Kenji Iohara
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Symmetries, Integrable Systems and Representations

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Dedicated to Prof. Michio Jimbo for his Sixtieth Anniversary

Preface

This is the joint proceedings of the two conferences:

1. **Infinite Analysis 11—Frontier of Integrability—**
   University of Tokyo, Japan in July 25th to 29th, 2011,
2. **Symmetries, Integrable Systems and Representations**
   Université Claude Bernard Lyon 1, France in December 13th to 16th, 2011.

As both of the conferences had been organized in the occasion of 60th anniversary of Prof. Michio Jimbo, the topics covered in this proceedings are very large. Indeed, it includes combinatorics, differential equations, integrable systems, probability, representation theory, solvable lattice models, special functions etc. We hope this volume might be interesting and useful both for young researchers and experienced specialists in these domains.

We shall mention about the financial supports we had; the conference at Tokyo was supported in part by Global COE programme "The research and training center for new development in mathematics" (Graduate School of Mathematical Science, University of Tokyo), and the conference at Lyon was supported by Institut Universitaire de France, GDR 3395 'Théorie de Lie algébrique et géométrique', GDRE 571 'Representation theory', Université Lyon 1 and Université Paris 6.

Lion, France

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Contents

A Presentation of the Deformed $W_{1+\infty}$ Algebra

N. Arbesfeld and O. Schiffmann

Abstract We provide a generators and relation description of the deformed $W_{1+\infty}$-algebra introduced in previous joint work of E. Vasserot and the second author. This gives a presentation of the (spherical) cohomological Hall algebra of the one-loop quiver, or alternatively of the spherical degenerate double affine Hecke algebra of $GL(\infty)$.

1 Introduction

In the course of their work on the cohomology of the moduli space of $U(r)$-instantons on $\mathbb{P}^2$ in relation to W-algebras and the AGT conjecture (see [6]) E. Vasserot and the second author introduced a certain one-parameter deformation $\mathbf{SH}^{\mathbf{c}}$ of the enveloping algebra of the Lie algebra $W_{1+\infty}$ of algebraic differential operators on $\mathbb{C}^*$. The algebra $\mathbf{SH}^{\mathbf{c}}$—which is defined in terms of Cherednik's double affine Hecke algebras—acts on the above mentioned cohomology spaces (with a central character depending on the rank n of the instanton space). For the same value of the central character, $\mathbf{SH}^{\mathbf{c}}$ is also strongly related to the affine W algebra of type $\mathfrak{gl}_n$, and has the same representation theory (of admissible modules) as the latter. The same algebra $\mathbf{SH}^{\mathbf{c}}$ arises again as the (spherical) cohomological Hall algebra of the quiver with one vertex and one loop, and as a degeneration of the (spherical) elliptic Hall algebra (see [6, Sects. 4, 8]. It also independently appears in the work of Maulik and Okounkov on the AGT conjecture, see [5].

The definition of $\mathbf{SH}^{\mathbf{c}}$ given in [6] is in terms of a stable limit of spherical degenerate double affine Hecke algebras, and does not yield a presentation by generators and relations. In this note, we provide such a presentation, which bears some resemblance with Drinfeld's new realization of quantum affine algebras and Yangians.

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Namely, we show that $\mathbf{SH^c}$ is generated by families of elements in degrees $-1, 0, 1$, modulo some simple quadratic and cubic relations (see Theorems 1, 2).

The definition of $\mathbf{SH^c}$ is recalled in Sect. 2. In the short Sect. 3 we briefly recall the links between $\mathbf{SH^c}$ and Cherednik algebras, resp. W-algebras. The presentation of $\mathbf{SH^c}$ is given in Sect. 4, and proved in Sect. 5. Although we have tried to make this note as self-contained as possible, there are multiple references to statements in [6] and the reader is advised to consult that paper (especially Sects. 1 and 8) for details.

2 Definition of $\mathbf{SH^c}$

2.1 Symmetric Functions and Sekiguchi Operators

Let κ be a formal parameter, and let us set $F = \mathbb{C}(\kappa)$. Let us denote by Λ_F the ring of symmetric polynomials in infinitely many variables with coefficients in F, i.e.

$$\Lambda_F = F[X_1, X_2, \ldots]^{\mathfrak{S}_\infty} = F[p_1, p_2, \ldots].$$

For λ a partition, we denote by J_λ the integral form of the Jack polynomial associated to λ and to the parameter $\alpha = 1/\kappa$. The integral form J_λ is characterized by the following relation:

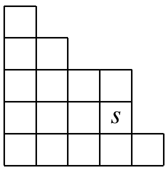
$$J_\lambda \in \bigoplus_{(1^n)<\mu\leqslant\lambda} F m_\mu + |\lambda|! m_{(1^n)}$$

where m_μ denotes the monomial symmetric function associated to a partition μ.

It is well-known that $\{J_\lambda\}$ forms a basis of Λ_F (see e.g. [7], or [6, Sects. 1.3, 1.6]). The polynomials J_λ arise as the joint spectrum of a family of commuting differential operators $\{D_{0,l}\}, l \geq 1$ called Sekiguchi operators. We will not need the expression of $D_{0,l}$ as a differential operator, but only their eigenvalues on the basis of Jack polynomials (which, of course, fully characterizes them):

$$D_{0,l}(J_\lambda) = \sum_{s\in\lambda} c(s)^{l-1} J_\lambda \tag{1}$$

where s runs through the set of boxes in the partition λ, and where $c(s) = x(s) - \kappa y(s)$ is the content of s. Here $x(s)$, $y(s)$ denote the x and y-coordinates of the box s, when λ is drawn according to the continental convention. For example, for the box s in the partition $(5, 4^2, 2, 1)$ depicted below



we have $x(s) = 3$ and $y(s) = 1$ hence $c(s) = 3 - \kappa$.

We denote by $D_{l,0} \in \mathrm{End}(\Lambda_F)$ the operator of multiplication by the power-sum function p_l.

2.2 The Algebras $\mathbf{SH}^+$ and $\mathbf{SH}^>$

Let $\mathbf{SH}^+$ be the unital subalgebra of $\mathrm{End}(\Lambda_F)$ generated by $\{D_{0,l}, D_{l,0} \mid l \geqslant 1\}$. For $l \geq 1$ we set $D_{1,l} = [D_{0,l+1}, D_{1,0}]$. This relation is still valid when $l = 0$, and we furthermore have

$$[D_{0,l}, D_{1,k}] = D_{1,k+l-1}, \quad l \geqslant 1, k \geqslant 0. \tag{2}$$

We denote by $\mathbf{SH}^>$ the unital subalgebra of $\mathbf{SH}^+$ generated by $\{D_{1,l} \mid l \geqslant 0\}$, and by $\mathbf{SH}^0$ the unital subalgebra of $\mathbf{SH}^+$ generated by the Sekiguchi operators $\{D_{0,l} \mid l \geqslant 1\}$. It is known (and easy to check from (1)) that the $D_{0,l}$ are algebraically independent, i.e. $\mathbf{SH}^0 = F[D_{0,1}, D_{0,2}, \ldots]$.

Observe that by (2), the operators $ad(D_{0,l})$ preserve the subalgebra $\mathbf{SH}^>$. This allows us to view $\mathbf{SH}^+$ as a semi-direct product of $\mathbf{SH}^0$ and $\mathbf{SH}^>$. In fact, the multiplication map induces an isomorphism

$$\mathbf{SH}^> \otimes \mathbf{SH}^0 \simeq \mathbf{SH}^+ \tag{3}$$

(see [6, Proposition 1.18]).

2.3 Grading and Filtration

The algebra $\mathbf{SH}^+$ carries an $\mathbb{N}$-grading, defined by setting $D_{0,l}$, $D_{1,k}$ in degrees zero and one respectively. This grading, which corresponds to the degrees as operators on polynomials will be called the *rank* grading. It also carries an $\mathbb{N}$-filtration compatible with the rank grading, induced from the filtration by the order of differential operators. It may alternatively be characterized as follows, see [6, Proposition 1.2]: $\mathbf{SH}^+[\leqslant d]$ is the space of elements $u \in \mathbf{SH}^>$ satisfying

$$ad(z_1) \circ \cdots \circ ad(z_{d+1})(u) = 0$$

for all $z_1, \ldots, z_{d+1} \in F[D_{1,0}, D_{2,0}, \ldots]$. We have $\mathbf{SH}^>[\leqslant 0] = F[D_{1,0}, D_{2,0}, \ldots]$. The following is proved in [6, Lemma 1.21]. Set $D_{r,d} = [D_{0,d+1}, D_{r,0}]$ for $r \geq 1, d \geq 0$.

