the photograph may be found in Volume 11 of the *Journal of the Indian Mathematical Society* or [65, p. 27].



It was not until today that I discovered at last what I had been so long searching for. The treasure hidden here is greater than that of the richest king in the world and to find it, the meaning of only one more sign had to be deciphered.

—Rabindranath Tagore, “The Hidden Treasure”

Preface

This is the fourth of five volumes that the authors are writing in their examination of all the claims made by S. Ramanujan in *The Lost Notebook and Other Unpublished Papers*. Published by Narosa in 1988, the treatise contains the "Lost Notebook," which was discovered by the first author in the spring of 1976 at the library of Trinity College, Cambridge. Also included in this publication are partial manuscripts, fragments, and letters that Ramanujan wrote to G.H. Hardy from nursing homes during 1917–1919. Although some of the claims examined in our fourth volume are found in the original lost notebook, most of the claims examined here are from the partial manuscripts and fragments. Classical analysis and classical analytic number theory are featured.

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Contents

1

Introduction

In contrast to our first three volumes [12–14] devoted to *Ramanujan's Lost Notebook and Other Unpublished Papers* [269], this volume does not focus on q-series. Number theory and classical analysis are in the spotlight in the present book, which is the fourth of five projected volumes, wherein the authors plan to discuss all the claims made by Ramanujan in [269]. As in our previous volumes, in the sequel, we liberally interpret *lost notebook* not only to include the original lost notebook found by the first author in the library at Trinity College, Cambridge, in March 1976, but also to include all of the material published in [269]. This includes letters that Ramanujan wrote to G.H. Hardy from nursing homes, several partial manuscripts, and miscellaneous papers. Some of these manuscripts are located at Oxford University, are in the handwriting of G.N. Watson, and are "copied from loose papers." However, it should be emphasized that the original manuscripts in Ramanujan's handwriting can be found at Trinity College Library, Cambridge.

We now relate some of the highlights in this volume, while at the same time offering our thanks to several mathematicians who helped prove some of these results.

Chapter 2 is devoted to two intriguing identities involving double series of Bessel functions found on page 335 of [269]. One is connected with the classical *circle problem*, while the other is conjoined to the equally famous *Dirichlet divisor problem.* The double series converge very slowly, and the identities were extremely difficult to prove. Initially, the second author and his collaborators, Sun Kim and Alexandru Zaharescu, were not able to prove the identities with the order of summation as prescribed by Ramanujan, i.e., the identities were proved with the order of summation reversed [57, 71]. It is possible that Ramanujan intended that the summation indices should tend to infinity "together." The three authors therefore also proved the two identities with the product of the summation indices tending to ∞ [57]. Finally, these authors proved Ramanujan's first identity with the order of summation as prescribed by Ramanujan [60]. It might be remarked here that the proofs under the three interpretations of the summation indices are entirely different; the authors

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did not use any idea from one proof in the proofs of the same identity under different interpretations. In Chap. 2, we provide proofs of the two identities with the order of summation indicated by Ramanujan in the first identity and with the order of summation reversed in the second identity. We also establish the identities when the product of the two indices of summation tends to infinity. In addition to thanking Sun Kim and Alexandru Zaharescu for their collaborations, the present authors also thank O-Yeat Chan, who performed several calculations to discern the convergence of these and related series.

It came as a huge surprise to us while examining pages in [269] when we espied famous formulas of N.S. Koshliakov and A.P. Guinand, although Ramanujan wrote them in slightly disguised forms. Moreover, we discovered that Ramanujan had found some consequences of these formulas that had not theretofore been found by any other authors. We are grateful to Yoonbok Lee and Jaebum Sohn for their collaboration on these formulas, which are the focus of Chap. 3.

Chapter 4, on the classical gamma function, features two sets of claims. We begin the chapter with some integrals involving the gamma function in the integrands. Secondly, we examine a claim that reverts to a problem [260] that Ramanujan submitted to the *Journal of the Indian Mathematical Society*, which was never completely solved. On page 339 in [269], Ramanujan offers a refinement of this problem, which was proved by the combined efforts of Ekaterina Karatsuba [177] and Horst Alzer [4].