Proposition 1 (i) *The associated graded algebra $gr\,\mathbf{SH}^+$ is equal to the free commutative polynomial algebra in the generators $D_{r,d} \in gr\,\mathbf{SH}^+[r,d]$, for $r \geqslant 0, d \geqslant 0$, $(r,d) \neq (0,0)$.*

(ii) *The associated graded algebra $gr\,\mathbf{SH}^>$ is equal to the free commutative polynomial algebra in the generators $D_{r,d} \in gr\,\mathbf{SH}^+[r,d]$, for $r \geqslant 1, d \geqslant 0$.*

We will need the following slight variant of the above result, which can easily be deduced from [6, Proposition 1.38]. For $r \geqslant 1$, set $D'_{r,d} = ad(D_{0,2})^d(D_{r,0})$. Then

$$D'_{r,d} \in r^{d-1} D_{r,d} \oplus \mathbf{SH}^{>}[r, \leqslant d-1]. \tag{4}$$

In particular, $gr\,\mathbf{SH}^{>}$ is also freely generated by the elements $D'_{r,d} \in gr\,\mathbf{SH}^{>}[r,d]$.

2.4 *The Algebra* $\mathbf{SH^c}$

Let $\mathbf{SH}^{<}$ be the opposite algebra of $\mathbf{SH}^{>}$. We denote the generator of $\mathbf{SH}^{>}$ corresponding to $D_{1,l}$ by $D_{-1,l}$. The algebra $\mathbf{SH^c}$ is generated by $\mathbf{SH}^{>}, \mathbf{SH}^0, \mathbf{SH}^{<}$ together with a family of central elements $\mathbf{c} = (c_0, c_1, \ldots)$ indexed by $\mathbb{N}$, modulo a certain set of relations involving the commutators $[D_{-1,k}, D_{1,l}]$ (see [6, Sect. 1. 8]). In order to write down these relations, we need a few notations. Set $\xi = 1 - \kappa$ and

$$G_0(s) = -\log(s), \qquad G_l(s) = \big(s^{-1} - 1\big)/l, \quad l \geqslant 1,$$

$$\varphi_l(s) = \sum_{q=1,-\xi,-\kappa} s^l \big(G_l(1-qs) - G_l(1+qs)\big), \quad l \geqslant 1,$$

$$\phi_l(s) = s^l G_l(1+\xi s).$$

We may now define $\mathbf{SH^c}$ as the algebra generated by $\mathbf{SH}^{>}, \mathbf{SH}^{<}, \mathbf{SH}^0$ and $F[c_0, c_1, \ldots]$ modulo the following relations:

$$[D_{0,l}, D_{1,k}] = D_{1,k+l-1}, \qquad [D_{-1,k}, D_{0,l}] = D_{-1,k+l-1}, \tag{5}$$

$$[D_{-1,k}, D_{1,l}] = E_{k+l}, \quad l,k \geqslant 0, \tag{6}$$

where the elements E_h are determined through the formulas

$$1 + \xi \sum_{l \geqslant 0} E_l s^{l+1} = \exp\Big(\sum_{l \geqslant 0} (-1)^{l+1} c_l \phi_l(s)\Big) \exp\Big(\sum_{l \geqslant 0} D_{0,l+1} \varphi_l(s)\Big). \tag{7}$$

Set $\mathbf{SH}^{0,\mathbf{c}} = \mathbf{SH}^0 \otimes F[c_0, c_1, \ldots]$. One can show that the multiplication map provides an isomorphism of F-vector spaces

$$\mathbf{SH}^{>} \otimes \mathbf{SH}^{0,\mathbf{c}} \otimes \mathbf{SH}^{<} \simeq \mathbf{SH^c}.$$

Putting the generators $D_{\pm 1,k}$ in degree ± 1 and the generators $D_{0,l}, c_i$ in degree zero induces an $\mathbb{Z}$-grading on $\mathbf{SH^c}$. One can show that the order filtration on $\mathbf{SH}^{>}, \mathbf{SH}^{<}$ can be extended to a filtration on the whole $\mathbf{SH^c}$, but we won't need this last fact.

3 Link to W-Algebras, Cherednik Algebras and Shuffle Algebras

3.1 Relation the Cherednik Algebras

Let ω be a new formal parameter and let $\mathbf{SH}^{\omega}$ be the specialization of $\mathbf{SH}$ at $c_0 = 0, c_i = -\kappa^i \omega^i$. Let $\mathbf{H}_n$ be Cherednik's degenerate (or trigonometric) double affine Hecke algebra with parameter κ (see [2]). Let $\mathbf{SH}_n \subset \mathbf{H}_n$ be its spherical subalgebra. The following result shows that $\mathbf{SH}^{\omega}$ may be thought of as the stable limit of $\mathbf{SH}_n$ as n goes to infinity (see [6, Sect. 1.7]):

Theorem *For any n there exists a surjective algebra homomorphism $\Phi_n : \mathbf{SH}^{\omega} \to \mathbf{SH}_n$ such that $\Phi_n(\omega) = n$. Moreover $\bigcap_n \mathrm{Ker}\,\Phi_n = \{0\}$.*

3.2 Realization as a Shuffle Algebra

Consider the rational function

$$g(z) = \frac{h(z)}{z}, \qquad h(z) = (z + 1 - \kappa)(z - 1)(z + \kappa).$$

Following [3], we may associate to $g(z)$ an $\mathbb{N}$-graded associative F-algebra $A_{g(z)}$, the *symmetric shuffle algebra of* $g(z)$ as follows. As a vector space,

$$A_{g(z)} = \bigoplus_{n \geqslant 0} A_{g(z)}[n], \qquad A_{g(z)}[n] = F[z_1, \dots, z_n]^{\mathfrak{S}_n}$$

with multiplication given by

$$P(z_1, \dots, z_r) \star Q(z_1, \dots, z_s) = \sum_{\sigma \in Sh_{r,s}} \sigma \cdot \Bigg(\prod_{\substack{1 \leqslant i \leqslant r \\ r+1 \leqslant j \leqslant r+s}} g(z_i - z_j) \cdot P(z_1, \dots, z_r) Q(z_{r+1}, \dots, z_{r+s}) \Bigg)$$

where $Sh_{r,s} \subset \mathfrak{S}_{r+s}$ is the set of (r, s) shuffles inside the symmetric group $\mathfrak{S}_{r+s}$. Let $S_{g(z)} \subseteq A_{g(z)}$ denote the subalgebra generated by $A_{g(z)}[1] = F[z_1]$. The restriction of the grading on $A_{g(z)}$ yields a grading $S_{g(z)} = \bigoplus_{n \geqslant 0} S_{g(z)}[n]$. The following is proved in [6, Cor. 6.4]:

Theorem *The assignment $S_{g(z)}[1] \ni z_1^l \mapsto D_{1,l}$, $l \geqslant 0$ induces an isomorphism of F-algebras*

$$S_{g(z)} \xrightarrow{\sim} \mathbf{SH}^{>}.$$

Remark The normalization used here differs slightly from [6]. Namely, the isomorphism in [6, Cor. 6.4] is between $\mathbf{SH}^{>}$ and the shuffle algebra associated to the rational function $\frac{1}{z}(z+x+y)(z-x)(z-y)$, where x and y are formal parameters satisfying $\kappa = -y/x$. In the present note, we have applied the transformation $z \mapsto z/x$, yielding the above isomorphism.

3.3 Relation to W-Algebras

Let $W_{1+\infty}$ be the universal central extension of the Lie algebra of all differential operators on $\mathbb{C}^*$ (see e.g. [4]). This is a $\mathbb{Z}$-graded and $\mathbb{N}$-filtered Lie algebra. The following result shows that **SH** may be thought of as a deformation of the universal enveloping algebra $U(W_{1+\infty})$ of $W_{1+\infty}$ (see [6, Appendix F]):

Theorem *The specialization of* $\mathbf{SH}^{\mathbf{c}}$ *at* $\kappa = 1$ *and* $c_i = 0$ *for* $i \geqslant 1$ *is isomorphic to* $U(W_{1+\infty})$.

More interesting is the fact that, for certain good choices of the parameters $c_0, c_1, \ldots$, a suitable completion of $\mathbf{SH}^{\mathbf{c}}$ is isomorphic to the current algebra of the (affine) W-algebra $W(\mathfrak{gl}_r)$ (see e.g. [1, Sect. 3.11]). We will not need this result, so we are a bit vague here and refer to [6, Sect. 8] for the full details. Fix an integer $r \geqslant 1$, $k \in \mathbb{C}$ and let $(\varepsilon_1, \ldots, \varepsilon_r)$ be new formal parameters. Let $\mathfrak{U}(W_k(\mathfrak{gl}_r))'$ be the formal current algebra of $W(\mathfrak{gl}_r)$ at level k, defined over the field $F(\varepsilon_1, \ldots, \varepsilon_r)$ (see [6, Sect. 8.4] for details). Let $\mathbf{SH}^{(r)}$ be the specialization of $\mathbf{SH}^{\mathbf{c}}$ to $\kappa = k + r$, $c_i = \varepsilon_1^i + \cdots + \varepsilon_r^i$ for $i \geqslant 0$. The following is proved in [6, Cor. 8.24], to which we refer for details.

Theorem *There is an embedding* $\mathbf{SH}^{(r)} \to \mathfrak{U}(W_k(\mathfrak{gl}_r))'$ *with a dense image, which induces an equivalence between the category of admissible* $\mathbf{SH}^{(r)}$*-modules and the category of admissible* $\mathfrak{U}(W_k(\mathfrak{gl}_r))'$*-modules.*

4 Presentation of $\mathbf{SH}^{+}$ and $\mathbf{SH}^{\mathbf{c}}$

4.1 Generators and Relations for $\mathbf{SH}^{+}$

Consider the F-algebra $\widetilde{\mathbf{SH}}^{+}$ generated by elements $\{\tilde{D}_{0,l} \mid l \geqslant 1\}$ and $\{\tilde{D}_{1,k} \mid k \geqslant 0\}$ subject to the following set of relations:

$$[\tilde{D}_{0,l}, \tilde{D}_{0,k}] = 0, \quad \forall l, k \geqslant 1, \tag{8}$$

$$[\tilde{D}_{0,l}, \tilde{D}_{1,k}] = \tilde{D}_{1,l+k-1}, \quad \forall l \geqslant 1, k \geqslant 0, \tag{9}$$

$$\big(3[\tilde{D}_{1,2},\tilde{D}_{1,1}]-[\tilde{D}_{1,3},\tilde{D}_{1,0}]+[\tilde{D}_{1,1},\tilde{D}_{1,0}]\big)+\kappa(\kappa-1)\big(\tilde{D}_{1,0}^2+[\tilde{D}_{1,1},\tilde{D}_{1,0}]\big)=0, \tag{10}$$

$$\big[\tilde{D}_{1,0},[\tilde{D}_{1,0},\tilde{D}_{1,1}]\big]=0. \tag{11}$$

Let $\widetilde{\mathbf{SH}}^0=F[\tilde{D}_{0,1},\tilde{D}_{0,2},\ldots]$ denote the subalgebra of $\widetilde{\mathbf{SH}}^+$ generated by $\tilde{D}_{0,l}$, $l\geqslant 1$, and let $\widetilde{\mathbf{SH}}^>$ be the subalgebra generated by $\tilde{D}_{1,k}$, $k\geqslant 0$. The algebras $\widetilde{\mathbf{SH}}^+$, $\widetilde{\mathbf{SH}}^0$, $\widetilde{\mathbf{SH}}^>$ are all $\mathbb{N}$-graded, where $\tilde{D}_{0,l}$ and $\tilde{D}_{1,k}$ are placed in degrees zero and one respectively. According to the terminology used for $\mathbf{SH}^+$, we call this grading the *rank grading*.

Theorem 1 *The assignment $\tilde{D}_{0,l}\mapsto D_{0,l}$, $\tilde{D}_{1,k}\mapsto D_{1,k}$ for $l\geqslant 1, k\geqslant 0$ induces an isomorphism of graded F-algebras*

$$\phi:\widetilde{\mathbf{SH}}^+\xrightarrow{\sim}\mathbf{SH}^+.$$

Obviously, the map ϕ restricts to isomorphisms $\widetilde{\mathbf{SH}}^0\simeq\mathbf{SH}^0$, $\widetilde{\mathbf{SH}}^>\simeq\mathbf{SH}^>$. Note however that $\widetilde{\mathbf{SH}}^>$ is *not* generated by the elements $\tilde{D}_{1,k}$ with the sole relations (10), (11). Theorem 1 is proved in Sect. 5.

4.2 Generators and Relations for $\mathbf{SH}^{\mathbf{c}}$

For the reader's convenience, we write down the presentation of $\mathbf{SH}^{\mathbf{c}}$, an immediate corollary of Theorem 1 above. Let $\widetilde{\mathbf{SH}}^{\mathbf{c}}$ be the algebra generated by elements $\{\tilde{D}_{0,l}\mid l\geqslant 1\}$, $\{\tilde{D}_{\pm1,k}\mid k\geqslant 0\}$ and $\{\tilde{\mathbf{c}}_i\mid i\geqslant 0\}$ subject to the following set of relations:

$$[\tilde{D}_{0,l},\tilde{D}_{0,k}]=0,\quad \forall l,k\geqslant 1, \tag{12}$$

$$[\tilde{D}_{0,l},\tilde{D}_{1,k}]=\tilde{D}_{1,l+k-1},\qquad [\tilde{D}_{-1,k},\tilde{D}_{0,l}]=\tilde{D}_{-1,l+k-1},\quad \forall l\geqslant 1,k\geqslant 0, \tag{13}$$

$$\big(3[\tilde{D}_{1,2},\tilde{D}_{1,1}]-[\tilde{D}_{1,3},\tilde{D}_{1,0}]+[\tilde{D}_{1,1},\tilde{D}_{1,0}]\big)+\kappa(\kappa-1)\big(\tilde{D}_{1,0}^2+[\tilde{D}_{1,1},\tilde{D}_{1,0}]\big)=0, \tag{14}$$

$$\begin{aligned}&\big(3[\tilde{D}_{-1,2},\tilde{D}_{-1,1}]-[\tilde{D}_{-1,3},\tilde{D}_{-1,0}]+[\tilde{D}_{-1,1},\tilde{D}_{-1,0}]\big)\\&\quad+\kappa(\kappa-1)\big(-\tilde{D}_{1,0}^2+[\tilde{D}_{-1,1},\tilde{D}_{-1,0}]\big)=0,\end{aligned} \tag{15}$$

$$\big[\tilde{D}_{1,0},[\tilde{D}_{1,0},\tilde{D}_{1,1}]\big]=0,\qquad \big[\tilde{D}_{-1,0},[\tilde{D}_{-1,0},\tilde{D}_{-1,1}]\big]=0, \tag{16}$$

$$[\tilde{D}_{-1,k},\tilde{D}_{1,l}]=\tilde{E}_{k+l},\quad l,k\geqslant 0, \tag{17}$$

where the $\tilde{E}_l$ are defined by the formula (7).

Theorem 2 *The assignment* $\tilde{D}_{0,l} \mapsto D_{0,l}, \tilde{D}_{\pm 1,k} \mapsto D_{\pm 1,k}$ *for* $l \geqslant 1, k \geqslant 0$ *and* $\tilde{\mathbf{c}}_i \mapsto \mathbf{c}_i$ *for* $i \geqslant 0$ *induces an isomorphism of* F*-algebras*

$$\phi : \widetilde{\mathbf{SH}}^{\mathbf{c}} \xrightarrow{\sim} \mathbf{SH}^{\mathbf{c}}.$$

Coupled with the Theorems in Sect. 3.3, this provides a potential'generators and relations' approach to the study of the category of admissible modules over the W-algebras $W_k(\mathfrak{gl}_r)$.

5 Proof of Theorem 1

5.1 First Reductions

Let us first observe that ϕ is a well-defined algebra map, i.e. that relations (8)–(11) hold in $\mathbf{SH}^+$. For (8), (9) this follows from the definition of $\mathbf{SH}^+$ and [6, (1.38)]. Equation (10) may be checked directly, e.g. from the Pieri rules (see [6, (1.26)]), or from the shuffle realization of $\mathbf{SH}^>$ (see Sect. 5.2 below). As for Eq. (11), we have by [6, (1.35)], $[[D_{1,1}, D_{1,0}], D_{1,0}] = [D_{2,0}, D_{1,0}] = 0$. The map ϕ is surjective by construction; in the rest of the proof, we show that it is injective as well.