Hypergeometric functions are featured in Chap. 5. This chapter contains two particularly interesting results. The first is an explicit representation for a quotient of two particular bilateral hypergeometric series, which was proved in a paper [50] by the second author and Wenchang Chu, whom we thank for his expert collaboration. We also appreciate correspondence with Tom Koornwinder about one particular formula on bilateral series that was crucial in our proof. Ramanujan's formula is so unexpected that no one but Ramanujan could have discovered it! The second is a beautiful continued fraction, for which Soon-Yi Kang, Sung-Geun Lim, and Sohn [175] found two entirely different proofs, each providing a different understanding of the entry. A further beautiful continued fraction of Ramanujan was only briefly examined in [175], but Kang supplied us with a very nice proof, which appears here for the first time.

Chapter 6 contains accounts of two incomplete manuscripts on Euler's constant γ, one of which was coauthored by the second author with Doug Bowman [46] and the other of which was coauthored by the second author with Tim Huber [55].

Sun Kim kindly collaborated with the second author on Chap. 7, on an unusual problem examined in a rough manuscript by Ramanujan on Diophantine approximation [56]. She also worked with the second author and Zaharescu on another partial manuscript providing the best possible Diophantine approximation to $e^{2/a}$, where a is any nonzero integer [61].

This manuscript was another huge surprise to us, for it had never been noticed by anyone, to the best of our knowledge, that Ramanujan had derived the best possible Diophantine approximation to $e^{2/a}$, which was first proved in print approximately 60 years after Ramanujan had found his proof. A third manuscript on Diophantine approximation in [269] turned out to be without substance, unless we have grossly misinterpreted Ramanujan's claims on page 343 of [269].

We next collect some results from number theory, not all of which are correct. At the beginning of Chap. 8, in Sect. 8.1, we relate that Ramanujan had anticipated the famous work of L.G. Sathe [275–278] and A. Selberg [281] on the distribution of primes, although Ramanujan did not state any specific theorems. In prime number theory, Dickman's function is a famous and useful function, but in Sect. 8.2, we see that Ramanujan had discovered Dickman's function at least 10 years before Dickman did in 1930 [106]. A.J. Hildebrand, a colleague of the second author, supplied a clever proof of Ramanujan's formula for, in standard notation, $\Psi(x, x^{\epsilon})$ and then provided us with a heuristic argument that might have been the approach used by Ramanujan. We then turn to a formula for $\zeta(\frac{1}{2})$, first given in Sect. 8 of Chap. 15 in Ramanujan's second notebook. In [269], Ramanujan offers an elegant reinterpretation of this formula, which renders an already intriguing result even more fascinating. Next, we examine a fragment on sums of powers that was very difficult to interpret; our account of this fragment is taken from a paper by D. Schultz and the second author [67]. One of the most interesting results in the chapter yields an unusual algorithm for generating solutions to Euler's diophantine equation $a^3 + b^3 = c^3 + d^3$. This result was established in different ways by Mike Hirschhorn in a series of papers [141, 158–160].

Chapter 9 is devoted to discarded fragments of manuscripts and partial manuscripts concerning the divisor functions $\sigma_k(n)$ and $d(n)$, respectively, the sum of the kth powers of the divisors of n, and the number of divisors of n. Some of this work is related to Ramanujan's paper [265]. An account of one of these fragments appeared in a paper that the second author coauthored with Prapanpong Pongsriiam [63].

In the next chapter, Chap. 10, we prove all of the results on page 196 of [269]. Two of the results evaluating certain Dirichlet series are especially interesting. A more detailed examination of these results can be found in a paper that the second author coauthored with Heng Huat Chan and Yoshio Tanigawa [47].

Chapter 11 contains some unusual old and new results on primes arranged in two rough, partial manuscripts. Ramanujan's manuscripts contain several errors, and we conjecture that this work predates his departure for England in 1914. Harold Diamond helped us enormously in both interpreting and correcting the claims made by Ramanujan in the two partial manuscripts examined in Chap. 11.

In Chap. 12, we discuss a manuscript that was either intended to be a paper by itself or, more probably, was slated to be the concluding portion of

Ramanujan's paper [263]. The results in this paper hark back to Ramanujan's early preoccupation with infinite series identities and the material in Chap. 14 of his second notebook [38, 268]. The second author had previously published an account of this manuscript [42]. Our account here includes a closer examination of two of Ramanujan's series by Johann Thiel, to whom we are very grateful for his contributions.