Using relation (9) it is easy to see that any monomial in the generators $\tilde{D}_{0,l}, \tilde{D}_{1,k}$ may be expressed as a linear combination of similar monomials, in which all $\tilde{D}_{0,l}$ appear on the right of all $\tilde{D}_{1,k}$. Hence the multiplication map $\widetilde{\mathbf{SH}}^> \otimes \widetilde{\mathbf{SH}}^0 \to \widetilde{\mathbf{SH}}^+$ is surjective. Since ϕ clearly restricts to an isomorphism $\widetilde{\mathbf{SH}}^0 \simeq \mathbf{SH}^0$ we only have to show, by (3), that ϕ restricts to an isomorphism $\widetilde{\mathbf{SH}}^> \simeq \mathbf{SH}^>$. Our strategy will be to construct a suitable filtration on $\widetilde{\mathbf{SH}}^>$ mimicking the order filtration of $\mathbf{SH}^>$ and to pass to the associated graded algebras.

5.2 Verification in Ranks One and Two

We begin by proving directly, using the shuffle realization of $\mathbf{SH}^>$, that ϕ is an isomorphism in ranks one and two. This is obvious in rank one since ϕ is a graded map and the only relation in rank one is (9).

Suppose $\sum \alpha_i D_{1,k_i} D_{1,l_i} = 0$ is a relation in rank two. The shuffle realization then implies $\sum \alpha_i z^{k_i} \star z^{l_i} = 0$ so that

$$h(z_1 - z_2)\Big(\sum \alpha_i z_1^{k_i} z_2^{l_i}\Big) = h(z_2 - z_1)\Big(\sum \alpha_i z_1^{l_i} z_2^{k_i}\Big).$$

Therefore $\sum \alpha_i z_1^{k_i} z_2^{l_i} = h(z_2 - z_1) P(z_1, z_2)$ where $P(z_1, z_2)$ is some symmetric polynomial in z_1, z_2. Hence $\sum \alpha_i z_1^{k_i} z_2^{l_i}$ is a linear combination of polynomials of the form $h(z_2 - z_1)(z_1^k z_2^l + z_1^l z_2^l k)$ so that $\sum \alpha_i D_{1,k_i} D_{1,l_i}$ is a linear combination of expressions of the form

$$\begin{aligned}
&3[D_{1,l+2}, D_{1,k+1}] - 3[D_{1,l+1}, D_{1,k+2}] \\
&\quad - [D_{1,l+3}, D_{1,k}] + [D_{1,l}, D_{1,k+3}] + [D_{1,l+1}, D_{1,k}] - [D_{1,l}, D_{1,k+1}] \\
&\quad + \kappa(\kappa-1)\big(D_{1,k}D_{1,l} + D_{1,l}D_{1,k} + [D_{1,l+1}, D_{1,k}] - [D_{1,l}, D_{1,k+1}]\big). \quad (18)
\end{aligned}$$

If I denotes the image of (10) under the action of $F[ad\,\tilde{D}_{0,2}, ad\,\tilde{D}_{0,3}, \ldots]$ then using (9) we see that each such expression lies in $\phi(I)$ so that ϕ is indeed an isomorphism in rank two.

We remark that the relations (18) may be written in a more standard way using the generating functions $D(z) = \sum_l D_{1,l} z^{-l}$ as follows:

$$k(z-w)D(z)D(w) = -k(w-z)D(w)D(z) \quad (19)$$

where $k(u) = (u - 1 + \kappa)(u + 1)(u - \kappa) = -h(-u)$. In particular, the defining relation (10) may be replaced by the above (19), of which it is a special case.

5.3 *The Order Filtration on* $\widetilde{\mathbf{SH}}^{>}$

We now turn to the definition of the analog, on $\widetilde{\mathbf{SH}}^{>}$, of the order filtration on $\mathbf{SH}^{>}$. We will proceed by induction on the rank r. For $r = 1, d \geqslant 0$, we set

$$\widetilde{\mathbf{SH}}^{>}[1, \leqslant d] = \bigoplus_{k \leqslant d} F\tilde{D}_{1,k}.$$

Assuming that $\widetilde{\mathbf{SH}}^{>}[r', \leqslant d']$ has been defined for all $r' < r$ we let $\widetilde{\mathbf{SH}}^{>}[r, \leqslant d]$ be the subspace spanned by all products

$$\widetilde{\mathbf{SH}}^{>}\big[r', \leqslant d'\big] \cdot \widetilde{\mathbf{SH}}^{>}\big[r'', \leqslant d''\big], \quad r' + r'' = r, d' + d'' = d$$

and by the spaces

$$ad(\tilde{D}_{1,l})\big(\widetilde{\mathbf{SH}}^{>}[r-1, \leqslant d - l + 1]\big), \quad l = 0, \ldots, d+1.$$

From the above definition, it is clear that $\widetilde{\mathbf{SH}}^{>}$ is a $\mathbb{Z}$-filtered algebra. Note that it is not obvious at the moment that $\widetilde{\mathbf{SH}}^{>}[r, \leqslant d] = \{0\}$ for $d < 0$. Because the associated graded $gr\,\mathbf{SH}^{>}$ is commutative, it follows by induction on the rank r that $\phi : \widetilde{\mathbf{SH}}^{>} \to \mathbf{SH}^{>}$ is a morphism of filtered algebras. We denote by $gr\,\widetilde{\mathbf{SH}}^{>}$ the associated graded of $\widetilde{\mathbf{SH}}^{>}$ and we let $\overline{\phi} : gr\,\widetilde{\mathbf{SH}}^{>} \to gr\,\mathbf{SH}$ be the induced map. The map $\overline{\phi}$ is graded with respect to both rank and order. Moreover $\overline{\phi}$ is an isomorphism in ranks 1 and 2 (indeed, that the filtration as defined above coincides with the order filtration in rank 2 can be seen directly from [6, (1.84)]). The rest of the proof of

Theorem 1 consists in checking that $\overline{\phi}$ is an isomorphism. Once more, we will argue by induction. So in the remainder of the proof, *we fix an integer $r \geqslant 3$ and assume that $\overline{\phi}$ is an isomorphism in ranks $r' < r$.*

5.4 Commutativity of the Associated Graded

By our assumption above, the algebra $gr\widetilde{\mathbf{SH}}^>$ is commutative in ranks less than r, that is $ab = ba$ whenever $\text{rank}(a) + \text{rank}(b) < r$. Our first task is to extend this property to the rank r.

Lemma 1 *The algebra $gr\widetilde{\mathbf{SH}}^>$ is commutative in rank r.*

Proof We have to show that for $a \in \widetilde{\mathbf{SH}}^>[r_1, \leqslant d_1], b \in \widetilde{\mathbf{SH}}^>[r_2, \leqslant d_2]$ and $r_1 + r_2 = r$ we have

$$[a, b] \in \widetilde{\mathbf{SH}}^>[r, \leqslant d_1 + d_2 - 1]. \tag{20}$$

We argue by induction on r_1. If $r_1 = 1$ then (20) holds by definition of the filtration. Now let $r_1 > 1$ and let us further assume that (20) is valid for all r_1', r_2' with $r_1' + r_2' = r$ and $r_1' < r_1$. We will now prove (20) for r_1, r_2, thereby completing the induction step. According to the definition of the filtration, there are two cases to consider:

Case 1 We have $a = a_1a_2$ with $a_1 \in \widetilde{\mathbf{SH}}^>[s', \leqslant d'], a_2 \in \widetilde{\mathbf{SH}}^>[s'', \leqslant d'']$ such that $s' + s'' = r_1, d' + d'' = d_1$. Then $[a, b] = a_1[a_2, b] + [a_1, b]a_2$. By our induction hypothesis on r, $[a_2, b] \in \widetilde{\mathbf{SH}}^>[s'' + r_2, \leqslant d'' + d_2 - 1]$ hence $a_1[a_2, b] \in \widetilde{\mathbf{SH}}^>[r, \leqslant d_1 + d_2 - 1]$. The term $[a_1, b]a_2$ is dealt with in a similar fashion.