Perhaps the most fascinating formula found in the three manuscripts on Fourier analysis in the handwriting of Watson is a transformation formula involving the Riemann Ξ-function and the logarithmic derivative of the gamma function in Chap. 13. We are pleased to thank Atul Dixit, who collaborated with the second author on several proofs of this formula. One of the hallmarks of Ramanujan's mathematics is that it frequently generates further interesting mathematics, and this formula is no exception. In a series of papers [108–111], Dixit found analogues of this formula and found new bonds with the Ξ-function, in particular, with the beautiful formulas of Guinand and Koshliakov.

The second of the aforementioned manuscripts features integrals that possess transformation formulas like those satisfied by theta functions. Two of the integrals were examined by Ramanujan in two papers [256, 258], [267, pp. 59–67, 202–207], where he considered the integrals to be analogues of Gauss sums, a view that we corroborate in Chap. 14. One of the integrals, to which page 198 of [269] is devoted, was not examined earlier by Ramanujan. Ping Xu and the second author established Ramanujan's claims for this integral in [69]; the account given in Chap. 14 is slightly improved in places over that in [69]. (The authors are grateful to Noam Elkies for a historical note at the end of Sect. 14.1.)

In the third manuscript, on Fourier analysis, which we discuss in Chap. 15, Ramanujan considers some problems on Mellin transforms.

The next three chapters pertain to some of Ramanujan's earlier published papers. We then consider miscellaneous collections of results in classical analysis and elementary mathematics in the next two chapters.

Chapter 21 is devoted to some strange, partially incorrect claims of Ramanujan that likely originate from an early part of his career.

In summary, the second author is exceedingly obliged to his coauthors Doug Bowman, O.-Yeat Chan, Wenchang Chu, Atul Dixit, Tim Huber, Sun Kim, Yoonbok Lee, Sung-Geun Lim, Prapanpong Pongsriiam, Dan Schultz, Jaebum Sohn, Ping Xu, and Alexandru Zaharescu for their contributions.

As with earlier volumes, Jaebum Sohn carefully read several chapters and offered many corrections and helpful comments, for which we are especially grateful. Michael Somos offered several proofs for Chap. 20 and numerous corrections in several other chapters. Mike Hirschhorn also contributed many useful remarks.

We offer our gratitude to Harold Diamond, Andrew Granville, A.J. Hildebrand, Pieter Moree, Kannan Soundararajan, Gérald Tenenbaum, and Robert Vaughan for their comments that greatly enhanced our discussion of the

Sathe–Selberg results and Dickman's function in Sects. 8.1 and 8.2, respectively. Atul Dixit uncovered several papers by Guinand and Koshliakov of which the authors had not previously been aware. Most of Sect. 8.7 was kindly supplied to us by Jean-Louis Nicolas, with M.Tip Phaovibul also providing valuable insights. Useful correspondence with Ron Evans about Mersenne primes is greatly appreciated.

We offer our sincere thanks to Springer's TEX experts, Suresh Kumar and Rajiv Monsurate, for much technical advice, and to Springer copy editor David Kramer for several corrections and helpful remarks.

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2

Double Series of Bessel Functions and the Circle and Divisor Problems

2.1 Introduction

In this chapter we establish identities that express certain finite trigonometric sums as double series of Bessel functions. These results, stated in Entries 2.1.1 and 2.1.2 below, are identities claimed by Ramanujan on page 335 in his lost notebook [269], for which no indications of proofs are given. (Technically, page 335 is *not* in Ramanujan's lost notebook; this page is a fragment published by Narosa *with* the original lost notebook.) As we shall see in the sequel, the identities are intimately connected with the famous *circle* and *divisor problems*, respectively. The first identity involves the ordinary Bessel function $J_1(z)$, where the more general ordinary Bessel function $J_\nu(z)$ is defined by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}, \qquad 0 < |z| < \infty, \qquad \nu \in \mathbb{C}. \quad (2.1.1)$$

The second identity involves the Bessel function of the second kind $Y_1(z)$ [314, p. 64, Eq. (1)], with $Y_\nu(z)$ more generally defined by

$$Y_\nu(z) := \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \quad (2.1.2)$$

and the modified Bessel function $K_1(z)$, with $K_\nu(z)$ [314, p. 78, Eq. (6)] defined, for $-\pi < \arg z < \frac{1}{2}\pi$, by

$$K_\nu(z) := \frac{\pi}{2} \frac{e^{\pi i\nu/2} J_{-\nu}(iz) - e^{-\pi i\nu/2} J_\nu(iz)}{\sin(\nu\pi)}. \quad (2.1.3)$$

If ν is an integer n, then it is understood that we define the functions by taking the limits as $\nu \to n$ in (2.1.2) and (2.1.3).

To state Ramanujan's claims, we need to next define

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$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{2.1.4}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

Entry 2.1.1 (p. 335). *Let $F(x)$ be defined by (2.1.4). If $0 < \theta < 1$ and $x > 0$, then*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}. \tag{2.1.5}$$

Entry 2.1.2 (p. 335). *Let $F(x)$ be defined by (2.1.4). Then, for $x > 0$ and $0 < \theta < 1$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}} \right\}, \tag{2.1.6}$$

where

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z). \tag{2.1.7}$$

Ramanujan's formulation of (2.1.5) is given in the form

$$\begin{aligned} &\left[\frac{x}{1}\right]\sin(2\pi\theta) + \left[\frac{x}{2}\right]\sin(4\pi\theta) + \left[\frac{x}{3}\right]\sin(6\pi\theta) + \left[\frac{x}{4}\right]\sin(8\pi\theta) + \cdots \\ &= \pi x\left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) + \frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\left\{\frac{J_1(4\pi\sqrt{m\theta x})}{\sqrt{m\theta}} - \frac{J_1(4\pi\sqrt{m(1-\theta)x})}{\sqrt{m(1-\theta)}}\right. \\ &\left. + \frac{J_1(4\pi\sqrt{m(1+\theta)x})}{\sqrt{m(1+\theta)}} - \frac{J_1(4\pi\sqrt{m(2-\theta)x})}{\sqrt{m(2-\theta)}} + \frac{J_1(4\pi\sqrt{m(2+\theta)x})}{\sqrt{m(2+\theta)}} - \cdots\right\}, \end{aligned} \tag{2.1.8}$$

"where $[x]$ denotes the greatest integer in x if x is not an integer and $x - \frac{1}{2}$ if x is an integer." His formulation of (2.1.6) is similar. Since Ramanujan employed the notation $[x]$ in a nonstandard fashion, we think it is advisable to introduce the alternative notation (2.1.4). As we shall see in the sequel, there is some evidence that Ramanujan did not intend the double sums to be interpreted as iterated sums, but as double sums in which the product mn of the summation indices tends to ∞.

Note that the series on the left-hand sides of (2.1.5) and (2.1.6) are finite, and discontinuous if x is an integer. To examine the right-hand side of (2.1.5), we recall [314, p. 199] that, as $x \to \infty$,

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) + O\left(\frac{1}{x^{3/2}}\right). \tag{2.1.9}$$

Hence, as $m, n \to \infty$, the terms of the double series on the right-hand side of (2.1.5) are asymptotically equal to

$$\frac{1}{\pi\sqrt{2}x^{1/4}m^{3/4}} \left(\frac{\cos\left(4\pi\sqrt{m(n+\theta)x} - \frac{3}{4}\pi\right)}{(n+\theta)^{3/4}} - \frac{\cos\left(4\pi\sqrt{m(n+1-\theta)x} - \frac{3}{4}\pi\right)}{(n+1-\theta)^{3/4}} \right).$$

Thus, if indeed the double series on the right side of (2.1.5) does converge, it converges conditionally and not absolutely. A similar argument clearly pertains to (2.1.6).

We now discuss in detail Entry 2.1.1; our discourse will then be followed by a detailed account of Entry 2.1.2.

It is natural to ask what led Ramanujan to the double series on the right side of (2.1.5). Let $r_2(n)$ denote the number of representations of the positive integer n as a sum of two squares. Recall that the famous *circle problem* is to determine the precise order of magnitude, as $x \to \infty$, for the "error term" $P(x)$, defined by

$$\sum_{0 \le n \le x}{}^{\prime} r_2(n) = \pi x + P(x), \tag{2.1.10}$$

where the prime $\prime$ on the summation sign on the left side indicates that if x is an integer, only $\frac{1}{2}r_2(x)$ is counted. Moreover, we define $r_2(0) = 1$. In [144], Hardy showed that $P(x) \neq O(x^{1/4})$, as x tends to ∞. (He actually showed a slightly stronger result.)