Case 2 We have $a = [\tilde{D}_{1,l}, a']$ with $a' \in \widetilde{\mathbf{SH}}^>[r_1 - 1, \leqslant d_1 - l + 1]$. Then $[a, b] = [[\tilde{D}_{1,l}, a'], b] = [\tilde{D}_{1,l}, [a', b]] - [a', [\tilde{D}_{1,l}, b]]$. By our induction hypothesis on r, $[a', b] \in \widetilde{\mathbf{SH}}^>[r_1 + r_2 - 1, \leqslant d_1 + d_2 - l]$ hence $[\tilde{D}_{1,l}, [a', b]] \in \widetilde{\mathbf{SH}}^>[r, \leqslant d_1 + d_2 - 1]$. Similarly, $[\tilde{D}_{1,l}, b] \in \widetilde{\mathbf{SH}}^>[r_2 + 1, \leqslant d_2 + l - 1]$. The inclusion $[a', [\tilde{D}_{1,l}, b]] \in \widetilde{\mathbf{SH}}^>[r, \leqslant d_1 + d_2 - 1]$ now follows from the induction hypothesis on r_1.

We are done. □

5.5 The Degree Zero Component

We now focus on the filtered piece of order $\leqslant 0$ of $\widetilde{\mathbf{SH}}^>$. We inductively define elements $\tilde{D}_{l,0}$ for $l \geqslant 2$ by

$$\tilde{D}_{l,0} = \frac{1}{l-1}[\tilde{D}_{1,1}, \tilde{D}_{l-1,0}].$$

From [6, (1.35)] we have $\phi(\tilde{D}_{l,0}) = D_{l,0}$. Since we assume are assuming that $\overline{\phi}$ is an isomorphism in ranks less than r, we have $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ whenever $l + l' < r$.

Lemma 2 *We have* $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ *for* $l + l' = r$.

Proof If $r = 3$ this reduces to the cubic relation (11). For $r = 4$ we have to consider

$$\begin{aligned}[\tilde{D}_{3,0}, \tilde{D}_{1,0}] &= \frac{1}{2}\big[[\tilde{D}_{1,1}, \tilde{D}_{2,0}], \tilde{D}_{1,0}\big] \\ &= \frac{1}{2}\big[\tilde{D}_{1,1}, [\tilde{D}_{2,0}, \tilde{D}_{1,0}]\big] - \frac{1}{2}\big[\tilde{D}_{2,0}, [\tilde{D}_{1,1}, \tilde{D}_{1,0}]\big] \\ &= -\frac{1}{2}[\tilde{D}_{2,0}, \tilde{D}_{2,0}] = 0.\end{aligned}$$

Now let us fix l, l' with $l + l' = r$. We have

$$\begin{aligned}[\tilde{D}_{l,0}, \tilde{D}_{l',0}] &= \frac{1}{l-1}\big[[\tilde{D}_{1,1}, \tilde{D}_{l-1,0}], \tilde{D}_{l',0}\big] \\ &= \frac{1}{l-1}\big[\tilde{D}_{1,1}, [\tilde{D}_{l-1,0}, \tilde{D}_{l',0}]\big] - \frac{1}{l-1}\big[\tilde{D}_{l-1,0}, [\tilde{D}_{1,1}, \tilde{D}_{l',0}]\big] \\ &= -\frac{l'}{l-1}[\tilde{D}_{l-1,0}, \tilde{D}_{l'+1,0}]. \qquad (21)\end{aligned}$$

If $r = 2k$ is even then by repeated use of (21) we get

$$[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = c[\tilde{D}_k, \tilde{D}_k] = 0$$

for some constant c. Next, suppose that $r = 2k + 1$ is odd, with $k \geqslant 2$. Applying $ad(\tilde{D}_{1,1})$ to $[\tilde{D}_{k+1,0}, \tilde{D}_{k-1,0}] = 0$ yields the relation

$$(k+1)[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] + (k-1)[\tilde{D}_{k+1,0}, \tilde{D}_{k,0}] = 0. \qquad (22)$$

Similarly, applying $ad(\tilde{D}_{2,1})$ to $[\tilde{D}_{k,0}, \tilde{D}_{k-1,0}] = 0$ and using the relation $[D_{k,1}, D_{l,0}] = kl D_{l+k,0}$ in $\mathbf{SH}^{>}$ (see [6, (1.91), (8.47)]) we obtain the relation

$$k[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] + (k-1)[\tilde{D}_{k,0}, \tilde{D}_{k+1,0}] = 0. \qquad (23)$$

Equations (22) and (23) imply that $[\tilde{D}_{k+2,0}, \tilde{D}_{k-1,0}] = [\tilde{D}_{k+1,0}, \tilde{D}_{k,0}] = 0$. The general case of $[\tilde{D}_{l,0}, \tilde{D}_{l',0}] = 0$ is now deduced, as in the case $r = 2k$, from repeated use of (21). □

Note that Lemma 2 above implies that $\widetilde{\mathbf{SH}}^{>}[r, \leqslant -1] = \{0\}$.

5.6 Completion of the Induction Step

Recall that $gr\,\mathbf{SH}^{>}$ is a free polynomial algebra in generators in the generators $D'_{s,d}$ for $s \geqslant 1, d \geqslant 0$. In order to prove that $\overline{\phi}$ is an isomorphism in rank r, it suffices, in virtue of Lemma 1, to show that the factor space

$$U_{r,d} = gr\,\widetilde{\mathbf{SH}}^{>}[r,d] \;\; / \;\left\{ \sum_{\substack{r'+r''=r \\ d'+d''=d}} gr\,\widetilde{\mathbf{SH}}^{>}\left[r',d'\right] \cdot gr\,\widetilde{\mathbf{SH}}^{>}\left[r'',d''\right] \right\}$$

is one dimensional for any $d \geqslant 0$. Let us set, for any $s \geqslant 1, d \geqslant 0$

$$\tilde{D}'_{s,d} = ad(\tilde{D}_{0,2})^d(\tilde{D}_{s,0}) \in \widetilde{\mathbf{SH}}^{>}[s, \leqslant d].$$

We will denote by the same symbol $\tilde{D}'_{s,d}$ the corresponding element of $gr\,\widetilde{\mathbf{SH}}^{>}[s,d]$. Note that $\tilde{D}'_{s,0} = \tilde{D}_{s,0}$ We claim that in fact $U_{r,d} = F\tilde{D}'_{r,d}$. Observe that $\phi(\tilde{D}'_{s,d}) = D'_{s,d}$ for any s, d, hence $\tilde{D}'_{s,d} \in U_{s,d}$ for any $s \leqslant r, d \geqslant 0$. Moreover, by our general induction hypothesis on r we have $U_{s,d} = F\tilde{D}'_{s,d}$ for any $s < r$ and $d \geqslant 0$.

We will prove that $U_{r,d} = F\tilde{D}'_{r,d}$ by induction on d. For $d = 0$, this comes from Lemma 2. So fix $d > 0$ and let us assume that $U_{r,l} = F\tilde{D}'_{r,l}$ for all $l < d$. By definition of the filtration on $\widetilde{\mathbf{SH}}^{>}$, $U_{r,d}$ is linearly spanned by the classes of the elements

$$\left[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}\right], \left[\tilde{D}_{1,1}, \tilde{D}'_{r-1,d}\right], \ldots, \left[\tilde{D}_{1,d+1}, \tilde{D}'_{r-1,0}\right].$$

By our induction hypothesis on d, the elements

$$\left[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d}\right], \left[\tilde{D}_{1,1}, \tilde{D}'_{r-1,d-1}\right], \ldots, \left[\tilde{D}_{1,d}, \tilde{D}'_{r-1,0}\right]$$

all belong to $F\tilde{D}'_{r,d-1} \oplus \widetilde{\mathbf{SH}}^{>}[r, \leqslant d-2]$. Applying $ad(\tilde{D}_{0,2})$, we see that

$$\left[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}\right] + \left[\tilde{D}_{1,1}, \tilde{D}'_{r-1,d}\right], \ldots, \left[\tilde{D}_{1,d}, \tilde{D}'_{r-1,1}\right] + \left[\tilde{D}_{1,d+1}, \tilde{D}'_{r-1,0}\right] \quad (24)$$

all belong to $F\tilde{D}'_{r,d} \oplus \widetilde{\mathbf{SH}}^{>}[r, \leqslant d-1]$. Next, applying $ad(\tilde{D}_{0,d+2})$ to the equality $[\tilde{D}_{1,0}, \tilde{D}_{r-1,0}] = 0$ yields

$$[\tilde{D}_{1,0}, \tilde{D}_{r-1,d+1}] + [\tilde{D}_{1,d+1}, \tilde{D}_{r-1,0}] = 0$$

which implies, by (4), that

$$\begin{aligned}\left[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}\right] &+ r^d[\tilde{D}_{1,d+1}, \tilde{D}_{r-1,0}] \\ &\in \left[\tilde{D}_{1,0}, \widetilde{\mathbf{SH}}^{>}[r-1, \leqslant d]\right] \subseteq \widetilde{\mathbf{SH}}^{>}[r, \leqslant d-1].\end{aligned} \quad (25)$$