In 1906, W. Sierpiński [288] proved that $P(x) = O(x^{1/3})$, as $x \to \infty$. After Sierpiński's work, most efforts toward obtaining an upper bound for $P(x)$ have ultimately rested upon the identity

$$\sum_{0 \le n \le x}{}^{\prime} r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}), \tag{2.1.11}$$

(2.1.9), and methods of estimating the resulting trigonometric series. Here, the prime $\prime$ on the summation sign on the left side has the same meaning as above. The identity (2.1.11) was first published and proved in Hardy's paper [144], [150, pp. 243–263]. In a footnote, Hardy [150, p. 245] remarks, "The form of this equation was suggested to me by Mr. S. Ramanujan, to whom I had communicated the analogous formula for $d(1)+d(2)+\cdots+d(n)$, where $d(n)$ is the number of divisors of n." Thus, it is possible that Ramanujan was the first to prove (2.1.11), although we do not know anything about his derivation.

Observe that the summands in the series on the right side of (2.1.11) are similar to those on the right side of (2.1.5). Moreover, the sums on the left side in each formula are finite sums over $n \leq x$. Thus, it seems plausible that there is a connection between these two formulas, and as we shall see, indeed there is. Ramanujan might therefore have derived (2.1.5) in anticipation of applying it to the circle problem.

In his paper [144], Hardy relates a beautiful identity of Ramanujan connected with $r_2(n)$, namely, for $a, b > 0$, [144, p. 283], [150, p. 263],

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{(n+a)b}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{(n+b)a}},$$

which is not given elsewhere in any of Ramanujan's published or unpublished work. If we differentiate the identity above with respect to b, let $a \to 0$, replace $2\pi\sqrt{b}$ by s, and use analytic continuation, we find that for $\operatorname{Re} s > 0$,

$$\sum_{n=1}^{\infty} r_2(n) e^{-s\sqrt{n}} = \frac{2\pi}{s^2} - 1 + 2\pi s \sum_{n=1}^{\infty} \frac{r_2(n)}{(s^2 + 4\pi^2 n)^{3/2}},$$

which was the key identity in Hardy's proof that $P(x) \neq O(x^{1/4})$, as $x \to \infty$.

In summary, there is considerable evidence that while Ramanujan was at Cambridge, he and Hardy discussed the *circle problem*, and it is likely that Entry 2.1.1 was motivated by these discussions.

Note that if the factors $\sin(2\pi n\theta)$ were missing on the left side of (2.1.5), then this sum would coincide with the number of integral points (n, l) with $n, l \geq 1$ and $nl \leq x$, where the pairs (n, l) satisfying $nl = x$ are counted with weight $\frac{1}{2}$. Hence,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) = {\sum_{1 \leq n \leq x}}' d(n), \tag{2.1.12}$$

where $d(n)$ denotes the number of divisors of n, and the prime $'$ on the summation sign indicates that if x is an integer, only $\frac{1}{2}d(x)$ is counted. Of course, similar remarks hold for the left side of (2.1.6). Therefore one may interpret the left sides of (2.1.5) and (2.1.6) as weighted divisor sums.

Berndt and A. Zaharescu [71] first proved Entry 2.1.1, but with the order of summation on the double sum *reversed* from that recorded by Ramanujan. The authors of [71] proved this emended version of Ramanujan's claim by first replacing Entry 2.1.1 with the following equivalent theorem.

Theorem 2.1.1. *For $0 < \theta < 1$ and $x > 0$,*

$$\begin{aligned}&\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) - \pi x\left(\frac{1}{2}-\theta\right)\\&= \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{n+\theta}\sin^2\left(\frac{\pi(n+\theta)x}{m}\right) - \frac{1}{n+1-\theta}\sin^2\left(\frac{\pi(n+1-\theta)x}{m}\right)\right).\end{aligned} \tag{2.1.13}$$

It should be emphasized that this reformulation fails to exist for Ramanujan's original formulation in Entry 2.1.1. After proving the aforementioned alternative version of Entry 2.1.1, the authors of [71] derived an identity involving the twisted character sums

$$d_\chi(n) = \sum_{k|n}\chi(k), \tag{2.1.14}$$

where χ is an odd primitive character modulo q. The following theorem on twisted character sums is proved in [71]; we have corrected the sign on the second expression on the right-hand side. The prime $'$ on the summation sign has the same meaning as it does in our discussions above, e.g., as in (2.1.10).