The collection of inclusions (24), (25) may be considered as a system of linear equations in $U_{r,d}$ modulo $F\tilde{D}'_{r,d}$ in the variables $[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}], \ldots, [\tilde{D}_{1,d+1}, \tilde{D}'_{r-1,0}]$

whose associated matrix

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 1 & -r^d \end{pmatrix}$$

is invertible. We deduce that $[\tilde{D}_{1,0}, \tilde{D}'_{r-1,d+1}], \ldots, [\tilde{D}_{1,d+1}, \tilde{D}'_{r-1,0}]$ all belong to the space $F\tilde{D}'_{r,d} \oplus \widetilde{\mathbf{SH}}^{>}[r, \geqslant d-1]$ as wanted. This closes the induction step on d. We have therefore proved that $U_{r,d} = F\tilde{D}'_{r,d}$ for all $d \geqslant 0$, and hence that $\overline{\phi}$ and ϕ is an isomorphism in rank r. This closes the induction step on r. Theorem 1 is proved.

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Generating Series of the Poincaré Polynomials of Quasihomogeneous Hilbert Schemes

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Abstract In this paper we prove that the generating series of the Poincaré polynomials of quasihomogeneous Hilbert schemes of points in the plane has a beautiful decomposition into an infinite product. We also compute the generating series of the numbers of quasihomogeneous components in a moduli space of sheaves on the projective plane. The answer is given in terms of characters of the affine Lie algebra $\widehat{sl}_m$.

1 Introduction

The Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points in the plane $\mathbb{C}^2$ parametrizes ideals $I \subset \mathbb{C}[x,y]$ of colength n: $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n$. There is an open dense subset of $(\mathbb{C}^2)^{[n]}$, that parametrizes the ideals, associated with configurations of n distinct points. The Hilbert scheme of n points in the plane is a nonsingular, irreducible,

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quasiprojective algebraic variety of dimension $2n$ with a rich and much studied geometry, see [9, 22] for an introduction.

The cohomology groups of $(\mathbb{C}^2)^{[n]}$ were computed in [6] and we refer the reader to the papers [5, 15–17, 24] for the description of the ring structure in the cohomology $H^*((\mathbb{C}^2)^{[n]})$.

There is a $(\mathbb{C}^*)^2$-action on $(\mathbb{C}^2)^{[n]}$ that plays a central role in this subject. The algebraic torus $(\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ by scaling the coordinates, $(t_1, t_2) \cdot (x, y) = (t_1 x, t_2 y)$. This action lifts to the $(\mathbb{C}^*)^2$-action on the Hilbert scheme $(\mathbb{C}^2)^{[n]}$.

Let $T_{\alpha,\beta} = \{(t^\alpha, t^\beta) \in (\mathbb{C}^*)^2 | t \in \mathbb{C}^*\}$, where $\alpha, \beta \geq 1$ and $gcd(\alpha, \beta) = 1$, be a one dimensional subtorus of $(\mathbb{C}^*)^2$. The variety $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ parametrizes quasihomogeneous ideals of colength n in the ring $\mathbb{C}[x, y]$. Irreducible components of $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ were described in [7]. Poincaré polynomials of irreducible components in the case $\alpha = 1$ were computed in [3]. For $\alpha = \beta = 1$ it was done in [12].

For a manifold X let $H_*(X)$ denote the homology group of X with rational coefficients. Let $P_q(X) = \sum_{i\geq 0} \dim H_i(X) q^{\frac{i}{2}}$. The main result of this paper is the following theorem (it was conjectured in [3]):

Theorem 1

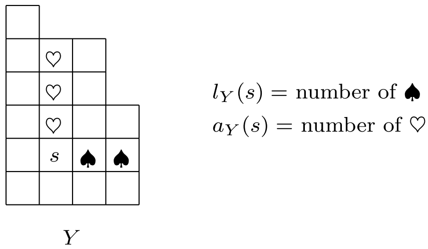
$$\sum_{n\geq 0} P_q\left(\left((\mathbb{C}^2)^{[n]}\right)^{T_{\alpha,\beta}}\right) t^n = \prod_{\substack{i\geq 1 \\ (\alpha+\beta)\nmid i}} \frac{1}{1-t^i} \prod_{i\geq 1} \frac{1}{1-qt^{(\alpha+\beta)i}}. \tag{1}$$

There is a standard method for constructing a cell decomposition of the Hilbert scheme $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ using the Bialynicki-Birula theorem. In this way the Poincaré polynomial of this Hilbert scheme can be written as a generating function for a certain statistic on Young diagrams of size n. However, it happens that this combinatorial approach doesn't help in a proof of Theorem 1. In fact, we get very nontrivial combinatorial identities as a corollary of this theorem, see Sect. 1.1.

We can describe the main geometric idea in the proof of Theorem 1 in the following way. The irreducible components of $((\mathbb{C}^2)^{[n]})^{T_{\alpha,\beta}}$ can be realized as fixed point sets of a $\mathbb{C}^*$-action on cyclic quiver varieties. Theorem 4 tells us that the Betti numbers of the fixed point set are equal to the shifted Betti numbers of the quiver variety. Then known results about cohomology of quiver varieties can be used for a proof of Theorem 1.

In principle, Theorem 4 has an independent interest. However, there is another application of this theorem. In [4] we studied the generating series of the numbers of quasihomogeneous components in a moduli space of sheaves on the projective plane. Combinatorially we managed to compute it only in the simplest case. Now using Theorem 4 we can give an answer in a general case, this is Theorem 5. We show that it proves our conjecture from [4].

Fig. 1 Arms and legs in a Young diagram



1.1 Combinatorial Identities

Here we formulate two combinatorial identities that follow from Theorem 1. We denote by $\mathcal{Y}$ the set of all Young diagrams. For a Young diagram Y let

$$r_l(Y) = \left|\left\{(i,j) \in Y | j = l\right\}\right|,$$
$$c_l(Y) = \left|\left\{(i,j) \in Y | i = l\right\}\right|.$$

For a point $s = (i, j) \in \mathbb{Z}^2_{\geq 0}$ let

$$l_Y(s) = r_j(Y) - i - 1,$$
$$a_Y(s) = c_i(Y) - j - 1,$$

see Fig. 1. Note that $l_Y(s)$ and $a_Y(s)$ are negative, if $s \notin Y$.

The number of boxes in a Young diagram Y is denoted by $|Y|$.

Theorem 2 *Let α and β be two arbitrary positive coprime integers. Then we have*

$$\sum_{Y \in \mathcal{Y}} q^{\sharp\{s \in Y | \alpha l(s) = \beta(a(s)+1)\}} t^{|Y|} = \prod_{\substack{i \geq 1 \\ (\alpha+\beta) \nmid i}} \frac{1}{1-t^i} \prod_{i \geq 1} \frac{1}{1 - q t^{(\alpha+\beta)i}}.$$

In the case $\alpha = \beta = 1$ another identity can be derived from Theorem 1. The q-binomial coefficients are defined by

$$\begin{bmatrix} M \\ N \end{bmatrix}_q = \frac{\prod_{i=1}^{M}(1-q^i)}{\prod_{i=1}^{N}(1-q^i)\prod_{i=1}^{M-N}(1-q^i)}.$$

By $\mathcal{P}$ we denote the set of all partitions. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$, let $|\lambda| = \sum_{i=1}^{r} \lambda_i$.