Theorem 2.1.2. *Let q be a positive integer, let χ be an odd primitive character modulo q, and let $d_\chi(n)$ be defined by (2.1.14). Then, for any $x > 0$,*

$$\begin{aligned}\sideset{}{'}\sum_{1\le n\le x} d_\chi(n) &= L(1,\chi)x + \frac{i\tau(\chi)}{2\pi}L(1,\bar\chi) + \frac{i\sqrt{x}}{\tau(\bar\chi)}\sum_{1\le h<q/2}\bar\chi(h)\\&\times\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{h}{q})x}\right)}{\sqrt{m(n+\frac{h}{q})}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\frac{h}{q})x}\right)}{\sqrt{m(n+1-\frac{h}{q})}}\right\},\end{aligned} \tag{2.1.15}$$

where $L(s,\chi)$ denotes the Dirichlet L-function associated with the character χ, and $\tau(\chi)$ denotes the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{q}\chi(m)e^{2\pi i m/q}. \tag{2.1.16}$$

Using Theorem 2.1.2, Berndt and Zaharescu [71] derived a representation for $\sum'_{0\le n\le x} r_2(n)$.

Corollary 2.1.1. *For any $x > 0$,*

$$\begin{aligned}&\sideset{}{'}\sum_{0\le n\le x} r_2(n) = \pi x\\&+ 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}}\right\}.\end{aligned} \tag{2.1.17}$$

A possible advantage in using (2.1.17) in the circle problem is that $r_2(n)$ does not occur on the right side of (2.1.17), as in (2.1.11). On the other hand, the double series is likely to be more difficult to estimate than a single infinite series.

The summands in (2.1.17) have a remarkable resemblance to those in (2.1.11). It is therefore natural to ask whether the two identities are equivalent. We next show that (2.1.11) and (2.1.17) are *formally* equivalent. The key to this equivalence is a famous result of Jacobi. Let χ be the nonprincipal Dirichlet character modulo 4. Then Jacobi's formula [167], [44, p. 56, Theorem 3.2.1] is given by

$$r_2(n) = 4 \sum_{\substack{d|n \\ d\,\mathrm{odd}}} (-1)^{(d-1)/2} =: 4d_\chi(n), \tag{2.1.18}$$

for all positive integers n. Therefore,

$$\begin{aligned}
&\sum_{k=1}^{\infty} r_2(k) \left(\frac{x}{k}\right)^{1/2} J_1(2\pi\sqrt{kx}) \\
&= 4\sum_{k=1}^{\infty} \sum_{\substack{d|k \\ d\text{ odd}}} (-1)^{(d-1)/2} \left(\frac{x}{k}\right)^{1/2} J_1(2\pi\sqrt{kx}) \\
&= 4\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left(\frac{J_1(2\pi\sqrt{m(4n+1)x})}{\sqrt{m(4n+1)}} - \frac{J_1(2\pi\sqrt{m(4n+3)x})}{\sqrt{m(4n+3)}} \right) \\
&= 2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left(\frac{J_1(4\pi\sqrt{m(n+\frac{1}{4})x})}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1(4\pi\sqrt{m(n+\frac{3}{4})x})}{\sqrt{m(n+\frac{3}{4})}} \right).
\end{aligned} \tag{2.1.19}$$

Hence, we have shown that (2.1.11) and (2.1.17) are versions of the same identity, *provided that the rearrangement of series in* (2.1.19) *is justified.* (J.L. Hafner [139] independently has also shown the formal equivalence of (2.1.11) and (2.1.17).)

In this chapter, we prove Entry 2.1.1 under two different interpretations, the first with the double series on the right-hand side summed in the order specified by Ramanujan, and the second with the double series on the right side interpreted as a double sum in which the product mn of the summation indices m and n tends to infinity. The former proof first appeared in a paper by Berndt, S. Kim, and Zaharescu [60], while the latter proof is taken from another paper [57] by the same trio of authors. We do not here give a proof of Entry 2.1.1 with the order of summation on the right-hand side of (2.1.5) reversed [71]. We emphasize that the three proofs of Entry 2.1.1 under different interpretations of the double sum on the right-hand side are entirely different; we are unable to use any portion or any idea of one proof in any of the other two proofs.