Theorem 3

$$\sum_{\lambda \in \mathcal{P}} \prod_{i \geq 1} \begin{bmatrix} \lambda_i - \lambda_{i+2} + 1 \\ \lambda_{i+1} - \lambda_{i+2} \end{bmatrix}_q t^{\frac{\lambda_1(\lambda_1-1)}{2} + |\lambda|} = \prod_{i \geq 1} \frac{1}{(1-t^{2i-1})(1-qt^{2i})}.$$

Here for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_r$, we adopt the convention $\lambda_{>r} = 0$.

1.2 Cyclic Quiver Varieties

Quiver varieties were introduced by H. Nakajima in [20]. Here we review the construction in the particular case of cyclic quiver varieties. We follow the approach from [21].

Let $m \ge 2$. We fix vector spaces $V_0, V_1, \dots, V_{m-1}$ and $W_0, W_1, \dots, W_{m-1}$ and we denote by

$$v = (\dim V_0, \dots, \dim V_{m-1}),\ w = (\dim W_0, \dots, \dim W_{m-1}) \in \mathbb{Z}^m_{\ge 0}$$

the dimension vectors. We adopt the convention $V_m = V_0$. Let

$$M(v,w) = \left(\bigoplus_{k=0}^{m-1} \mathrm{Hom}(V_k, V_{k+1})\right) \oplus \left(\bigoplus_{k=0}^{m-1} \mathrm{Hom}(V_k, V_{k-1})\right) \oplus \left(\bigoplus_{k=0}^{m-1} \mathrm{Hom}(W_k, V_k)\right) \oplus \left(\bigoplus_{k=0}^{m-1} \mathrm{Hom}(V_k, W_k)\right).$$

The group $G_v = \prod_{k=0}^{m-1} GL(V_k)$ acts on $M(v,w)$ by

$$g \cdot (B_1, B_2, i, j) \mapsto \left(gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}\right).$$

The map $\mu\colon M(v,w) \to \bigoplus_{k=0}^{m-1} \mathrm{Hom}(V_k, V_k)$ is defined as follows

$$\mu(B_1, B_2, i, j) = [B_1, B_2] + ij.$$

Let

$$\mu^{-1}(0)^s = \left\{ (B,i,j) \in \mu^{-1}(0) \,\middle|\, \begin{array}{c} \text{if a collection of subspaces } S_k \subset V_k \\ \text{is } B\text{-invariant and contains } \mathrm{Im}(i), \text{ then } S_k = V_k \end{array} \right\}.$$

The action of G_v on $\mu^{-1}(0)^s$ is free. The quiver variety $\mathfrak{M}(v,w)$ is defined as the quotient

$$\mathfrak{M}(v,w) = \mu^{-1}(0)^s / G_v,$$

see Fig. 2.

The variety $\mathfrak{M}(v,w)$ is irreducible (see e.g. [21]).

We define the $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^m$-action on $\mathfrak{M}(v,w)$ as follows:

$$(t_1, t_2, e_k) \cdot (B_1, B_2, i_k, j_k) = \left(t_1B_1, t_2B_2, e_k^{-1} i_k, t_1t_2e_k j_k\right).$$

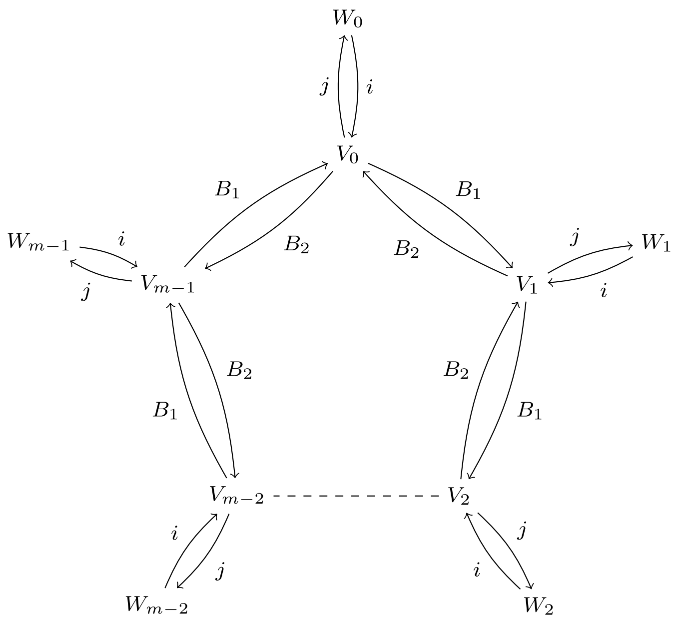


Fig. 2 Cyclic quiver variety $\mathfrak{M}(v, w)$

1.3 $\mathbb{C}^*$*-Action on* $\mathfrak{M}(v, w)$

In this section we formulate Theorem 4 that is a key step in the proofs of Theorems 1 and 5.

Let α and β be any two positive coprime integers, such that $\alpha + \beta = m$. Define the integers $\lambda_0, \lambda_1, \ldots, \lambda_{m-1} \in [-(m-1), 0]$ by the formula $\lambda_k \equiv -\alpha k \mod m$. We define the one-dimensional subtorus $\widetilde{T}_{\alpha,\beta} \subset (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^m$ by

$$\widetilde{T}_{\alpha,\beta} = \left\{\left(t^{\alpha}, t^{\beta}, t^{\lambda_0}, t^{\lambda_1}, \ldots, t^{\lambda_{m-1}}\right) \in \left(\mathbb{C}^*\right)^2 \times \left(\mathbb{C}^*\right)^m | t \in \mathbb{C}^*\right\}.$$

For a manifold X we denote by $H_*^{BM}(X)$ the homology group of possibly infinite singular chains with locally finite support (the Borel-Moore homology) with rational coefficients. Let $P_q^{BM}(X) = \sum_{i \geq 0} \dim H_i^{BM}(X) q^{\frac{i}{2}}$.

Theorem 4 *The fixed point set* $\mathfrak{M}(v, w)^{\widetilde{T}_{\alpha,\beta}}$ *is compact and*

$$P_q^{BM}\left(\mathfrak{M}(v, w)\right) = q^{\frac{1}{2}\dim \mathfrak{M}(v,w)} P_q\left(\mathfrak{M}(v, w)^{\widetilde{T}_{\alpha,\beta}}\right).$$

1.4 Quasihomogeneous Components in the Moduli Space of Sheaves

Here we formulate our result that relates the numbers of quasihomogeneous components in a moduli space of sheaves with characters of the affine Lie algebra $\widehat{sl}_m$.

The moduli space $\mathcal{M}(r,n)$ is defined as follows (see e.g. [22]):

$$\mathcal{M}(r,n)=\left\{(B_1,B_2,i,j)\;\middle|\;\begin{matrix}(1)\ [B_1,B_2]+ij=0\\ (2)\ \text{(stability) There is no subspace}\\ S\subsetneq\mathbb{C}^n \text{ such that } B_\alpha(S)\subset S\ (\alpha=1,2)\\ \text{and } \mathrm{Im}(i)\subset S\end{matrix}\right\}\Big/GL_n(\mathbb{C}),$$

where $B_1,B_2\in\mathrm{End}(\mathbb{C}^n)$, $i\in\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n)$ and $j\in\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)$ with the action of $GL_n(\mathbb{C})$ given by

$$g\cdot(B_1,B_2,i,j)=\left(gB_1g^{-1},gB_2g^{-1},gi,jg^{-1}\right),$$

for $g\in GL_n(\mathbb{C})$.

The variety $\mathcal{M}(r,n)$ has another description as the moduli space of framed torsion free sheaves on the projective plane, but for our purposes the given definition is better. We refer the reader to [22] for details. The variety $\mathcal{M}(1,n)$ is isomorphic to $(\mathbb{C}^2)^{[n]}$ (see e.g. [22]).

Define the $(\mathbb{C}^*)^2\times(\mathbb{C}^*)^r$-action on $\mathcal{M}(r,n)$ by

$$(t_1,t_2,e)\cdot\left[(B_1,B_2,i,j)\right]=\left[\left(t_1B_1,t_2B_2,ie^{-1},t_1t_2ej\right)\right].$$

Consider two positive coprime integers α and β and a vector

$$\boldsymbol{\omega}=(\omega_1,\omega_2,\dots,\omega_r)\in\mathbb{Z}^r$$

such that $0\le\omega_i<\alpha+\beta$. Let $T^{\boldsymbol{\omega}}_{\alpha,\beta}$ be the one-dimensional subtorus of $(\mathbb{C}^*)^2\times(\mathbb{C}^*)^r$ defined by

$$T^{\boldsymbol{\omega}}_{\alpha,\beta}=\left\{\left(t^\alpha,t^\beta,t^{\omega_1},t^{\omega_2},\dots,t^{\omega_r}\right)\in\left(\mathbb{C}^*\right)^2\times\left(\mathbb{C}^*\right)^r|t\in\mathbb{C}^*\right\}.$$

In [4] we studied the numbers of the irreducible components of $\mathcal{M}(r,n)^{T^{\boldsymbol{\omega}}_{\alpha,\beta}}$ and found an answer in the case $\alpha=\beta=1$. Now we can solve the general case.

We define the vector $\boldsymbol{\rho}=(\rho_0,\rho_1,\dots,\rho_{\alpha+\beta-1})\in\mathbb{Z}^{\alpha+\beta}_{\ge0}$ by $\rho_i=\sharp\{j|\omega_j=i\}$ and the vector $\boldsymbol{\mu}\in\mathbb{Z}^{\alpha+\beta}_{\ge0}$ by $\mu_i=\rho_{-i\alpha\bmod\alpha+\beta}$.

Let E_k,F_k,H_k, $k=1,2,\dots,\alpha+\beta$, be the standard generators of $\widehat{sl}_{\alpha+\beta}$. Let $\mathcal{V}$ be the irreducible highest weight representation of $\widehat{sl}_{\alpha+\beta}$ with the highest weight $\boldsymbol{\mu}$. Let $x\in\mathcal{V}$ be the highest weight vector. We denote by $\mathcal{V}_p$ the vector subspace of $\mathcal{V}$ generated by vectors $F_{i_1}F_{i_2}\dots F_{i_p}x$. The character $\chi_{\boldsymbol{\mu}}(q)$ is defined by

$$\chi_{\boldsymbol{\mu}}(q)=\sum_{p\ge0}(\dim\mathcal{V}_p)q^p.$$

We denote by $h_0(X)$ the number of connected components of a manifold X.

Theorem 5

$$\sum_{n\geq 0} h_0\big(\mathcal{M}(r,n)^{T^{\omega}_{\alpha,\beta}}\big)q^n = \chi_{\boldsymbol{\mu}}(q).$$

In [14] the authors found a combinatorial formula for characters of $\widehat{sl}_m$ in terms of Young diagrams with certain restrictions. In [8] the same combinatorics is used to give a formula for certain characters of the quantum continuous gl_∞. Comparing these two combinatorial formulas it is easy to see that Conjecture 1.2 from [4] follows from Theorem 5.

Remark 1 There is a small mistake in Conjecture 1.2 from [4]. The vector $\mathbf{a}' = (a'_0, a'_1, \ldots, a'_{\alpha+\beta-1})$ should be defined by $a'_i = a_{-\alpha i \bmod \alpha+\beta}$. The rest is correct.

1.5 Organization of the Paper

We prove Theorem 4 in Sect. 2. Then using this result we prove Theorem 1 in Sect. 3. In Sect. 4 we derive the combinatorial identities as a corollary of Theorem 1. Finally, using Theorem 4 we prove Theorem 5 in Sect. 5.

2 Proof of Theorem 4

In this section we prove Theorem 4. The Grothendieck ring of quasiprojective varieties is a useful technical tool and we remind its definition and necessary properties in Sect. 2.1.

2.1 Grothendieck Ring of Quasiprojective Varieties

The Grothendieck ring $K_0(\nu_{\mathbb{C}})$ of complex quasiprojective varieties is the abelian group generated by the classes $[X]$ of all complex quasiprojective varieties X modulo the relations:

1. if varieties X and Y are isomorphic, then $[X]=[Y]$;
2. if Y is a Zariski closed subvariety of X, then $[X]=[Y]+[X\backslash Y]$.

The multiplication in $K_0(\nu_{\mathbb{C}})$ is defined by the Cartesian product of varieties: $[X_1]\cdot[X_2]=[X_1\times X_2]$. The class $[\mathbb{A}^1_{\mathbb{C}}]\in K_0(\nu_{\mathbb{C}})$ of the complex affine line is denoted by $\mathbb{L}$.

We need the following property of the ring $K_0(\nu_{\mathbb{C}})$. There is a natural homomorphism of rings $\theta\colon \mathbb{Z}[z]\to K_0(\nu_{\mathbb{C}})$, defined by $\theta(z)=\mathbb{L}$. This homomorphism is an inclusion (see e.g. [18]).

2.2 Proof of Theorem 4

Let $r=\sum_{i=0}^{m-1}\theta_i$. For an arbitrary $\boldsymbol{v}\in\mathbb{Z}^r$ let $\Gamma^{\boldsymbol{v}}_{\alpha,\beta}\subset T^{\boldsymbol{v}}_{\alpha,\beta}$ be the subgroup of roots of 1 of degree m. Let

$$\boldsymbol{\theta}=(\underbrace{0,\dots,0}_{w_0\text{ times}},\underbrace{\lambda_1,\dots,\lambda_1}_{w_1\text{ times}},\dots,\underbrace{\lambda_{m-1},\dots,\lambda_{m-1}}_{w_{m-1}\text{ times}})\in\mathbb{Z}^r.$$

Lemma 1 1. *We have the following decomposition into irreducible components*

$$\mathcal{M}(r,n)^{\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}}=\coprod_{\substack{v\in\mathbb{Z}^m_{\geq 0}\\ \sum v_k=n}}\mathfrak{M}(v,w). \tag{2}$$

2. *The $T^{\boldsymbol{\theta}}_{\alpha,\beta}$-action on the left-hand side of* (2) *corresponds to the $\widetilde{T}_{\alpha,\beta}$-action on the right-hand side of* (2).

Proof Let Γ_m be the group of roots of unity of degree m. By definition, a point $[(B_1,B_2,i,j)]\in\mathcal{M}(r,n)$ is fixed under the action of $\Gamma^{\boldsymbol{\theta}}_{\alpha,\beta}$ if and only if there exists a homomorphism $\lambda\colon\Gamma_m\to GL_n(\mathbb{C})$ satisfying the following conditions:

$$\begin{aligned}\zeta^\alpha B_1&=\lambda(\zeta)^{-1}B_1\lambda(\zeta),\\ \zeta^\beta B_2&=\lambda(\zeta)^{-1}B_2\lambda(\zeta),\\ i\circ\operatorname{diag}\left(\zeta^{\theta_1},\zeta^{\theta_2},\dots,\zeta^{\theta_r}\right)^{-1}&=\lambda(\zeta)^{-1}i,\\ \operatorname{diag}\left(\zeta^{\theta_1},\zeta^{\theta_2},\dots,\zeta^{\theta_r}\right)\circ j&=j\lambda(\zeta),\end{aligned} \tag{3}$$

where $\zeta=e^{\frac{2\pi\sqrt{-1}}{m}}$. Suppose that $[(B_1,B_2,i,j)]$ is a fixed point. Then we have the weight decomposition of $\mathbb{C}^n$ with respect to $\lambda(\zeta)$, i.e. $\mathbb{C}^n=\bigoplus_{k\in\mathbb{Z}/m\mathbb{Z}}V'_k$, where $V'_k=\{v\in\mathbb{C}^n|\lambda(\zeta)\cdot v=\zeta^k v\}$. We also have the weight decomposition of $\mathbb{C}^r$, i.e. $\mathbb{C}^r=\bigoplus_{k\in\mathbb{Z}/m\mathbb{Z}}W'_k$, where $W'_k=\{v\in\mathbb{C}^r|\operatorname{diag}(\zeta^{\theta_1},\dots,\zeta^{\theta_r})\cdot v=\zeta^k v\}$. From conditions (3) it follows that the only components of B_1, B_2, i and j that might survive are:

$$\begin{aligned}B_1\colon V'_k&\to V'_{k-\alpha},\\ B_2\colon V'_k&\to V'_{k-\beta},\\ i\colon W'_k&\to V'_k,\\ j\colon V'_k&\to W'_k.\end{aligned}$$

Let us denote $V'_{-\alpha k\bmod m}$ by V_k and $W'_{-\alpha k\bmod m}$ by W_k. Then the operators B_1,B_2,i,j act as follows: $B_{1,2}\colon V_k\to V_{k\pm1}, i\colon W_k\to V_k, j\colon V_k\to W_k$. The first